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Abstract

Central limit theorems are obtained for the PARMSR (perturbation analysis Robbins-Monro single run) algorithm with averaging, updated either after every regenerative cycle or after every fixed-length observation period, for one-dependent regenerative processes. These stochastic approximation algorithms with averaging turn out to have identical limiting behavior, i.e., the same convergence rate and the same limit covariance matrix, when the convergence is expressed in terms of the total observation time of the system (or the total computing budget in the case of a simulation). Under certain assumptions, these algorithms are asymptotically efficient, in the sense that both their convergence rate and limit covariance are optimal. The strong convergence rate of the usual PARMSR algorithm updated after every fixed length observation period is established using a limit theorem on double array martingales. This is the key step for obtaining the asymptotic efficiency of the algorithms with averaging and has interest in its own right.

Key words. perturbation analysis, asymptotic efficiency, central limit theorems, stochastic approximation, recursive estimation, queueing theory.

Résumé

Nous obtenons des théorèmes de limite centrale pour des algorithmes d'approximation stochastique de type "PARMSR (perturbation analysis Robbins-Monro single run)" avec lissage par la moyenne, pour des systèmes regénératifs au sens large, et où le paramètre à optimiser est mis à jour soit à la fin de chaque cycle regénératif, soit à intervalle fixe. Nous montrons que les propriétés de convergence de ces algorithmes sont les mêmes, i.e., ils ont le même taux de convergence et la même matrice de covariance asymptotique, lorsque la convergence est exprimée en fonction du budget total de calcul (i.e., de la durée totale d'observation du système). Sous certaines hypothèses, ces algorithmes ont un taux de convergence et une matrice de covariance asymptotique optimaux.

1 Introduction

Consider a discrete-time stochastic process $\{J_i(\theta),\ i \geq 0\}$, where $\theta \in \mathbb{R}^l$ is an l-dimensional control parameter, and suppose that we want to minimize the performance measure

$$\overline{J}(\theta) = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \mathsf{E}[J_i(\theta)] \tag{1.1}$$

with respect to θ. Perturbation analysis (PA) offers viable means to estimate $f(\theta) \stackrel{\triangle}{=} d\overline{J}(\theta)/d\theta$ by observing a single sample path of the system (see, e.g., [14], [19], and other references therein). It is natural to treat the current estimate as a noise-contaminated observation of $f(\theta)$, and put it in a stochastic approximation (SA) algorithm to recursively estimate the optimal parameter, while the system is running. When the Robbins-Monro (RM) algorithm (see [29]) is applied, this is referred to as a PARMSR (perturbation analysis Robbins-Monro single run) algorithm by Suri[34] and Suri and Leung[35]. Considerable effort has been devoted, in the recent years, to studying the convergence of the PARMSR algorithm in the field of discrete event dynamic systems (DEDSs); see, e.g., [8]–[10], [13], [21], [22], [24], [36], and [37], among others.

When a PARMSR algorithm is implemented, one concern is how to choose the step-sizes. Consider a classical SA algorithm

$$\theta_{n+1} = \theta_n - a_n f_{n+1} \tag{1.2}$$

with an arbitrary initial value θ_0, where $a_n = A^*/n$ for some matrix A^*,

$$f_{n+1} = f(\theta_n) + \varepsilon_{n+1}$$

is an unbiased estimate of $f(\theta_n)$, θ_n is the nth estimate for the optimizer θ^0, and ε_{n+1} is the observation error at the $(n+1)$-th step. It is well-known that under certain conditions on the regression function $f(\theta)$, on the noise sequence $\{\varepsilon_n,\ n \geq 1\}$, and on the matrix A^*, $\sqrt{n}(\theta_n - \theta^0)$ is asymptotically $N(0,\ S^*)$, i.e., centered normal with some limiting covariance matrix S^*. The trace of S^* is minimized by taking $A^* = M_1^{-1}$, where $M_1 = d\,f(\theta^0)/d\,\theta$ is the Hessian matrix of $\overline{J}(\theta)$ at θ^0. This optimal covariance matrix is $S^* = M_1^{-1} S_0^* (M_1^{-1})'$, where the prime means "transpose" and S_0^* is the asymptotic covariance matrix of $(1/\sqrt{n}) \sum_{j=1}^{n} \varepsilon_j$. But since M_1 is generally unknown, this optimal scheme is usually impracticable.

This has motivated the introduction of SA algorithms with averaging, using a slowly varying gain sequence $\{a_n,\ n \geq 1\}$ which decreases at a rate slower than $1/n$ (see [3], [5], [28], and [39]). One of these algorithms uses (1.2) as usual, then retains the following estimator of the optimizer at step n:

$$\bar{\theta}_{n+1} = \frac{1}{n+1} \sum_{i=1}^{n+1} \theta_i.$$

Under some conditions, $\sqrt{n}(\bar{\theta}_n - \theta^0)$ is asymptotically $N(0,\ S^*)$. A major advantage of the averaged algorithm is that there is no need to know M_1^{-1}. The conditions under which the above results have been proved, however, do not hold in the PARMSR setup. For example, conditional on θ_n, the error ε_{n+1} generally has nonzero expectation β_n, where $\beta_n \not\to 0$, and is correlated with the previous errors $e_n, e_{n-1}, \ldots$.

In this paper, we study the asymptotic efficiency of the PARMSR algorithm with averaging, in the context where the process $\{J_i(\theta),\ i \geq 0\}$ is one-dependent regenerative. This covers a wide class of systems. The PARMSR algorithm observes the process $\{J_i(\theta)\}$, say, for L_n steps with $\theta = \theta_n$, uses this information to obtain a PA gradient estimator f_{n+1}, computes the next parameter value θ_{n+1} from (1.2), continues running the system for another L_{n+1} steps with $\theta = \theta_{n+1}$ to estimate f_{n+2}, and so on. Let $N_n = \sum_{j=1}^{n} L_j$ be the cumulative *computing budget* for the first n steps of the PARMSR algorithm. We shall express the convergence speed of the algorithm in terms of N_n (as done, e.g., in [24]). When $\sqrt{N_n}(\bar{\theta}_n - \theta^0)$ is asymptotically $N(0,\ S^*)$, with $S^* = M_1^{-1} S_0^* (M_1^{-1})'$, where S_0^* is the asymptotic covariance matrix of $(1/\sqrt{N_n}) \sum_{j=1}^{n} \varepsilon_j$ (which is the best possible limit covariance matrix for a given gradient estimator), then for that particular gradient estimator we say that the PARMSR algorithm is *asymptotically optimal*, or *asymptotically efficient* as in [3, 5, 39]. Most limit theorems in the literature of stochastic approximation are, however, expressed in terms of n, the number of iterations, rather than in terms of N_n (see, e.g., [7] and [20]).

We analyze the following two cases:

- (R) The parameter θ_n is updated after each regenerative cycle (so, L_n is random and represents the length of the nth regenerative cycle). In this regenerative case, we use θ_{n-1} instead of θ_n to obtain f_{n+1}, as explained in Section 2.3.

- (F) θ_n is updated after every L steps in the system's evolution (so $L_n = L$, a positive constant).

For case (R), the limiting behavior is relatively easy to analyze since the main part of the observation noise can be decomposed into two martingale difference sequences. Then, standard results on stochastic approximation are applicable. For case (F), the analysis is much more difficult, primarily because the standard conditions on the observation noise, assumed in, e.g., [3], [28] and [39], do not hold. These authors require the observation noise to satisfy the properties of martingale differences, or of stationary ϕ-mixing processes, or of the infinite sum of a martingale difference sequence. But for the PARMSR algorithm with fixed-length observation period, the observation noise has a very complicated dynamic, as shown in previous convergence studies; see, e.g., [9], [10], [21], [22], [24], [36], and [37]. In this paper, we first obtain the strong convergence rate of the usual PARMSR algorithm (without averaging), using a limit theorem on double array martingales taken from [6] and [17]. We then apply this result to obtain the asymptotic efficiency of the PARMSR algorithm with averaging for case (F).

Our main results say that for both (R) and (F), the PARMSR algorithm with averaging is asymptotically efficient. Both cases thus have the same convergence rate and the same limit covariance matrix, in terms of N_n. Moreover, for (F), this limit covariance is independent of the updating frequency L. Our emphasis in this paper is on the case (F). The implementation in this case is much easier, because there is no need to recognize the regeneration points, so it depends much less on the structure of the system. For case (R), the algorithm must identify the regeneration points explicitly. This is usually hard for complex systems. See [30]–[31] on identification of regeneration points for queueing networks.

The rest of the paper is organized as follows. The asymptotic efficiency of the PARMSR algorithm with averaging for case (R) is analyzed in Section 2. For case (F), we begin with $L = 1$ for simplicity of writing. We establish the strong convergence rate of the PARMSR algorithm in Section 3 and obtain the asymptotic efficiency of the algorithm with averaging in Section 4. We then extend the results to the case where $L \geq 1$ in Section 5.

2 Asymptotic Efficiency of the PARMSR Algorithm with Averaging for Case (R)

2.1 Model and Problem Formulation

We begin with the construction of controlled one-dependent regenerative processes. Let

$$\Big\{\{X_i^{(m)},\ i \geq 1\},\ \{\theta_i^{(m)},\ i \geq 1\},\ \{J_i^{(m)},\ i \geq 1\},\ \eta_m,\ \ m \geq 1\Big\} \tag{2.1}$$

be a controlled 4-tuple stochastic process, defined on some common probability space $\{\Omega,\ \mathcal{F},\ P\}$, where for each $m \geq 1$, $\{X_i^{(m)},\ i \geq 1\}$ is a d-dimensional state process, $\{\theta_i^{(m)},\ i \geq 1\}$ is an l-dimensional control parameter process belonging to a compact set $D \subseteq \mathbb{R}^l$, $\{J_i^{(m)},\ i \geq 1\}$ is a $\mathbb{R}^1$-valued cost function process, and η_m is an integer-valued random variable (r.v.). Define

$$\mathcal{F}_i^{(m)} = \sigma\{X_1^{(m)},\ \cdots,\ X_i^{(m)}\}, \tag{2.2}$$

$$\mathcal{F}^{(m)} = \sigma\{X_i^{(r)},\ \ 1 \le i \le \eta_m,\ 1 \le r \le m\}, \tag{2.3}$$

$$\widetilde{\mathcal{F}}_i^{(m)} = \sigma\{\mathcal{F}_i^{(m)} \cup \mathcal{F}^{(m-1)}\}, \quad \forall\ m \ge 1, \tag{2.4}$$

where for all $r \le 0$ we define $\mathcal{F}^{(r)} = \sigma\{X_0,\ \theta_0,\ J_0\}$, a σ-algebra containing the initial information. For each $m \ge 1$, we assume that

(i) for any $i \ge 1$, $\theta_i^{(m)}$ is $\widetilde{\mathcal{F}}_i^{(m)}$-measurable;

(ii) η_m is a stopping time with respect to $\{\mathcal{F}_i^{(m)},\ i \ge 1\}$;

(iii) for any $i \ge 1$, $J_i^{(m)}$ is $\mathcal{F}_i^{(m)}$-measurable.

By (2.1), we can construct a controlled 4-tuple stochastic process

$$\Big(\{X_i,\ i \ge 0\},\ \{\widetilde{\theta}_i,\ i \ge 0\},\ \{J_i,\ i \ge 0\},\ \{\eta_m,\ m \ge 1\}\Big) \tag{2.5}$$

as follows. Define $k_m = 0$ for $m \le 0$ and

$$k_m = \sum_{j=1}^{m} \eta_j \quad \text{for}\ \ m \ge 1, \tag{2.6}$$

and let

$$\left.\begin{aligned} X_{k_m+i} &= X_i^{(m+1)} \\ \widetilde{\theta}_{k_m+i} &= \theta_i^{(m+1)} \\ J_{k_m+i} &= J_i^{(m+1)} \end{aligned}\right\} \qquad \text{for}\ \ 1 \le i \le \eta_{m+1},\ m \ge 0, \tag{2.7}$$

where X_0, $\widetilde{\theta}_0$, and J_0 are initial values.

If for some $m \ge 0$, $\theta_i^{(m+1)} = \theta$ for all $i \ge 1$, where θ is a deterministic parameter in D, we use

$$\Big(\{X_i^{(m+1)}(\theta),\ i \ge 1\},\ \{J_i^{(m+1)}(\theta),\ i \ge 1\},\ \eta_{m+1}(\theta)\Big)$$

to represent

$$\Big(\{X_i^{(m+1)},\ i \ge 1\},\ \{\theta_i^{(m+1)},\ i \ge 1\},\ \{J_i^{(m+1)},\ i \ge 1\},\ \eta_{m+1}\Big).$$

The controlled 4-tuple stochastic process (2.1) is assumed to be *one-dependent regenerative* in the sense that the following condition (A0) is satisfied.

(A0). For any fixed $\theta \in D$ and $m \ge 1$, $\Big(\{X_i^{(m+1)}(\theta),\ i \ge 1\},\ \{J_i^{(m+1)}(\theta),\ i \ge 1\},\ \eta_{m+1}(\theta)\Big)$ is independent of $\mathcal{F}^{(m-1)}$ and η_m, and has the same distribution as $\Big(\{X_i^{(2)},\ i \ge 1\},\ \{J_i^{(2)},\ i \ge 1\},\ \eta_2\Big)$ conditional on $\theta_j^{(2)} = \theta$ for all $j \ge 1$.

If $\theta_i^{(m)} = \theta$ for all $i \geq 1,\ m \geq 1$, we obtain a 3-tuple stochastic process

$$(\{X_i(\theta),\ i \geq 0\},\ \{J_i(\theta),\ i \geq 0\},\ \{\eta_m(\theta),\ m \geq 1\})$$

by defining

$$\left.\begin{aligned} X_{k_m(\theta)+i}(\theta) &= X_i^{(m+1)}(\theta) \\ J_{k_m(\theta)+i}(\theta) &= J_i^{(m+1)}(\theta) \end{aligned}\right\} \quad \text{for} \quad 1 \leq i \leq \eta_{m+1}(\theta),\ m \geq 0, \tag{2.8}$$

where $X_0(\theta)$ and $J_0(\theta)$ are the initial values, $k_m(\theta) = 0$ for $m \leq 0$, and

$$k_m(\theta) = \sum_{j=1}^{m} \eta_j(\theta) \quad \text{for} \quad m \geq 1. \tag{2.9}$$

For a fixed $\theta \in D$, the process $\{X_i(\theta),\ i \geq 0\}$ is *one-dependent regenerative* in the common sense (see, e.g., [32]). The $\{k_m(\theta),\ m \geq 1\}$ are called the *regeneration points* or *regeneration times.* If for all $i \geq 1$ and $\theta \in D$, $X_{k_1(\theta)+i}(\theta)$ and $X_i(\theta)$ also have the same distribution, then the process is called *non-delayed regenerative*. If $\{X_i(\theta),\ k_{m-1}(\theta) < i \leq k_m(\theta)\}$ are also independent of $\{X_{k_m(\theta)+i}(\theta),\ i > 0\}$ for all $m \geq 1$, then the process is called *classically regenerative*. We require that $\{J_i(\theta),\ i \geq 0\}$ is a one-dependent regenerative process with the same regenerative points $\{k_m(\theta),\ m \geq 0\}$ for $\{X_i(\theta),\ i \geq 0\}$. This can be achieved by choosing, e.g., $J_i(\theta) = \phi(X_i(\theta),\ \theta)$ for all $i \geq 0$, where $\phi(\cdot,\ \cdot)$ is a measurable mapping. By Proposition V.1.1 in [1], $\{\phi(X_i(\theta),\ \theta),\ i \geq 0\}$ is a regenerative process with the same regeneration points $\{k_m(\theta),\ m \geq 0\}$.

By the splitting technique from [2] and [26], it is well-known that Harris-recurrent Markov chains (HRMCs) are one-dependent regenerative processes. We refer the reader to [1], [25], [27], and [32] for appropriate background on HRMCs and one-dependent regenerative processes. Under standard rate conditions, Sigman [31] shows that open queueing networks can be modeled by HRMCs and gives explicit regeneration points. Similar results hold for closed queueing networks (see [30]).

The performance measure of interest is the steady-state average

$$\overline{J}(\theta) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} \mathsf{E}[J_i(\theta)], \quad \theta \in D,$$

assuming that this limit exists. Let $\eta_m(\theta) = k_m(\theta) - k_{m-1}(\theta)$ be the length of the mth regenerative cycle, for $m \geq 1$. Without loss of generality, we shall assume henceforth that $\{X_i(\theta),\ i \geq 0\}$ is non-delayed regenerative. If

$$\mathsf{E}[\eta_1(\theta)] < \infty \qquad \text{and} \qquad \mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} |J_i(\theta)|\right] < \infty$$

for all $\theta \in D$, then $\overline{J}(\theta)$ is well defined on D and we have

$$\overline{J}(\theta) = \frac{1}{\mathsf{E}[\eta_1(\theta)]}\mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} J_i(\theta)\right] \tag{2.10}$$

from the renewal-reward theorem (see, e.g., [1] and [38]). The problem is how to find θ^0 such that $\overline{J}(\theta^0) = \min_{\theta \in D} \overline{J}(\theta)$. We are mainly interested in the situation where $\overline{J}(\theta)$ or its gradient are too hard to compute exactly, but where gradient estimators can be computed, either on-line or by simulation.

2.2 Strong Consistency of the Gradient Estimators Using Infinitesimal Perturbation Analysis

We now turn to the sample path gradient $dJ_i(\theta)/d\theta \in R^l$, which is called the *infinitesimal perturbation analysis* (IPA) gradient. The computation of $\{d\,J_i(\theta)/d\theta,\ i \geq 1\}$ has been widely studied; see, e.g., [14], [19] and references therein.

In this paper, we focus exclusively on these systems where $\{d\,J_i(\theta)/d\theta,\ \ i \geq 1\}$ inherits a regenerative structure from $\{J_i(\theta),\ i \geq 1\}$. This means that the regeneration points of $\{d\,J_i(\theta)/d\theta,\ \ i \geq 1\}$ coincide with those of $\{J_i(\theta),\ i \geq 1\}$. A typical condition under which the regenerative structure of $\{J_i(\theta),\ i \geq 1\}$ is preserved by $\{d\,J_i(\theta)/d\theta,\ \ i \geq 1\}$ is that there exists a state of degree one; see [14] and [16]. In the context of queueing systems, suppose that $\{J_i(\theta),\ i \geq 1\}$ is a HRMC and satisfies a recursion of the general form $J_{i+1}(\theta) = \phi(X_{i+1}(\theta),\ \theta)$, $X_{i+1}(\theta) = (W^*_{i+1}(\theta),\ u_{i+1}(\theta))$, where $\phi(\cdot,\ \cdot)$ is a measurable mapping, $\{W^*_i(\theta),\ i \geq 1\}$ is the sequence of waiting times, and $\{u_i(\theta),\ i \geq 1\}$ is the input sequence including the i.i.d. interarrival times and the i.i.d. service times. Then, the regeneration of $\{d\,J_i(\theta)/d\theta,\ \ i \geq 1\}$ is determined by that of $\{d\,X_i(\theta)/d\theta,\ \ i \geq 1\}$. If there is an open set C_X that $\{X_i(\theta),\ i \geq 1\}$ visits infinitely often, and such that $d\,W^*_i(\theta)/d\theta = 0$ when $X_i(\theta) \in C_X$, then the times at which $X_i(\theta)$ visits C_X are regeneration points for $\{X_i(\theta),\ d\,X_i(\theta)/d\theta,\ i \geq 1\}$, provided that $X_i(\theta)$ is absolutely continuous with respect to θ for all $\theta \in D$. We refer the reader to [15] for more details.

In practice, $(1/t)\sum_{i=1}^{t} d\,J_i(\theta)/d\theta$, $t \geq 1$ serve as estimators for $f(\theta)$. To obtain the strong consistency of the estimators, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \frac{d\,J_i(\theta)}{d\theta} = f(\theta), \quad a.s.,$$

we impose the following conditions.

(A1). $\{J_i(\theta),\ i \geq 1\}$ are absolutely continuous with respect to θ on D.

(A2). $\{d\,J_i(\theta)/d\,\theta,\quad i \geq 1\}$ and $\{J_i(\theta),\ i \geq 1\}$ are one-dependent regenerative processes with the same regeneration points $\{k_m(\theta),\ m \geq 0\}$. There is a sequence of one-dependent and identically distributed random variables (r.v.'s) $\{Z_m(\theta),\quad m \geq 1\}$ such that

$$\|\frac{d\,J_{k_m+i}(\theta)}{d\,\theta}\| \leq Z_{m+1}(\theta), \quad 1 \leq i \leq \eta_{m+1}(\theta),\ \theta \in D.$$

(A3). For some $\xi_0 \geq 1$, $\sup_{\theta \in D} \mathsf{E}\left[Z_1(\theta)^{\xi_0}\right] < \infty$ and $\sup_{\theta \in D} \mathsf{E}\left[\eta_1(\theta)^{\xi_0}\right] < \infty$.

Lemma 2.1 *Suppose that conditions (A0)-(A3) are satisfied with $\xi_0 = 2$. Then the IPA derivative estimators are strongly consistent and*

$$f(\theta) = \frac{1}{\mathsf{E}[\eta_1(\theta)]}\mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} \frac{d\ J_i(\theta)}{d\,\theta}\right], \qquad \forall\, \theta \in D. \tag{2.11}$$

Proof. The proof is along the same lines as in [14]-[16].

□

2.3 PARMSR with Update After Every Regenerative Cycle

Since $\eta_1(\theta) \geq 1$, finding a root of $f(\theta)$ is equivalent to finding a root of $f(\theta)\overline{\eta}(\theta)$, where $\overline{\eta}(\theta) = \mathsf{E}[\eta_1(\theta)]$. If explicit regeneration points for the model under consideration are known as a prerequisite, by Lemma 2.1 we can use the information contained in a regenerative cycle to obtain an unbiased estimate for $f(\theta)\overline{\eta}(\theta)$. This motivates the following projected RM algorithm.

$$\theta_{n+1} = \Pi_D(\theta_n - a_n f_{n+1}) = \begin{cases} \theta_n - a_n f_{n+1}, & \text{if } \theta_n - a_n f_{n+1} \in D; \\ \theta_n, & \text{if } \theta_n - a_n f_{n+1} \notin D, \end{cases} \tag{2.12}$$

$$f_{n+1} = \sum_{i=1}^{\eta_{n+1}(\theta_{n-1})} \frac{d\,J_{k_n+i}(\theta_{n-1})}{d\,\theta}, \tag{2.13}$$

$$k_n = \sum_{i=1}^{n} \eta_i(\theta_{i-2}), \tag{2.14}$$

where $\theta_{-1},\ \theta_0 \in D$ are initial values, a_n is a step-size, f_{n+1} is the $(n+1)$-th estimate for $f(\theta_{n-1})\overline{\eta}(\theta_{n-1})$, and Π_D is a projection operator. When D is a closed convex set, we can also define $\Pi_D(x)$ to be the nearest boundary point whenever $x \notin D$. It is worth noticing that the decision parameters throughout the evolution of the $(n+1)$-th regenerative cycle are fixed at θ_{n-1}, rather than θ_n. This is because of the one-dependent nature of our model, and will simplify our convergence analysis of the algorithm. Note that k_n as defined here is not quite the same as $k_n(\theta)$ defined in (2.9) for a fixed θ, because θ changes between the regeneration cycles. In the setting of (2.1), we have

$$\left.\begin{aligned} X_i^{(m+1)} &= X_i^{(m+1)}(\theta_{m-1}) \\ \theta_i^{(m+1)} &= \theta_{m-1} \\ J_i^{(m+1)} &= J_i^{(m+1)}(\theta_{m-1}) \\ \eta_{m+1} &= \eta_{m+1}(\theta_{m-1}) \end{aligned}\right\} \quad \text{for } \ 1 \le i \le \eta_{m+1}(\theta_{m-1}), \ m \ge 0, \tag{2.15}$$

where k_m is defined by (2.6).

For most practical problems, the parameter θ cannot take an arbitrary value. Specifically, we suppose that D is a bounded set in R^l. In the context of queueing systems, for example, D may represent the stability region such that the standard load condition can be fulfilled. In the case where D is unbounded, then a stochastic approximation procedure with randomly varying truncations can be employed to deal with such a problem (see, e.g., [3]–[5], and [7]).

The PARMSR algorithm without averaging for case (R) is composed of (2.12)–(2.14) (cf. [8], [13], [22], and [23]). The observation noise ε_{n+1} can be expressed as

$$\varepsilon_{n+1} = f_{n+1} - f(\theta_n)\overline{\eta}(\theta_n). \tag{2.16}$$

Our aim in this paper is to obtain a central limit theorem for $\{\bar{\theta}_n,\ n \ge 1\}$, where

$$\bar{\theta}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1}\theta_i = \frac{1}{n+1}\left(\theta_{n+1} + n\bar{\theta}_n\right). \tag{2.17}$$

The PARMSR algorithm with averaging for case (R) consists of (2.12)–(2.14) and (2.17).

2.4 Limiting Behavior of the PARMSR Algorithm with Averaging for Case (R)

We first list the following conditions that will be used later on.

(A4). There are constants $\overline{a} > 0$ and $\nu \in (\frac{1}{2},\ 1)$ such that $0 < a_n \le \overline{a}n^{-\nu}$ for all $n \ge 1$; $a_j = \overline{a}$ for all $j \le 0$; $\sum_{n=1}^{\infty} a_n = \infty$; $0 \le a_{n+1}^{-1} - a_n^{-1} \xrightarrow[n\to\infty]{} 0$.

(A5). D is a bounded set and $f(\theta)$ is bounded on D. The optimizer θ^0 is an interior point of D. There are a stable matrix M_1 (all eigenvalues of M_1 have positive real parts) and positive constants r_0 and c_1 such that

$$\|f(\theta) - M_1(\theta - \theta^0)\| \le c_1 \|\theta - \theta^0\|^2, \quad \text{whenever } \|\theta - \theta^0\| \le r_0.$$

(A6). There exists a continuously differentiable function $v : R^l \to R$ such that $v(\theta^0) = 0$ and for all $\Delta_1 > 0$,

$$\sup_{\theta \in D,\, \Delta_1 \le \|\theta - \theta^0\|} (f(\theta))' v_\theta(\theta) > 0.$$

(A7). $f(\theta)$ is Lipschitz with modulus B_1 on D, i.e.,

$$\|f(x_1) - f(x_2)\| \le B_1 \|x_1 - x_2\|, \quad \forall\, x_1,\ x_2 \in D.$$

(A8). $\overline{\eta}(\theta)$ and $H(\theta)$ are continuous at θ^0, where $H(\theta)$ is defined as

$$H(\theta) = \mathsf{E}\left[\left(\sum_{i=1}^{\eta_1(\theta)} d\,J_i(\theta)/d\,\theta\right)\left(\sum_{i=1}^{\eta_1(\theta)} d\,J_i(\theta)/d\,\theta\right)'\right].$$

Let us comment on conditions (A1)–(A8). To establish the strong consistency of the IPA gradient estimators, conditions (A1)–(A3) are standard; see, e.g., [14]-[16] for details. The slow gain condition on the step-sizes is put in condition (A4). Conditions (A5) and (A6) are standard in the context of stochastic approximation. Condition (A7) requires that the regression function $f(\theta)$ is sufficiently smooth. We note that if our model is a classically regenerative process, then the Lipschitz conditions on $f(\theta)$ and $\overline{\eta}(\theta)$ required in Theorems 2.1-2.3 can be dropped since in this situation, θ_{n-1} in (2.13) may be replaced by θ_n and we do not need to decompose ε_{n+1} into two parts (see (2.18)–(2.20)). Condition (A8) requires the continuity of $\overline{\eta}(\theta)$ and $H(\theta)$ at the optimizer θ^0. This is a mild condition.

Theorem 2.1 *Suppose that (i) conditions (A0)-(A6) hold, with $\xi_0 \ge 4$ in (A3); (ii) $\overline{\eta}(\theta)$ is Lipschitz with modulus B_2. Then $\|\theta_n - \theta^0\| = o(a_n^\delta)$, a.s., $\forall\, \delta \in [0,\ 1 - 1/(2\nu))$, where ν is given by condition (A4).*

Proof. We first decompose ε_{n+1} into two parts

$$\varepsilon_{n+1} = \varepsilon_{n+1}^{(1)} + \varepsilon_{n+1}^{(2)}, \tag{2.18}$$

where

$$\varepsilon_{n+1}^{(1)} = \sum_{i=1}^{\eta_{n+1}(\theta_{n-1})} \frac{d\, J_{k_n+i}(\theta_{n-1})}{d\theta} - f(\theta_{n-1})\overline{\eta}(\theta_{n-1}), \tag{2.19}$$

$$\varepsilon_{n+1}^{(2)} = (f(\theta_{n-1}) - f(\theta_n))\overline{\eta}(\theta_{n-1}) + f(\theta_n)(\overline{\eta}(\theta_{n-1}) - \overline{\eta}(\theta_n)). \tag{2.20}$$

With the definition (2.15), we define $\{\mathcal{F}^{(m)},\ m \geq 0\}$ as in (2.3), where $\mathcal{F}^{(m)} = \sigma\{X_0,\ J_0,\ \theta_0,\ \theta_{-1}\}$ for all $m \leq 0$. By Lemma 2.1 and the one-dependence assumption, it is seen that $\{\varepsilon_{2n}^{(1)},\ \mathcal{F}^{(2n)},\ n \geq 1\}$ and $\{\varepsilon_{2n-1}^{(1)},\ \mathcal{F}^{(2n-1)},\ n \geq 1\}$ are martingale difference sequences. By condition (A2), it follows from (2.19) that

$$\|\varepsilon_{n+1}^{(1)}\| \leq \eta_{n+1}(\theta_{n-1})Z_{n+1}(\theta_{n-1}) + \sup_{\theta\in D}\|f(\theta)\| \sup_{\theta\in D}\overline{\eta}(\theta), \tag{2.21}$$

which yields

$$\sup_n \mathsf{E}\left[\|\varepsilon_{n+1}^{(1)}\|^2 \mid \mathcal{F}^{(n-1)}\right] < \infty, \tag{2.22}$$

by Schwarz inequality and condition (A3). Then using the local convergence theorem of martingales (see, e.g., [11] and [33]), it is seen that $\sum_{n=1}^{\infty} a_{2n-1}^{(1-\delta)}\varepsilon_{2n}^{(1)} < \infty$ and $\sum_{n=1}^{\infty} a_{2n-2}^{(1-\delta)}\varepsilon_{2n-1}^{(1)} < \infty$ a.s., which implies that

$$\sum_{i=1}^{\infty} a_{i-1}^{1-\delta}\varepsilon_i^{(1)} < \infty, \quad a.s. \tag{2.23}$$

By Lemma 2 of [36] and condition (A3), one derives

$$a_{n-1}^{(1-\delta)/2}Z_n(\theta_{n-2}) \xrightarrow[n\to\infty]{} 0 \text{ and } a_{n-1}^{(1-\delta)/2}\eta_n(\theta_{n-2}) \xrightarrow[n\to\infty]{} 0 \ a.s., \quad \forall\, \delta \in [0,\ 1 - 1/(2\nu)). \tag{2.24}$$

Using condition (A2) and (2.24), it follows from (2.12)–(2.13) that

$$\|\theta_n - \theta_{n-1}\| \leq a_{n-1}\|f_n\| \leq a_{n-1}^{\delta} \cdot a_{n-1}^{1-\delta}\eta_n(\theta_{n-2})Z_n(\theta_{n-2}) = o(a_n^{\delta}) \quad a.s.,$$

which gives

$$\|\varepsilon_{n+1}^{(2)}\| \leq \|\theta_n - \theta_{n-1}\| \left(B_1 \sup_{\theta\in D}\overline{\eta}(\theta) + B_2 \sup_{\theta\in D}\|f(\theta)\|\}\right) = O(\|\theta_n - \theta_{n-1}\| = o(a_n^{\delta}) \quad a.s., \tag{2.25}$$

via (2.20) and the Lipschitz conditions.

By (2.23) and (2.25), the desired result follows from Theorem 3.2.1 of [7].

□

Theorem 2.2 *Suppose that conditions (A0)-(A8) hold with some* $\xi_0 > 4/\nu$ *and that* $\overline{\eta}(\theta)$ *is Lipschitz on D. Then*

$$\frac{\theta_n - \theta^0}{\sqrt{a_n}} \xrightarrow[n \to \infty]{d} N(0,\ S_1),$$

where $\xrightarrow{d}$ *means convergence in distribution and* $N(0,\ S_1)$ *is the multivariate normal distribution with covariance matrix*

$$S_1 = \int_0^\infty e^{-\overline{\eta}(\theta^0)M_1 t} S e^{-\overline{\eta}(\theta^0)M_1' t} dt$$

and $S = H(\theta^0)$.

Proof. By Theorem 2.1, $\theta_n \xrightarrow[n \to \infty]{} \theta^0$ a.s.. Then, there is a finite time n_0 such that

$$\theta_{n+1} = \theta_n - a_n f_{n+1} \quad \forall\ n \geq n_0. \tag{2.26}$$

Define $\varphi_{0,0} = I$, $\varphi_{n,n+1} = I$, and

$$\varphi_{n,k} = (I + a_n A_n) \cdots (I + a_k A_k)\ \forall\ n \geq k. \tag{2.27}$$

where $\{A_n,\ n \geq 0\}$ is a sequence of deterministic matrices such that $\lim_{n\to\infty} A_n = -M_1$. Then it is standard to derive that

$$\|\varphi_{n,k}\| \leq c_0 \exp\left(-c\sum_{j=k}^{n} a_j\right), \tag{2.28}$$

$$\sup_n \sum_{i=1}^{n} a_i \|\varphi_{n,i+1}\|^{r_1} < \infty, \quad \forall\ r_1 > 0, \tag{2.29}$$

where c_0 and c are some positive constants. From (2.26) one derives

$$\begin{aligned}
&\frac{\theta_{n+1} - \theta^0}{\sqrt{a_{n+1}}} \\
= \;& \varphi_{n,n_0}\frac{\theta_{n_0} - \theta^0}{\sqrt{a_{n_0}}} - \sum_{i=n_0}^{n} \varphi_{n,i+1}\sqrt{a_i}\varepsilon_{i+1}^{(1)} - \sum_{i=n_0}^{n} \varphi_{n,i+1}\sqrt{a_i}\varepsilon_{i+1}^{(2)} \\
& - \sum_{i=n_0}^{n} \varphi_{n,i+1} o(a_i)\sqrt{a_i}\varepsilon_{i+1} - \sum_{i=n_0}^{n} \varphi_{n,i+1}\sqrt{a_i}(1 + o(a_i))O(\|\theta_n - \theta^0\|^2),
\end{aligned} \tag{2.30}$$

where $O(\|\theta_n - \theta^0\|^2)$ means a vector with the same order as $\|\theta_n - \theta\|^2$. In the proof of Theorem 2 of [3], it is shown that $\mathsf{E}[\|\theta_n - \theta^0\|^2] = O(a_n)$, which yields that the last term on the right side of (2.30) converges to zero in probability via (2.28) and (2.29). By the conditions (A2), (A3), and Lemma 2 of [36], it can be shown that

$$\sqrt{a_n}\varepsilon_{n+1} \xrightarrow[n \to \infty]{} 0 \quad a.s., \tag{2.31}$$

which yields that the fourth term on the right side of (2.30) converges to zero a.s.. Similarly, we can prove that the first and the third terms converge to zero a.s.. Then, the desired theorem follows from standard martingale arguments.

□

Theorem 2.3 *Under the conditions of Theorem 2.2, the PARMSR algorithm with averaging for case (R) satisfies*

$$\sqrt{n}(\overline{\theta}_n - \theta^0) \xrightarrow[n \to \infty]{d} N(0,\ S_2),$$

where $S_2 = \overline{\eta}(\theta^0)^{-2} M_1^{-1} S (M_1^{-1})'$.

Proof. The proof uses similar arguments as that of Theorem 2 in [3], combined with a decomposition as in the proof of our Theorem 2.1. Since it is very long and technical, it is omitted.

□

For the first n iterations of the PARMSR algorithm (2.12)-(2.14), the total computing budget is $N_n = \sum_{i=1}^{n} \eta_i(\theta_{i-2})$. By the continuity of $\overline{\eta}(\theta)$ at θ^0 and the law of large numbers for martingales, it is seen that

$$\frac{N_n}{n} \xrightarrow[n \to \infty]{} \overline{\eta}(\theta^0), \quad a.s. \tag{2.32}$$

By (2.32) and Theorem 2.3 we arrive at the following corollary.

Corollary 2.1 *Let the conditions of Theorem 2.2 be satisfied. Then, the PARMSR algorithm with averaging for case (R) satisfies*

$$\sqrt{N_n}(\overline{\theta}_n - \theta^0) \xrightarrow[n \to \infty]{d} N(0,\ S^*),$$

where $S^* = \overline{\eta}(\theta^0)^{-1} M_1^{-1} S (M_1^{-1})'$. *This algorithm is asymptotically optimal.*

3 Convergence Rate of the PARMSR Algorithm with $L = 1$

From now on, we shall explore the limiting behavior of the PARMSR algorithm with update after every fixed length observation period, i.e., $L_n = L$. For simplicity of writing, we begin with the case of $L = 1$. In this section, we establish a strong convergence rate, which will be used for proving the asymptotic optimality of the algorithm with averaging in the next section. We extend our results to $L > 1$ in Section 5.

As in the proof of Theorem 2.1, for the convergence of $\sum_{n=1}^{\infty} a_n^{1-\delta} \varepsilon_{n+1}^{(1)}$, it suffices to prove the convergence of $\sum_{n=1}^{\infty} a_{2n-1}^{1-\delta} \varepsilon_{2n}^{(1)}$ and $\sum_{n=1}^{\infty} a_{2n-2}^{1-\delta} \varepsilon_{2n-1}^{(1)}$, where $\{\varepsilon_{2n}^{(1)},\ \mathcal{F}^{(2n)},\ n \geq 1\}$ and $\{\varepsilon_{2n-1}^{(1)},\ \mathcal{F}^{(2n-1)},\ n \geq 1\}$ are martingale difference sequences. Such a technique is standard when one wants to extend some results on classically regenerative processes to the one-dependent regenerative processes. Hence, no generality is lost by supposing that $\{J_n(\theta),\ n \geq 1\}$ is a classically regenerative process, for simplicity of writing.

The parameter θ is updated by the projected RM algorithm (2.12), where f_{n+1} is the $(n+1)$-th step estimate for $f(\theta)$. The computation of f_{n+1} is based on the *perturbation propagation rule* of IPA, though the control parameters are changed frequently. The concrete form of f_{n+1} may be very complicated, and needs to be analyzed on a case by case basis. Let η_{m+1} be the $(m+1)$-th regenerative cycle length. In the $(m+1)$-th regenerative cycle, suppose that f_{k_m+i} has the form

$$f_{k_m+i} = Y_{k_m+i}(\theta_{k_m},\ \theta_{k_m+1},\ \cdots,\ \theta_{k_m+i-1}), \quad \text{for } 1 \leq i \leq \eta_{m+1}, \tag{3.1}$$

where $k_m = \sum_{i=1}^{m} \eta_i$. If the control parameters are fixed at θ throughout the $(m+1)$-th regenerative cycle, we denote the regenerative cycle length by $\eta_{m+1}(\theta)$, and for $1 \leq i \leq \eta_{m+1}$, $\theta \in D$, we have

$$Y_{k_m+i}(\theta,\ \theta,\ \ldots,\ \theta) = \frac{d\, J_{k_m+i}(\theta)}{d\,\theta}. \tag{3.2}$$

For later use, we define the r.v.

$$D_{m,i} = \left. \frac{d\, J_{k_m+i}(\theta)}{d\,\theta} \right|_{\theta=\theta_{k_m}} = Y_{k_m+i}(\theta_{k_m},\ \ldots,\ \theta_{k_m}),$$

which is the value of f_{k_m+i} obtained if we assume that θ_{k_m+j} is fixed at θ_{k_m} for all $j \geq 0$. The PARMSR algorithm with observation period 1 consists of (2.12), (3.1), and (3.2) (see, e.g., [9], [10], [36] and [37]). The observation noise can be written as

$$\varepsilon_{n+1} = f_{n+1} - f(\theta_n). \tag{3.3}$$

When this algorithm is employed, in the setting of (2.1) we have

$$\theta_i^{(m)} = \theta_{k_m+i} \qquad \text{for } 1 \leq i \leq \eta_{m+1}, \tag{3.4}$$

where the control parameter process $\{\theta_n,\ n \geq 1\}$ is defined by the PARMSR algorithm. Note that (3.1) means that f_{k_m+i} depends on the parameters $\theta_{k_m}, \theta_{k_m+1}, \cdots, \theta_{k_m+i-1}$, and on the randomness involved in $\{X_1^{(m+1)},\ X_2^{(m+1)},\ \cdots,\ X_i^{(m+1)}\}$.

We use the following conditions.

(A9). There exist a parameter $\theta^* \in D$ and a sequence of one-dependent and identically distributed r.v.'s $\{Z_{m+1}^{(0)},\ m \geq 0\}$, with $\mathsf{E}[(Z_1^{(0)})^{6/\nu}] < \infty$, such that for all $\theta \in D$ and $1 \leq i \leq \eta_{m+1}$, a.s.,

$$\eta_{m+1} \leq \eta_{m+1}(\theta^*), \tag{3.5}$$

$$\|Y_{k_m+i}(\theta_{k_m},\ \theta_{k_m+1},\ \cdots,\ \theta_{k_m+i-1})\| \leq Z_{m+1}(\theta^*), \tag{3.6}$$

$$\begin{aligned} &\|Y_{k_m+i}(\theta_{k_m},\ \theta_{k_m+1},\ \cdots,\ \theta_{k_m+i-1}) - Y_{k_m+i}(\theta,\ \theta,\ \cdots,\ \theta)\| \\ &\qquad \leq Z_{m+1}^{(0)} \max_{1 \leq i \leq \eta_{m+1}} \|\theta_{k_m+i} - \theta\|. \end{aligned} \tag{3.7}$$

(A10). There are two positive constants α_0 and γ_1 such that $P\{\eta_{m+1} \neq \eta_{m+1}(\theta_{k_m})|\mathcal{F}^{(m)}\} \leq \alpha_0 a_{k_m}^{\gamma_1}$ where the filtration $\mathcal{F}^{(m)}$ is defined by (2.3).

(A11). $\xi_0 = \max\{2/\zeta,\ 4/\nu,\ 2p_1\}$, where ν is given by (A4), $p_1 > 1$ and $\zeta > 0$ are some constants such that

$$\nu\left(1 + \gamma_1(1 - \frac{1}{p_1})\right) > 1, \quad 0 < \zeta < \delta'\nu, \quad \gamma_1(1 - \frac{1}{p_1}) \geq \frac{1}{2},\ \delta' \in (0,\ \frac{1}{2}].$$

Note that γ_1 is given by (A10).

By putting some conditions on the service times and the interarrival times, conditions (A9) and (A10) have been verified for the $GI/G/1$ queueing systems; see, e.g., [9], [10], [36] and [37] for details. Suppose that the distribution of the interarrival times has a bounded density function, then (A10) can be verified with some constants $\alpha_0 > 0$ and $\gamma_1 \in (0,\ 1)$, where γ_1 can be arbitrarily close to 1. We also note that Chong and Ramadge[10] have checked conditions (A9) and (A10) for several classically regenerative systems, though the convergence rate of the PARMSR algorithms have not been studied there.

Before stating our main results in this section, we introduce two lemmas.

Lemma 3.1 *Suppose that $\{z_i\}$ is a sequence of r.v.s with the same distribution. Then for any $r > 0$, $\mathsf{E}|z_1|^r < \infty$ implies*

$$\lim_{n\to\infty} n^{-1/r} z_n = 0 \quad a.s.$$

Furthermore, if $\{z_i,\ i \geq 1\}$ are mutually independent, then the converse is true.

Proof. The proof follows essentially from the Borel-Cantelli lemma and Corollary 4.1.3 in [12], pp. 90-91.

□

Lemma 3.2 *Suppose that $\{z_n, \mathcal{B}_n^*\}$ is an l-dimensional martingale difference sequence satisfying*

$$\sup_n \mathsf{E}[\|z_{n+1}\|^2|\mathcal{B}_n^*] < \infty, \quad \|z_n\| = o(h(n)), \ a.s., \ h(n) \leq h(n+1), \qquad \forall\, n \geq 0,$$

and that $g_{n,i}$ is an $\mathcal{B}_i^*$-measurable $l \times l$-dimensional random matrix, for $1 \leq i \leq n$, which satisfies

$$\sum_{i=1}^{n} \|g_{n,i}\|^2 \leq \overline{g} < \infty, \quad a.s., \ \forall\, n \geq 1,$$

where $h(n)$ and $\overline{g}$ are positive constants. Then, as $n \to \infty$,

$$\max_{1 \leq i \leq n} \|\sum_{j=1}^{i} g_{n,j} z_{j+1}\| = o(h(n+1) \log n) \quad a.s.$$

Proof. The proof can be found in Guo, Huang, and Hannan[17]. See also [6].

□

Theorem 3.1 *Suppose that conditions (A0)–(A10) hold with $\xi_0 \geq \max\{4,\ 2p_1\}$, $\nu(1+\gamma_1(1-1/p_1)) > 1$ where $p_1 > 1$ is some constant, and γ_1 is given by condition (A10). Then $\theta_n \xrightarrow[n \to \infty]{} \theta^0$ a.s..*

Proof. The key idea lies in verifying that $\sum_{n=1}^{\infty} a_n \varepsilon_{n+1}$ converges a.s.. Then the desired result follows from Theorem 3.1 in [4] (see also Theorem 2.4.1 in [7]).

(i). We first show that $\sum_{m=1}^{\infty} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \varepsilon_{k_m+i}$ converges a.s.. From (3.1)–(3.3) it is easy to see that

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \varepsilon_{k_m+i} \\ = &\sum_{m=1}^{\infty} \sum_{i=1}^{\eta_{m+1}} (a_{k_m+i-1} - a_{k_m}) \varepsilon_{k_m+i} + \sum_{m=1}^{\infty} a_{k_m} \sum_{i=1}^{\eta_{m+1}(\theta_{k_m})} (D_{m,i} - f(\theta_{k_m})) \end{aligned}$$

$$+\sum_{m=1}^{\infty} a_{k_m} \sum_{i=1}^{\eta_{m+1}} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) + \sum_{m=1}^{\infty} a_{k_m} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i})$$

$$+\sum_{m=1}^{\infty} a_{k_m} \sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}} (D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} > \eta_{m+1}(\theta_{k_m})\}$$

$$-\sum_{m=1}^{\infty} a_{k_m} \sum_{i=\eta_{m+1}+1}^{\eta_{m+1}(\theta_{k_m})} (D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} < \eta_{m+1}(\theta_{k_m})\}. \tag{3.8}$$

As in Lemma 3.4 of [37], we can prove that each term on the right hand side of (3.8) converges a.s.. The details are omitted here.

(ii). By the result of step (i), analogous to Lemma 3.5 of [37], we can prove the almost sure convergence of $\sum_{n=1}^{\infty} a_n \varepsilon_{n+1}$.

□

Theorem 3.2 *Suppose that conditions (A0)-(A11) hold. Then, for the PARMSR algorithm with observation period $L = 1$, the following strong convergence rate is achieved:*

$$\|\theta_n - \theta^0\| = o(a_n^{1/2-\delta'}) \quad a.s.,$$

where δ' is any constant satisfying condition (A11).

Remark 3.1 *We note that δ' in Theorem 3.2 is independent of ν. Thus $1/2-\delta'$ can be arbitrarily close to $1/2$, if condition (A3) is fulfilled with very large ξ_0. This is different from Theorem 2.1 in Section 2 and Theorem 2.2 in [37], where $\|\theta_n - \theta^0\| = o(a_n^{\delta})$ a.s. with $\delta \in [0,\ 1-1/(2\nu))$ depending on ν. In particular, in the latter setup, if ν is close to $1/2$, then δ must be close to zero.*

For $n \geq 0$, define

$$\psi_{n,i} = \begin{cases} (I - a_n M_1) \cdots (I - a_i M_1) & \text{for } i \leq n; \\ I & \text{for } i = n+1; \\ 0 & \text{for } i \geq n+2; \end{cases} \tag{3.9}$$

$$\sigma(n) = \max\{m : \ k_m \leq n\}, \quad \tau(n) = \sigma(n) + 1. \tag{3.10}$$

To prove Theorem 3.2, we use the following lemma.

Lemma 3.3 *If conditions (A0)-(A11) are satisfied, with δ' satisfying (A11), then*

$$a_{n+1}^{\delta'-1/2} \sum_{j=0}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \xrightarrow[n \to \infty]{} 0, \quad a.s.$$

Proof. Since $-M_1$ is stable, it is standard to derive that (see, e.g., [3] and [7]):

$$\|\psi_{n,k}\| \leq c_0 \exp\left(-c \sum_{j=k}^{n} a_j\right), \tag{3.11}$$

$$\sup_n \sum_{i=1}^{n} a_i \exp\left(-rc \sum_{j=i+1}^{n} a_j\right) < \infty, \quad \sup_n \sum_{i=1}^{n} a_i \|\varphi_{n,i+1}\|^r < \infty, \quad \forall\, r > 0, \tag{3.12}$$

$$\frac{a_i}{a_n} = \exp\left(o(1) \sum_{s=i}^{n-1} a_s\right), \quad \forall\, n \geq i, \tag{3.13}$$

where c_0 and c are some constants and $o(1) \xrightarrow[i \to \infty]{} 0$.

Using (3.1)-(3.3) and (3.10), we have

$$\begin{aligned}
& a_{n+1}^{\delta'-1/2} \sum_{j=0}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \\
= \;& a_{n+1}^{\delta'-1/2} \sum_{j=k_{\sigma(n)}+1}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - f(\theta_{k_m+i-1})) \\
= \;& a_{n+1}^{\delta'-1/2} \sum_{j=k_{\sigma(n)}+1}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\
& + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - D_{m,i}) \\
& + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (a_{k_m+i-1} - a_{k_m}) \psi_{n,k_m+i} (D_{m,i} - f(\theta_{k_m})) \\
& + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m} (\psi_{n,k_m+i} - \psi_{n,k_m}) (D_{m,i} - f(\theta_{k_m})) \\
& + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} a_{k_m} \psi_{n,k_m} w_{m+1}
\end{aligned}$$

$$
\begin{aligned}
&+a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1} a_{k_m}\psi_{n,k_m}\sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}>\eta_{m+1}(\theta_{k_m})\}\\
&-a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1} a_{k_m}\psi_{n,k_m}\sum_{i=\eta_{m+1}+1}^{\eta_{m+1}(\theta_{k_m})}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}<\eta_{m+1}(\theta_{k_m})\},
\end{aligned}
\tag{3.14}
$$

where

$$
w_{m+1}=\sum_{i=1}^{\eta_{m+1}(\theta_{k_m})}(D_{m,i}-f(\theta_{k_m})). \tag{3.15}
$$

We will show that each term on the right hand side of the second equality in (3.14) converges to zero a.s.. Then, the desired result follows from (3.14).

(i). By condition (A4), there is a constant α_1 such that

$$
\max_{1\le i\le\eta_{m+1}}\left(\frac{a_{k_m}}{a_{k_m+i}}\right)\le 1+\alpha_1 a_{k_m}\eta_{m+1}(\theta^*)\quad\text{and}\quad \max_{1\le i\le\eta_{m+1}}|a_{k_m+i}-a_{k_m}|\le\alpha_1 a_{k_m}^2\eta_{m+1}(\theta^*). \tag{3.16}
$$

Using (3.6), it follows from (3.3) that

$$
\|\varepsilon_{k_m+i}\|\le W_{m+1}^{(0)}\triangleq\max_{\theta\in D}\|f(\theta)\|+Z_{m+1}(\theta^*),\quad\text{for }\ 1\le i\le\eta_{m+1}. \tag{3.17}
$$

By Lemma 3.1 and conditions (A3) and (A11), one has

$$
\frac{1}{\sqrt{m^v}}\eta_{m+1}(\theta^*)W_{m+1}^{(0)}\xrightarrow[m\to\infty]{}0\quad a.s., \tag{3.18}
$$

which yields

$$
\lim_{n\to\infty}\sqrt{a_{k_{\sigma(n)}}}\eta_{\sigma(n)+1}(\theta^*)W_{\sigma(n)+1}^{(0)}\xrightarrow[n\to\infty]{}0\quad a.s. \tag{3.19}
$$

By (3.11), (3.16) and (3.17), we have

$$
\begin{aligned}
\frac{a_{n+1}^{\delta'}}{\sqrt{a_{n+1}}}\left\|\sum_{j=k_{\sigma(n)}+1}^{n}a_j\psi_{n,j+1}\varepsilon_{j+1}\right\| &\le \frac{a_{n+1}^{\delta'}}{\sqrt{a_{n+1}}}c_0 a_{k_{\sigma(n)}}W_{\sigma(n)+1}^{(0)}\eta_{\sigma(n)+1}(\theta^*)\\
&\le c_0 a_{n+1}^{\delta'}\left(\frac{a_{k_{\sigma(n)}}}{a_{k_{\sigma(n)+1}}}\right)^{1/2}a_{k_{\sigma(n)}}^{1/2}\eta_{\sigma(n)+1}(\theta^*)W_{\sigma(n)+1}^{(0)}\\
&\le c_0 a_{n+1}^{\delta'}(1+\alpha_1 a_{k_{\sigma(n)}}\eta_{\sigma(n)+1}(\theta^*))^{1/2}\sqrt{a_{k_{\sigma(n)}}}\eta_{\sigma(n)+1}(\theta^*)W_{\sigma(n)+1}^{(0)}\\
&\xrightarrow[n\to\infty]{}0\quad\text{a.s.},
\end{aligned}
$$

which implies that the first term on the right hand side of the second equality in (3.14) converges to zero as $n \to \infty$, a.s.

(ii). Using (3.5) and (3.6), it follows from (2.12) and (3.1) that

$$\|\theta_{k_m+i} - \theta_{k_m}\| \le \sum_{j=1}^{i} \|\theta_{k_m+j} - \theta_{k_m+j-1}\| \le a_{k_m} W_{m+1}, \quad \text{for } 1 \le i \le \eta_{m+1}, \tag{3.20}$$

where

$$W_{m+1} = \eta_{m+1}(\theta^*) Z_{m+1}(\theta^*). \tag{3.21}$$

By condition (A7), (3.16) and (3.20), it is derived that

$$\begin{aligned}
& a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i}(f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \right\| \\
\le\ & B_1 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \frac{1}{\sqrt{a_{n+1}}} a_{k_m} W_{m+1} \\
\le\ & B_1 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \left(\frac{a_{k_m+i}}{a_{n+1}}\right)^{1/2} \max_{1 \le i \le \eta_{m=1}} \left(\frac{a_{k_m}}{a_{k_m+i}}\right)^{1/2} \sqrt{a_{k_m}} W_{m+1} \\
\le\ & B_1 c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j\right) \\
& \cdot (1 + \alpha_1 a_{k_m} \eta_{m+1}(\theta^*))^{1/2} \sqrt{a_{k_m}} W_{m+1} \\
& \xrightarrow[n \to \infty]{} 0 \quad a.s.,
\end{aligned} \tag{3.22}$$

since by (3.12)-(3.13)

$$\begin{aligned}
& \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j\right) \\
\le\ & \sum_{i=0}^{n} a_i \exp\left(-c \sum_{j=i+1}^{n} a_j\right) \exp\left(o(1) \sum_{j=i+1}^{n} a_j\right) \quad = \quad O(1) \quad a.s.,
\end{aligned}$$

and by (3.18)

$$(1 + \alpha_1 a_{k_m} \eta_{m+1}(\theta^*))^{1/2} \sqrt{a_{k_m}} W_{m+1} \xrightarrow[m \to \infty]{} 0 \quad a.s.$$

Thus, the second term on the right hand side of the second equality in (3.14) converges to zero a.s., as $n \to \infty$.

(iii). By (3.7) and (3.20) we get

$$\begin{aligned}\|f_{k_m+i} - D_{m,i}\| &\leq Z_{m+1}^{(0)} \max_{1\leq i\leq \eta_{m+1}} \|\theta_{k_m+i} - \theta_{k_m}\| \\ &\leq a_{k_m} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) Z_{m+1}(\theta^*), \quad \text{for } 1 \leq i \leq \eta_{m+1}. \end{aligned} \tag{3.23}$$

Similar to (3.18), by Lemma 3.1 it follows that

$$\sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) Z_{m+1}(\theta^*) \xrightarrow[m\to\infty]{} 0 \quad \text{a.s.}. \tag{3.24}$$

By the same argument as in (3.22), it is seen that

$$\begin{aligned} & a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - D_{m,i}) \right\| \\ \leq\ & a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \left(\frac{a_{k_m+i}}{a_{n+1}}\right)^{1/2} \max_{1\leq i\leq \eta_{m+1}} \left(\frac{a_{k_m}}{a_{k_m+i}}\right)^{1/2} \\ & \cdot \sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) Z_{m+1}(\theta^*) \\ \leq\ & c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j\right) \\ & \cdot (1 + \alpha_1 a_{k_m} \eta_{m+1}(\theta^*))^{1/2} \sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) Z_{m+1}(\theta^*) \\ & \xrightarrow[n\to\infty]{} 0 \quad a.s., \end{aligned} \tag{3.25}$$

which implies that the third term on the right hand side of the second inequality in (3.14) converges to zero as $n \to \infty$, a.s.

(iv). Analogous to (3.22), by (3.16) it can be shown that

$$\begin{aligned} & a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (a_{k_m+i-1} - a_{k_m}) \psi_{n,k_m+i} (D_{m,\,i} - f(\theta_{k_m})) \right\| \\ \leq\ & \frac{\alpha_1}{a_{n+1}^{1/2-\delta'}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m}^2 \|\psi_{n,k_m+i}\| \eta_{m+1}(\theta^*) W_{m+1}^{(0)} \\ \leq\ & \alpha_1 c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \left(\frac{a_{k_m+i}}{a_{n+1}}\right)^{1/2} \max_{1\leq i\leq \eta_{m+1}} \left(\frac{a_{k_m}^{3/2}}{a_{k_m+i-1}\sqrt{a_{k_m+i}}}\right) \\ & \cdot \sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) W_{m+1}^{(0)} \end{aligned}$$

$$\leq \alpha_1 c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j\right)$$
$$\cdot (1+\alpha_1 a_{k_m}\eta_{m+1}(\theta^*))^{3/2}\sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) W_{m+1}^{(0)}$$
$$\xrightarrow[n\to\infty]{} 0, \quad a.s. \tag{3.26}$$

This means that the fourth term on the right hand side of the second inequality in (3.14) converges to zero as $n \to \infty$, a.s.

(v). By the definition (3.9), we have

$$\psi_{n,k_m+i} - \psi_{n,k_m} = \sum_{j=1}^{i} a_{k_m+j-1}\psi_{n,k_m+j} M_1 = \psi_{n,k_m+i} \sum_{j=1}^{i} a_{k_m+j-1}\psi_{k_m+i-1,k_m+j} M_1. \tag{3.27}$$

Similar to (3.26), it is derived that

$$a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m}(\psi_{n,k_m+i} - \psi_{n,k_m})(D_{m,i} - f(\theta_{k_m})) \right\|$$
$$\leq c_0 \|M_1\| a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m}^2 \|\psi_{n,k_m+i}\| \eta_{m+1}(\theta^*) W_{m+1}^{(0)}$$
$$\leq c_0^2 \|M_1\| a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j\right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j\right)$$
$$\cdot (1+\alpha_1 a_{k_m}\eta_{m+1}(\theta^*))^{3/2}\sqrt{a_{k_m}} Z_{m+1}^{(0)} \eta_{m+1}(\theta^*) W_{m+1}^{(0)}$$
$$\xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}, \tag{3.28}$$

which means that the fifth term on the right hand side of the second inequality in (3.14) converges to zero as $n \to \infty$, a.s.

(vi). We now consider the convergence of the sixth term. First, by the definition (3.9) it is seen that

$$\sum_{m=0}^{\tau(n)} \psi_{n,k_m} a_{k_m} w_{m+1} = \sum_{m=0}^{n+1} a_{k_m} \psi_{n,k_m} w_{m+1}. \tag{3.29}$$

Let $h(n) = n^{\zeta}, \forall\, n \geq 1$, where ζ is some constant satisfying $0 < \zeta < \delta'\nu$. Then by condition (A11) and Lemma 2.1 we have

$$\frac{\|w_{n+1}\|}{h(n)} \leq \frac{W_{n+1}^{(0)} \eta_{n+1}(\theta^*)}{n^{\zeta}} \xrightarrow[n\to\infty]{} 0, \quad a.s. \tag{3.30}$$

By (3.11) and (3.13), it is derived that

$$\begin{aligned}\frac{1}{a_{n+1}}\sum_{m=0}^{n+1}a_{k_m}^2\|\psi_{n,k_m}\|^2 &\leq \frac{1}{a_{n+1}}\sum_{m=0}^{n+1}a_{k_m-1}a_{k_m}\|\psi_{n,k_m}\|^2\\ &\leq \frac{1}{a_{n+1}}\sum_{i=0}^{n+1}a_{i-1}\|\psi_{n,i}\|^2 a_i\\ &\leq c_0^2\sum_{i=0}^{n+1}\exp\left(-2c\sum_{j=i}^{n}a_j\right)a_{i-1}\exp\left(o(1)\sum_{j=i}^{n}a_j\right)\\ &= O(1), \end{aligned} \tag{3.31}$$

which, incorporating with (3.30) and Lemma 3.2, leads to

$$\frac{1}{\sqrt{a_{n+1}}}\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}w_{m+1} = O(1)+o(h(n)\log n). \tag{3.32}$$

By (3.32), it follows from (3.29) that

$$\begin{aligned}a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\tau(n)}a_{k_m}\psi_{n,k_m}w_{m+1} &= O(a_{n+1}^{\delta})+o\left(a_{n+1}^{\delta'}h(n)\log n\right)\\ &= o\left(\frac{1}{n^{\delta'\nu-\zeta}}\log n\right)\\ &= o(1) \quad \text{a.s.,}\end{aligned}$$

which implies that

$$a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1}a_{k_m}\psi_{n,k_m}w_{m+1}\xrightarrow[n\to\infty]{}0, \quad a.s.,$$

since by (3.11), (3.16), and (3.18)–(3.19) we have

$$\begin{aligned}&\frac{a_{n+1}^{\delta}}{\sqrt{a_{n+1}}}\|\psi_{n,k_{\sigma(n)}}a_{k_{\sigma(n)}}w_{\sigma(n)+1}+\psi_{n,k_{\tau(n)}}a_{k_{\tau(n)}}w_{\tau(n)+1}\|\\ \leq\ & c_0a_{n+1}^{\delta}\Big((1+\alpha_1a_{k_{\sigma(n)}}\eta_{\sigma(n)+1}(\theta^*))^{1/2}\sqrt{a_{k_{\sigma(n)}}}\eta_{\sigma(n)+1}(\theta^*)W_{\sigma(n)+1}^{(0)}\\ &\qquad+(\frac{a_n}{a_{n+1}})^{1/2}\sqrt{a_{k_{\tau(n)}}}\eta_{\tau(n)+1}(\theta^*)W_{\tau(n)+1}^{(0)}\Big)\\ &\xrightarrow[n\to\infty]{}0, \quad a.s.\end{aligned}$$

(vii). Set

$$V_{m+1}^{(0)}=\sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}>\eta_{m+1}(\theta_{k_m})\},\quad \forall\ m\geq 0.$$

By (3.30), we get

$$\frac{V_m^{(0)}-\mathsf{E}[V_m^{(0)}|\mathcal{F}^{(m-1)}]}{h(m)}\xrightarrow[m\to\infty]{}0,\quad a.s.,\tag{3.33}$$

which incorporating with (3.31) and Lemma 3.2 yields

$$\frac{1}{\sqrt{a_{n+1}}}\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}(V_{m+1}^{(0)}-\mathsf{E}[V_{m+1}^{(0)}|\mathcal{F}^{(m)}])=O(1)+o(h(n)\log n)=o(h(n)\log n),\quad a.s.\tag{3.34}$$

By Hölder inequality and condition (A10) for any $p_1>1$ we have

$$\begin{aligned}\|\mathsf{E}[V_{m+1}^{(0)}|\mathcal{F}^{(m)}]\| &\leq \mathsf{E}[\eta_{m+1}(\theta^*)W_{m+1}^{(0)}I\{\eta_{m+1}>\eta_{m+1}(\theta_{k_m})\}|\mathcal{F}^{(m)}]\\ &\leq \left(\mathsf{E}\left[(W_{m+1}^{(0)}\eta_{m+1}(\theta^*))^{p_1}\right]\right)^{1/p_1}P\{\eta_{m+1}>\eta_{m+1}(\theta_{k_m})|\mathcal{F}^{(m)}\}^{1-1/p_1}\\ &\leq \alpha_0^{1-1/p_1}\left(\mathsf{E}\left[(W_{m+1}^{(0)}\eta_{m+1}(\theta^*))^{p_1}\right]\right)^{1/p_1}a_{k_m}^{\gamma_1(1-1/p_1)},\end{aligned}\tag{3.35}$$

which leads to

$$\begin{aligned}&\frac{1}{a_{n+1}^{1/2-\delta}}\|\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}\mathsf{E}[V_{m+1}^{(0)}|\mathcal{F}^{(m)}]\|\\ \leq\ &\alpha_0^{1-1/p_1}\sup_m\left\{\left(\mathsf{E}\left[(W_{m+1}^{(0)}\eta_{m+1}(\theta^*))^{p_1}\right]\right)^{1/p_1}\right\}\frac{1}{a_{n+1}^{1/2-\delta}}\sum_{m=0}^{n+1}\|\psi_{n,k_m}\|a_{k_m-1}a_{k_m}^{\gamma_1(1-1/p_1)}\\ \leq\ &O(1)a_{n+1}^{\delta'}\sum_{i=-1}^{n+1}a_i\exp\left(-c\sum_{j=i+1}^{n}a_j\right)\left(\frac{a_{i+1}}{a_{n+1}}\right)^{1/2}a_{i+1}^{\gamma_1(1-1/p_1)-1/2}\\ =\ &o(1)\quad a.s.,\end{aligned}\tag{3.36}$$

provided $\gamma_1(1-1/p_1)\geq 1/2$.

Combining (3.34) and (3.36) gives

$$\frac{1}{a_{n-1}^{1/2-\delta}}\|\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}V_{m+1}^{(0)}\|=o(1)+o(a_{n+1}^{\delta'}h(n)\log n)=o(1)\quad a.s.,$$

where $0<\zeta<\delta'\nu$.

(viii). By the same way as in (vii), it can be shown that the last term on the right hand side of the second equality in (3.14) converges to zero, a.s.

□

Proof of Theorem 3.2:

By Theorem 3.1, there is a finite r.v. n_0^* such that after n_0^* steps the algorithm (2.12) will become the usual RM algorithm, i.e.,

$$\theta_{n+1} = \theta_n - a_n(f(\theta_n) + \varepsilon_{n+1}), \quad \forall\, n \geq n_0^*. \tag{3.37}$$

Then, for a deterministic integer n_0, on the event $\{n \geq n_0 \geq n_0^*\}$, one has

$$\begin{aligned} \frac{\theta_{n+1} - \theta^0}{a_{n+1}^{1/2-\delta'}} &= a_{n+1}^{\delta'-1/2}\psi_{n,n_0}(\theta_{n_0} - \theta^0) - a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\psi_{n,j+1}a_j\varepsilon_{j+1} \\ &\quad - a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\psi_{n,j+1}a_j(f(\theta_j) - M_1(\theta_j - \theta^0)). \end{aligned} \tag{3.38}$$

Let $r \leq r_0$, where r_0 is given by condition (A5). Define

$$\sigma^* = \begin{cases} \inf\{j: \ \ j > n_0, \ \|\theta_j - \theta^0\| \geq r\}, \\ 0, \quad \text{if} \ \ \|\theta_{n_0} - \theta^0\| \geq r. \end{cases} \tag{3.39}$$

By (3.11) and (3.13), it is easy to see that

$$\begin{aligned} \|a_{n+1}^{\delta'-1/2}\psi_{n,n_0}(\theta_{n_0} - \theta^0)\| &\leq c_0 a_{n_0}^{\delta'-1/2}\left(\frac{a_{n_0}}{a_{n+1}}\right)^{1/2-\delta'} \exp\left(-c\sum_{j=n_0}^{n} a_j\right)\|\theta_{n_0} - \theta^0\| \\ &\leq O(1)\exp\left(-c\sum_{j=n_0}^{n} a_j\right)\exp\left(o(1)\sum_{j=n_0}^{n} a_j\right) \\ &= o(1) \quad \text{as} \ \ n \to \infty. \end{aligned} \tag{3.40}$$

Let c_0^* be a constant such that

$$\sup_{j\geq 1}\left\{\left(\frac{a_j}{a_{j+1}}\right)^{1/2-\delta'}\right\} \leq c_0^*.$$

By condition (A5), (3.11) and (3.13) we have

$$\begin{aligned} & a_{n+1}^{\delta'-1/2}\left\|\sum_{j=n_0}^{n}\psi_{n,j+1}a_j(f(\theta_j) - M_1(\theta_j - \theta^0))I\{\sigma^* > n+1, \ \alpha_{i_0} \leq n_0\}\right\| \\ \leq \ & c_0c_1a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\exp\left(-c\sum_{s=j+1}^{n} a_s\right)a_j\|\theta_j - \theta^0\|^2 I\{\sigma^* > j, \ \alpha_{i_0} \leq n_0\} \end{aligned}$$

$$\leq c_0 c_1 r \sum_{j=n_0}^{n} \exp\left(-c \sum_{s=j+1}^{n} a_s\right) a_j \left(\frac{a_j}{a_{n+1}}\right)^{1/2-\delta'} \frac{\|\theta_j - \theta^0\| I\{\sigma^* > j,\ \alpha_{i_0} \leq n_0\}}{a_j^{1/2-\delta'}}$$

$$\leq c_0^* c_0 c_1 r \sum_{j=n_0}^{n} \exp\left(-\frac{c}{2} \sum_{s=j+1}^{n} a_s\right) a_j \frac{\|\theta_j - \theta^0\| I\{\sigma^* > j,\ \alpha_{i_0} \leq n_0\}}{a_j^{1/2-\delta'}}, \tag{3.41}$$

if n_0 is large enough.

By Lemma 3.3, (3.40)-(3.41), it follows from (3.38) that

$$\|\theta_{n+1} - \theta^0\| I\{\sigma^* > n+1,\ \alpha_{i_0} \leq n_0\} a_{n+1}^{\delta'-1/2},$$

$$\leq o(1) + c_0^* c_0 c_1 r \sum_{j=n_0}^{n} \exp\left(-\frac{c}{2} \sum_{s=j+1}^{n} a_s\right) a_j \|\theta_j - \theta^0\|^2 I\{\sigma^* > j,\ \alpha_{i_0} \leq n_0\} a_j^{\delta'-1/2}, \tag{3.42}$$

which, incorporating with the Bellman-Gronwall inequality, leads to

$$\|\theta_n - \theta^0\| I\{\sigma^* > n,\ \alpha_{i_0} \leq n_0\} a_n^{\delta'-1/2}$$

$$\leq o(1) + o(1) \sum_{j=n_0}^{n-1} \exp\left(-\frac{c}{2} \sum_{s=j+1}^{n} a_s\right) a_j \exp\left(c_0^* c_0 c_1 r \sum_{s=j+1}^{n-1} a_s\right)$$

$$\leq o(1) + o(1) \sum_{j=n_0}^{n-1} a_j \exp\left(-\frac{c}{4} \sum_{s=j+1}^{n-1} a_s\right)$$

$$= o(1), \tag{3.43}$$

where r is sufficiently small such that $c_0^* c_0 c_1 r \leq c/4$. From (3.43), it is readily seen that $\|\theta_n - \theta^0\| = o(a_n^{1/2-\delta'})$ as $n \to \infty$, a.s..

□

4 Asymptotic Efficiency of the PARMSR Algorithm with Averaging for $L = 1$

By combining (2.12),(3.1) and (2.17) we obtain the PARMSR algorithm with averaging, with observation period $L = 1$. Our main result in this section is as follows.

Theorem 4.1 *Suppose that conditions (A0)–(A11) hold with* $\nu\gamma_1(1-1/p_1) > 1/2$ *and* $0 < \delta' < 1/2(1-1/(2\nu))$. *Then, for the PARMSR algorithm with averaging, composed of (2.12),(3.1) and (2.17), one has*

$$\sqrt{n}(\overline{\theta}_n - \theta^0) \xrightarrow[n\to\infty]{d} N(0,\ S_3),$$

where $S_3 = \overline{\eta}(\theta^0)^{-1}M_1^{-1}S(M_1^{-1})'$.

In order to prove Theorem 4.1, we need several lemmas. From the definition (3.9), we have

$$\psi_{n,j} = \psi_{n-1,j} - a_n M_1 \psi_{n-1,j} = I - M_1 \sum_{i=j}^{n} a_i \psi_{i-1,j},$$

which gives

$$a_{j-1}\sum_{i=j}^{n}\psi_{i-1,j} = \sum_{i=j}^{n} a_i\psi_{i-1,j} + \sum_{i=j}^{n}(a_{j-1}-a_i)\psi_{i-1,j} = M_1^{-1} + G_{n,j}, \tag{4.1}$$

where

$$G_{n,j} \triangleq -M_1^{-1}\psi_{n,j} + \sum_{i=j}^{n}(a_{j-1}-a_i)\psi_{i-1,j}, \quad \forall\ n \geq j. \tag{4.2}$$

Lemma 4.1 *Let the conditions of Theorem 4.1 be satisfied. Then*

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{j=1}^{n-1}(M_1^{-1} + G_{n,j+1})(f(\theta_j) - M_1(\theta_j - \theta^0)) = 0, \quad a.s.$$

Proof. For any $\delta' \in (0,\ 1/2(1-1/(2\nu)))$, by condition (A4) and Theorem 3.2 it follows that

$$\sum_{i=1}^{\infty}\frac{1}{\sqrt{i}}\|\theta_i - \theta^0\|^2 = \sum_{i=1}^{\infty}\frac{1}{\sqrt{i}}o(a_i^{1-2\delta'}) = \sum_{i=1}^{\infty}\frac{1}{\sqrt{i}}o\left(\frac{1}{i^{\nu(1-2\delta')}}\right) < \infty \text{ a.s.},$$

which gives

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{j=1}^{n-1}\|\theta_j - \theta^0\|^2 = 0 \quad \text{a.s.}, \tag{4.3}$$

via the Kronecker lemma (see, e.g., [12]).

It is shown in Lemma 1 of [3] that for all $n \geq j \geq 1$, $G_{n,j}$ defined by (4.2) are bounded. By Theorem 3.1, condition (A5) and (4.3), it is derived that

$$\frac{1}{\sqrt{n}}\|\sum_{j=1}^{n-1}(M_1^{-1} + G_{n,j+1})(f(\theta_j) - M_1(\theta_j - \theta^0))\| = \frac{O(1)}{\sqrt{n}}\sum_{j=1}^{n-1}\|\theta_j - \theta^0\|^2 \xrightarrow[n\to\infty]{} 0 \quad a.s.,$$

from which the lemma follows.

□

Lemma 4.2 *Suppose that conditions (A0)-(A11) are satisfied. Then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i \xrightarrow[n\to\infty]{d} N(0,\ S_4),$$

where $S_4 = \overline{\eta}(\theta^0)^{-1}S$.

Proof. As in (3.8), $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i$ can be decomposed into the following terms

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i &= \frac{1}{\sqrt{n}}\sum_{j=k_{\sigma(n)}+1}^{n}\varepsilon_j + \frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}}(f_{k_m+i} - D_{m,i}) \\
&+\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}}(f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\
&+\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i} - f(\theta_{k_m}))I\{\eta_{m+1} > \eta_{m+1}(\theta_{k_m})\} \\
&-\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=\eta_{m+1}+1}^{\eta_{m+1}(\theta_{k_m})}(D_{m,i} - f(\theta_{k_m}))I\{\eta_{m+1} < \eta_{m+1}(\theta_{k_m})\} \\
&+\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1} w_{m+1},
\end{aligned} \tag{4.4}$$

where w_{m+1} is defined by (3.15).

We first show that each of the first five terms on the right hand side of (4.4) converges to zero a.s., as $n\to\infty$. Then we show that the last term converges to $N(0,\ S_4)$ in distribution. Hence the desired result follows.

(i). Similar to (3.19), it is easy to show that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{j=k_{\sigma(n)}+1}^{n}\varepsilon_j = 0, \quad a.s.$$

(ii). By (3.23) it is derived that

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\mathsf{E}\left[\left\|\sum_{i=1}^{\eta_{m+1}}(f_{k_m+i} - D_{m,i})\right\|\ \middle|\mathcal{F}^{(m)}\right]$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} a_{k_m} \mathsf{E}\left[\eta_{m+1}(\theta^*) Z_{m+1}(\theta^*) Z_{m+1}^{(0)}\right]$$

$$\leq \sum_{m=1}^{\infty} \frac{\overline{a}}{m^{\nu+1/2}} \mathsf{E}\left[\eta_{m+1}(\theta^*) Z_{m+1}(\theta^*) Z_{m+1}^{(0)}\right] < \infty, \quad a.s.,$$

which implies

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) < \infty \quad a.s., \tag{4.5}$$

by the local convergence theorem of martingales (see, e.g., [11], [33]). By the Kronecker lemma, it follows from (4.5) that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=0}^{n} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) = 0 \quad a.s.,$$

which gives

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) = 0 \quad a.s., \tag{4.6}$$

since $\sigma(n) \leq n, \ \forall\, n \geq 1$.

(iii). By (3.20), similar to (4.6) we can prove that the third term on the right side of (4.4) converges to zero as $n \to \infty$, a.s.

(iv). By (3.35) and condition (A4), we derive that

$$\mathsf{E}\left\| \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}} (D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} > \eta_{m+1}(\theta_{k_m})\} \right\|$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \alpha_0^{1-1/p_1} \left(\mathsf{E}\left[(W_{m+1}^{(0)} \eta_{m+1}(\theta^*))^{p_1}\right]\right)^{1/p_1} \overline{a}^{\gamma_1(1-1/p_1)} m^{-\gamma_1(1-1/p_1)\nu}$$

$$< \infty \tag{4.7}$$

if $\nu\gamma_1(1 - 1/p_1) > 1/2$. By (4.7), it is easy to prove that the fourth term on the right hand side of (4.4) converges to zero as $n \to \infty$, a.s.

(v). Similar to (iv), we can prove that the fifth term on the right hand side of (4.4) converges to zero a.s., as $n \to \infty$.

(vi). By the central limit theorem for martingales (see, e.g., [12] and [18]), it is seen that

$$\frac{1}{\sqrt{n}} \sum_{m=0}^{n} w_{m+1} \xrightarrow[n \to \infty]{d} N(0,\ S). \tag{4.8}$$

We now show that
$$\frac{\sigma(n)}{n} \xrightarrow[n\to\infty]{} \frac{1}{\overline{\eta}(\theta^0)}, \quad a.s. \tag{4.9}$$
One can decompose
$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\eta_i &= \frac{1}{n}\sum_{i=1}^{n}(\eta_i - \eta_i(\theta_{k_{i-1}})) + \frac{1}{n}\sum_{i=1}^{n}(\eta_i(\theta_{k_{i-1}}) - \overline{\eta}(\theta_{k_{i-1}})) \\ &+\frac{1}{n}\sum_{i=1}^{n}(\overline{\eta}(\theta_{k_{i-1}}) - \overline{\eta}(\theta^0)) + \overline{\eta}(\theta^0).\end{aligned} \tag{4.10}$$
By conditions (A9) and (A10), we have
$$\begin{aligned}\mathsf{E}\left[\sum_{i=1}^{\infty}\frac{1}{i}\|\eta_i - \eta_i(\theta_{k_{i-1}})\|\right] &\leq \sum_{i=1}^{\infty}\frac{1}{i}\mathsf{E}\left[\eta_i(\theta^*)I\{\eta_i \neq \eta_i(\theta_{k_{i-1}})\}\right] \\ &\leq \sum_{i=1}^{\infty}\frac{1}{i}\sqrt{\mathsf{E}[\eta_1(\theta^*)^2]}\sqrt{P\{\eta_i \neq \eta_i(\theta_{k_{i-1}})\}} \\ &\leq \sqrt{\alpha_0}\sqrt{\mathsf{E}[\eta_1(\theta^*)^2]}\overline{a}^{-\gamma_1/2}\sum_{i=1}^{\infty}\frac{1}{i}i^{-\gamma_1\nu/2} \\ &< \infty,\end{aligned}$$
which, combining with the Kronecker lemma, yields
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\eta_i - \eta_i(\theta_{k_{i-1}})) < \infty, \quad a.s. \tag{4.11}$$

By the local convergence theorem of martingales and the Kronecker lemma, it is derived that
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\eta_i(\theta_{k_{i-1}}) - \overline{\eta}(\theta_{k_{i-1}})) = 0, \quad a.s. \tag{4.12}$$

By Theorem 3.1, $\theta_n \xrightarrow[n\to\infty]{} \theta^0$ a.s., which implies
$$\frac{1}{n}\sum_{i=1}^{n}(\overline{\eta}(\theta_{k_{i-1}}) - \overline{\eta}(\theta^0)) \xrightarrow[n\to\infty]{} 0, \quad a.s. \tag{4.13}$$
via the continuity of $\overline{\eta}(\theta)$ at θ^0.

By (4.11)-(4.13), it follows from (4.10) that
$$\frac{1}{n}\sum_{i=1}^{n}\eta_i \xrightarrow[n\to\infty]{} \overline{\eta}(\theta^0), \quad a.s. \tag{4.14}$$

From the definition (3.10), we have

$$n < \sum_{i=1}^{\sigma(n)+1} \eta_i \le n + \eta_{\sigma(n)+1},$$

which gives

$$1 < \left(\frac{\sigma(n)}{n} + \frac{1}{n}\right) \frac{1}{\sigma(n)+1} \sum_{i=1}^{\sigma(n)+1} \eta_i \le 1 + \frac{\sigma(n)+1}{n} \frac{1}{\sigma(n)+1} \eta_{\sigma(n)+1}(\theta^*). \tag{4.15}$$

This yields (4.9) by (4.14) and Lemma 3.1.

To prove

$$\frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} w_{m+1} \xrightarrow[n \to \infty]{} N(0,\ S_4), \tag{4.16}$$

we need a central limit theorem for stopped martingales. We note that if the Kolmogorov inequality is replaced by the Doob's inequality (see, e.g., [12]), the proof of Theorem 9.4.1 in [12] goes through for the stopped martingales. Then, by (4.8) and (4.9), (4.16) follows.

□

Lemma 4.3 *If the conditions of Theorem 4.1 are fulfilled, then*

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} G_{n,j}\varepsilon_j \xrightarrow[n \to \infty]{P} 0,$$

where $\xrightarrow{P}$ *means convergence in probability.*

Proof. Similar to (4.4), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{j=1}^{n} G_{n,j}\varepsilon_j \\
= \ & \frac{1}{\sqrt{n}} \sum_{j=k_{\sigma(n)}+1}^{n} G_{n,j}\varepsilon_j + \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} G_{n,k_m+i}(f_{k_m+i} - D_{m,i}) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} G_{n,k_m+i}(f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=\eta_{m+1}(\theta_{k_m})+1}^{\eta_{m+1}} G_{n,k_m+i}(D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} > \eta_{m+1}(\theta_{k_m})\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=\eta_{m+1}+1}^{\eta_{m+1}(\theta_{k_m})} G_{n,k_m+i}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}<\eta_{m+1}(\theta_{k_m})\}\\
&+\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}(\theta_{k_m})} G_{n,k_m}(D_{m,i}-f(\theta_{k_m}))\\
&+\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}(\theta_{k_m})} (G_{n,k_m+i}-G_{n,k_m})(D_{m,i}-f(\theta_{k_m})). \qquad (4.17)
\end{aligned}
$$

Since $G_{n,j}$ is bounded for all $n \geq j \geq 1$, it follows from the proof of Lemma 4.2 that each of the first five terms on the right hand side of (4.17) converges to zero a.s., as $n \to \infty$. In what follows we prove that as the $n \to \infty$ the sixth term converges to zero in probability, while the last term converges to zero, a.s.

(i). By the definition (4.2), we get

$$
\begin{aligned}
\mathsf{E}\left[\left\|\frac{1}{\sqrt{n}}\sum_{m=0}^{\tau(n)} G_{n,k_m}w_{m+1}\right\|^2\right] &\leq \frac{1}{n}\mathsf{E}\left[\sum_{m=0}^{\tau(n)}\|G_{n,k_m}\|^2\mathsf{E}[\|w_{m+1}\|^2|\mathcal{F}^{(m)}]\right]\\
&\leq \mathsf{E}\left[(\eta_1(\theta^*)W_1^{(0)})^2\right]\frac{1}{n}\sum_{i=0}^{n+1}\|G_{n,i}\|^2 \xrightarrow[n\to\infty]{} 0 \qquad (4.18)
\end{aligned}
$$

via Lemma 1 of [3].

By (4.18), it is easy to derive that

$$
\frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1} G_{n,k_m}w_{m+1} \xrightarrow[n\to\infty]{P} 0.
$$

(ii). By (3.9) and (4.1), we have

$$
\begin{aligned}
&\|G_{n,j}-G_{n,j-1}\|\\
=\;& \|(a_{j-1}-a_{j-2})\sum_{s=j}^{n}\psi_{s-1,j}+a_{j-2}\sum_{s=j}^{n}(\psi_{s-1,j}-\psi_{s-1,j-1})-a_{j-2}\psi_{j-2,j-1}\|\\
\leq\;& \alpha_1 a_{j-1}a_{j-2}\sum_{s=j}^{n}\|\psi_{s-1,j}\|+a_{j-2}\sum_{s=j}^{n}a_{j-1}\|M_1\|\;\|\psi_{s-1,j}\|+a_{j-2}\\
\leq\;& a_{j-1}\left((\alpha_1+\|M_1\|)\sup_j\{\frac{a_{j-2}}{a_{j-1}}\}\;\sup_j\{a_{j-1}\sum_{s=j}^{n}\|\psi_{s-1,j}\|\}+\sup_j\{\frac{a_{j-2}}{a_{j-1}}\}\right)\\
\leq\;& c_3 a_{j-1}, \qquad \forall\, n\geq j\geq 1, \qquad (4.19)
\end{aligned}
$$

where α_1 and c_3 are some constants.

By (4.19) and (3.23), it follows that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\left\|\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}(\theta_{k_m})}(G_{n,k_m+i}-G_{n,k_m})(D_{m,i}-f(\theta_{k_m}))\right\| \\
\leq\ & \frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}(\theta_{k_m})}\|\sum_{j=1}^{i}(G_{n,k_m+j}-G_{n,k_m+j-1})\|\cdot\|D_{m,i}-f(\theta_{k_m})\| \\
\leq\ & \frac{c_3}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}a_{k_m}\eta_{m+1}(\theta^*)^2W_{m+1}^{(0)}\xrightarrow[n\to\infty]{}0, \quad a.s.,
\end{aligned}
$$

via similar arguments as for (4.6).

Thus, the proof of Lemma 4.3 is completed.

□

Proof of Theorem 4.1:

By (4.1), from (3.37) it follows that

$$
\begin{aligned}
\sqrt{n}(\overline{\theta}_n-\theta^0) &= o(1)+\frac{1}{\sqrt{n}}\sum_{i=n_0^*}^{n}(\theta_i-\theta^0) \\
&= o(1)+\frac{1}{\sqrt{n}}\sum_{i=n_0^*}^{n}\psi_{i-1,n_0^*}(\theta_{n_0^*}-\theta^0)-\frac{1}{\sqrt{n}}\sum_{i=n_0^*}^{n}\sum_{j=n_0^*}^{i-1}\psi_{i-1,j+1}a_j\varepsilon_{j+1} \\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=n_0^*}^{n}\sum_{j=n_0^*}^{i-1}\psi_{i-1,j+1}a_j(f(\theta_j)-M_1(\theta_j-\theta^0)) \\
&= o(1)+\frac{1}{\sqrt{n}}\frac{1}{a_{n_0^*-1}}(M_1^{-1}+G_{n,n_0^*})(\theta_{n_0^*}-\theta^0) \\
&\quad -\frac{1}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}(M_1^{-1}+G_{n,j+1})(f(\theta_j)-M_1(\theta_j-\theta^0)) \\
&\quad -\frac{M_1^{-1}}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}\varepsilon_{j+1}-\frac{1}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}G_{n,j+1}\varepsilon_{j+1}. \qquad (4.20)
\end{aligned}
$$

Since $G_{n,j}$, for all $n \geq i \geq 1$, are bounded uniformly, it is easy to see that

$$\frac{1}{\sqrt{n}}\frac{1}{a_{n_0^*-1}}(M_1^{-1}+G_{n,n_0^*})(\theta_{n_0^*}-\theta^0)\xrightarrow[n\to\infty]{}0, \quad a.s. \qquad (4.21)$$

By (4.21) and Lemmas 4.1-4.3, the desired result follows from (4.20).

□

5 Asymptotic Efficiency of the PARMSR Algorithm with Averaging for $L \geq 1$

In this section we extend the results of Sections 3 and 4 to $L \geq 1$. We use the projected RM algorithm (2.12) to update the control parameter, where f_{n+1} is the $(n+1)$-th estimate for $f(\theta_n)$ based on IPA. Set

$$\widetilde{\theta}_n = \theta_{\lfloor n/L \rfloor}, \quad \forall\, n \geq 0. \tag{5.1}$$

By the *perturbation propagation rule*, the $(n+1)$-th estimator for $f(\theta_n)$ is defined by

$$f_{n+1} = \frac{1}{L} \sum_{i=1}^{L} \beta_{nL+i}, \tag{5.2}$$

where β_i is an estimator of $d\, J_i(\theta)/d\,\theta, \quad \forall\, i \geq 1$. The expression for β_i is, in general, complicated, and depends on the concrete situation. Let η_{m+1} be the $(m+1)$-th regenerative cycle length. As in (3.1), we assume that β_{k_m+i} has the form of

$$\beta_{k_m+i} = Y_{k_m+i}(\widetilde{\theta}_{k_m},\ \widetilde{\theta}_{k_m+i},\ \cdots,\ \widetilde{\theta}_{k_m+i-1}), \quad \forall\, 1 \leq i \leq \eta_{m+1}, \tag{5.3}$$

where $k_m = \sum_{i=1}^{m} \eta_i$. In the setting of (2.1), we have

$$\theta_i^{(m+1)} = \widetilde{\theta}_{k_m+i} \qquad \text{for all } \ 1 \leq i \leq \eta_{m+1},\ m \geq 0,$$

which is in accordance with the definition (2.7).

The PARMSR algorithm with observation period L (without averaging) consists of (2.12),(5.2), and (5.3) (see, e.g., [22], [35], [36], and [37]). The PARMSR algorithm with averaging, with observation period L, consists of (2.12),(5.2),(5.3), and (2.17). The observation noise can be expressed as (3.3).

Theorem 5.1 *(i) Suppose that the conditions of Theorems 3.1 and 3.2 are satisfied with θ_n replaced by $\widetilde{\theta}_n$, $\forall\, n \geq 1$ in conditions (A9) and (A10). Then these theorems remain true for the PARMSR algorithm with observation period L (without averaging). (ii) Suppose that the conditions of Theorem 4.1 are satisfied with θ_n replaced by $\widetilde{\theta}_n$, $\forall\, n \geq 1$ in conditions (A9) and (A10). Then, the PARMSR algorithm with averaging and with observation period L obeys*

$$\sqrt{n}(\overline{\theta}_n - \theta^0) \xrightarrow[n \to \infty]{d} N(0,\ S_5), \tag{5.4}$$

where $S_5 = L^{-1}\overline{\eta}(\theta^0)^{-1} M_1^{-1} S (M_1^{-1})'$.

Proof. Let

$$\tilde{\varepsilon}_n = \beta_n - f(\tilde{\theta}_{n-1}), \quad \forall\, n \geq 1. \tag{5.5}$$

The proofs in Sections 3 and 4 can be applied to the present setting if we replace θ_n, ε_n, f_n by $\tilde{\theta}_n$, $\tilde{\varepsilon}_n$, and β_n, respectively. Details are omitted here (cf. [36] and [37]). We only mention the key point for (5.4). By the definitions (3.3), (5.2) and (5.5), it is seen that

$$\begin{aligned} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i &= \frac{1}{\sqrt{n}}\sum_{i=0}^{n}\left(\frac{1}{L}\sum_{j=1}^{L}\beta_{iL+j} - f(\theta_{i-1})\right) = \frac{1}{L\sqrt{n}}\sum_{i=1}^{(n+1)L}(\beta_i - f(\tilde{\theta}_{i-1})) \\ &= \frac{1}{L\sqrt{n}}\sum_{i=1}^{(n+1)L}\tilde{\varepsilon}_i. \end{aligned} \tag{5.6}$$

By the same proof as that of Lemma 4.2, it can be shown that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\varepsilon}_i \xrightarrow[n\to\infty]{d} N(0,\ S_4),$$

which, combining with (5.6), gives

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i \xrightarrow[n\to\infty]{d} N(0,\ S_4^*),$$

where $S_4^* = L^{-1}S_4 = L^{-1}\overline{\eta}(\theta^0)^{-1}S$. The rest of the proof works the same way as that of Theorem 4.1.

□

For the first n SA iterations, the total computing budget of the algorithm with observation period L is $N_n = nL$. Theorem 5.1 yields the following central limit theorem, in terms of N_n.

Corollary 5.1 *Suppose that the conditions of Theorem 5.1 (ii) are satisfied. Then, the PARMSR algorithm with averaging and with observation period L satisfies*

$$\sqrt{N_n}(\overline{\theta}_n - \theta^0) \xrightarrow[n\to\infty]{d} N(0,\ S^*),$$

where S^* is defined in Corollary 2.1.

It follows from Corollaries 2.1 and 5.1 that the PARMSR algorithms with averaging, updated either after every regenerative cycle or after every L steps of the process $\{J_i\}$, have the same convergence rate and the same limit covariance matrix, for arbitrary $L \geq 1$, as a function of the computing budget.

It should be pointed out that the small-sample or transient behavior of these algorithms may, however, be quite different. We also recall that our analysis of the asymptotic behavior is based on what happens after no projection of θ_n on D occurs anymore. Thus, our results do not apply if the optimizer θ^0 lies on the boundary of D. Moreover, if θ^0 is very close to the boundary, it may take a long while before no projection occurs, and this may affect the convergence speed.

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