Open-loop and feedback Nash equilibrium in scalar linear-state differential games with impulse control

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Abstract: We consider a deterministic two-player linear-state differential game, where Player 1 uses piecewise continuous controls, while Player 2 implements impulse controls. When the impulse instants are not the decision variables for Player 2, but provided exogenously, we recover the classical result that both open-loop and feedback Nash equilibria coincide for this class of games. When the number and timing of impulse instants are decision variables of Player 2, we show that the classical result no longer holds, that is, open-loop and feedback Nash equilibria are different.

We show that the impulse level is constant in both equilibria. More importantly, in the open-loop case, we show that the equilibrium number of impulses is at most three, while there can be at most two impulses in the feedback case.

Keywords: Linear-state differential games, impulse control, quasi-variational inequalities, Nash equilibrium

1   Introduction

Differential games are used to study competitive strategic interactions between multiple agents (players) over time; see Başar and Olsder (1999), Haurie et al. (2012), Başar et al. (2018). In the differential games literature, it is widely assumed that the players make their decisions at each instant of time or choose strategies that are piecewise continuous functions of time (also referred to as ordinary controls from here on). When one or more players choose actions only at certain specific time instants (also referred to as impulse controls from here on), the game problem is known as differential games with impulse controls. Zero-sum differential games where one player uses ordinary controls and the other uses impulse controls have been developed to study pursuit-evasion (Chikrii and Matichin (2005), Chikrii et al. (2007)), option pricing (El Farouq et al. (2010)) and related problems. The strategic interactions taking place in pollution regulation, for instance, between a polluting firm and a regulator (Ferrari and Koch (2019)), and exchange rate management (Aïd et al. (2020)) have been studied using two-player impulse differential games by considering that both players use impulse controls only.

The equilibrium of a differential game depends on the information that is available to the players when they make their decisions, see Başar and Olsder (1999). In the open-loop information structure, players' strategies depend on time and the initial state (a known parameter) while in the feedback information structure, players strategies' are functions of time and state values. A well-known result in the class of deterministic linear-state differential games (LSDGs) with ordinary controls is that open-loop Nash equilibria (OLNE) and feedback Nash equilibria (FNE) coincide (Dockner et al. (2000)). This implies that a precommitment by the players to an action profile over time does not make them worse off than when they adapt their strategies to the state of the system. To the best of our knowledge, the literature does not provide a comparative analysis of open-loop and feedback Nash equilibria for differential games with impulse controls.

LSDGs have been extensively studied in the literature; see, e.g., Başar and Olsder (1999), Dockner et al. (2000), Engwerda (2005), Haurie et al. (2012). Their popularity stems from their tractability, that is, the equilibrium strategies and outcomes can be determined analytically. One drawback of this class of games is that, by definition, the model cannot include nonlinear terms in the state variables.[1] However, the fact that there is no restriction on the form of the control variables that enter the players' objective functionals or the dynamics renders LSDGs appealing in some applications of differential games (see, Jørgensen et al. (2003)). In this article, we consider a LSDG model with linear dynamics and quadratic cost functions for the players. The more general case can be obtained as an extension of our model by devising a numerical procedure to characterize the OLNE and FNE.

In this paper, we aim at (i) characterizing OLNE and FNE in LSDGs with impulse controls when the impulse instants are given; (ii) characterizing FNE when the impulse instants are endogenous (the open-loop case was studied in Sadana et al. (2019)); and (iii) verifying if OLNE and FNE coincide in LSDGs with impulse controls.

Our contributions are summarized as follows:

1. When the timing of impulses is fixed (or given exogenously), we provide analytical characterization of OLNE and FNE in Theorem 1 and Theorem 2, respectively. Further, we show in Theorem 3 that both equilibria coincide for this class of games.
2. When the number and timing of the impulses are also decision variables (or to be determined endogenously) of Player 2, besides the size of the impulse, we derive analytical expressions for OLNE in Theorem 4, and FNE in Theorem 5 and Theorem 7.
3. In the endogenous case, we show in Theorem 4 that the equilibrium number of impulses in the OLNE is at most three, whereas in the FNE, in Theorem 7, we show that there can be at most two impulses. In particular, when the instantaneous and terminal costs are both increasing or

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[1]It is possible to have a particular type of interaction between control and state variables and still retain the features of the class of LSDGs (see Dockner et al. (2000)).

decreasing in state, we show that there can be at most one impulse in the feedback case, whereas there can be at most three impulses in the open-loop case. Moreover, we show that in the open-loop case, the equilibrium impulse timing of Player 2 depends on Player 1's problem parameters. In the feedback case, we show that such a dependency does not exist.

4. We provide generalization of our results for other cost structures in Theorem 8, and show that our results remain qualitatively unaltered for the multi-dimensional extension of our scalar LSDG model.

5. On the application side, we use our model to study the strategic decision making of two players, one of whom values the state positively and the other values the state negatively. To illustrate, we consider a firm (Player 1) that invests continuous effort to improve the security (state) of the system and the hacker (Player 2) attacks the firm to lower the system's security.

This paper is organized as follows: In Section 1.1, we review the literature on impulse controls and differential games with impulse controls. In Section 2, we introduce our model. In Section 3, we compare the open-loop and feedback equilibria assuming that the impulse instants are known a priori while, in Section 4, we characterize the two equilibria when the impulse instants are endogenous. Further, in Section 5, we provide a numerical example to illustrate that OLNE and FNE differ in LSDGs when impulse instants are determined endogenously in the game. Some general results obtained by considering other cost structures and the multi-dimensional extension of our model are given in Section 6. Section 7 concludes.

1.1 Literature review

In problems involving one decision maker, impulse controls have been quite naturally used in instances involving a fixed (or transaction) cost, as in, e.g., cash management (Baccarin (2009)), exchange rate intervention (Bertola et al. (2016)), inventory control problems (Berovic and Vinter (2004)), demand throttling to manage server congestion (Perera et al. (2020)), price management in retail energy markets (Basei (2019)), forest management (Alvarez (2004)) and investments in product innovation (Chahim et al. (2017)). Some of the papers dealing with deterministic impulse controls include Berovic and Vinter (2004), Chahim et al. (2012), Leander et al. (2015), Reddy et al. (2016), Chahim et al. (2017) and Grames et al. (2019).

Deterministic zero-sum differential games with impulse controls have been studied in Chikrii and Matichin (2005), Chikrii et al. (2007), El Farouq et al. (2010) and El Asri, Brahim (2013). For stochastic zero-sum impulse-control differentiable games with one player using an ordinary control, and the other using an impulse control, see Azimzadeh (2019). In differential games with impulse control, the player who acts at discrete time instants solves an impulse control problem. The Hamiltonian Maximum Principle (see Blaquière (1977a), Blaquière (1977b)) and the Bensoussan-Lions quasi-variational inequalities (see Bensoussan and Lions (1982), Bensoussan and Lions (1984)) provide a framework to determine the time and level of such interventions. Recent works that use quasi-variational inequalities (QVI) to determine the equilibrium in stochastic games with impulse control include Aïd et al. (2020) and Azimzadeh (2019). In deterministic settings, QVIs are used in El Farouq et al. (2010).

The closest paper to our work is Aïd et al. (2020) where Nash equilibrium is obtained for stochastic nonzero-sum impulse games using the QVIs under the feedback information structure. However, they assumed that both players use threshold-type impulse controls only, that is, impulse controls are used when the state leaves the boundaries of a region. In contrast to their model, our game problem involves one player using ordinary controls and the other using impulse controls. Basei et al. (2019) study the N-person extension of the two-player game given in Aïd et al. (2020), and its corresponding mean field game. Aïd et al. (2020) also studied a LSDG model to derive analytical solutions.

Given that problems in regulation and cybersecurity (Taynitskiy et al. (2019)) involve impulse controls, nonzero-sum differential games with impulse controls are useful for many diverse applications. Recently, Sadana et al. (2019) considered a class of finite-horizon two-player nonzero-sum linear-state

differential games, where one player uses an ordinary control, while the other intervenes only at some instants of time in the game, that is, implements an impulse control. To illustrate, a game in which a firm continuously makes marketing, production, and security decisions, and a hacker attacks the firm occasionally fits the model in Sadana et al. (2019). When there are no fixed costs for Player 2 at the impulse instants and all the impulses are interior, i.e., impulse cannot occur at the initial and final time, Sadana et al. (2019) determined a unique OLNE using the Hamiltonian Maximum Principle. In this article, we determine both the OLNE and FNE by allowing for interior impulse instants, and also consider fixed costs in our model.[2] We also provide a comparative analysis of OLNE obtained using Hamiltonian Maximum Principle and FNE derived from the QVIs for scalar deterministic nonzero-sum linear-state differential games with impulse controls.

2   Model

In this section, we introduce a scalar deterministic finite-horizon two-player nonzero-sum linear-state differential game model, where Player 1 uses ordinary controls while Player 2 uses impulse controls.

Let $T < \infty$ be the duration of the game. For Player 1, control action at time $t \in [0, T]$ is denoted by $u(t) \in \Omega_u \subset \mathbb{R}$, where Ω_u is a compact and convex subset of $\mathbb{R}$. We assume that $u : [0, T] \to \Omega_u$ is a piecewise continuous function of time and denotes the strategy profile of Player 1. The set of strategy profiles of Player 1 is denoted by $\mathcal{U}$. Player 2 intervenes or takes actions only at certain isolated time instants (or impulse instants) during the time period $[0, T]$. We denote by $\{\tau_1, \tau_2, \cdots, \tau_k\}$, $k \in \mathbb{N}$ (the set of natural numbers), the set of intervention instants of Player 2, which satisfy the monotone increasing sequence property, that is

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq T. \tag{1}$$

The state of the system evolves according to a scalar linear differential equation during the non-impulse instants of time as follows:

$$\dot{x}(t) = Ax(t) + Bu(t), \ t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}, \ x(0^-) = x_0, \tag{2}$$

where $x(t)$ denotes the state of the system at time $t \in [0, T]$, $x_0 \in \mathbb{R}$ denotes initial state of the system, which is assumed to be given and 0^- denotes the time instant just before 0, and $A \in \mathbb{R}$ and $B \in \mathbb{R}\backslash\{0\}$ are constants. During the impulse instant τ_i $(i = 1, 2, \cdots, k)$, Player 2 induces a jump in the state variable according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i, \tag{3}$$

where $v_i \in \Omega_v$ denotes the control action of Player 2 at impulse instant τ_i, and Ω_v denotes the control set of Player 2, which is assumed to be a compact and convex subset of $\mathbb{R}$. Here, $Q \in \mathbb{R}\backslash\{0\}$ is a constant.

The time instants before and after the impulse instant τ_i are denoted by τ_i^- and τ_i^+, respectively. Further, $x(\tau_i^-) = \lim_{t \uparrow \tau_i} x(t)$ and $x(\tau_i^+) = \lim_{t \downarrow \tau_i} x(t)$ are the state variables evaluated before and after the impulse instant τ_i. We assume that the state is left continuous at points of discontinuity, that is, $x(\tau_i^-) = x(\tau_i)$. The strategy of Player 2 is denoted by $\tilde{v} := (\{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k)\}, k) \in \mathcal{V}$, where $\mathcal{V}$ denotes the strategy set. We note that the number of impulses $k \in \mathbb{N}$ is also a decision variable of Player 2, where $k < \infty$. Clearly, Player 1 influences the evolution of the system during non-impulse instants (2) whereas Player 2's control results in jump in the state variable (3) at impulse instants.

Player 1 uses a strategy $u(.) \in \mathcal{U}$ to maximize the objective

$$J_1(x_0, u(.), \tilde{v}) = \int_0^T \frac{1}{2}\left(2w_1 x(t) + R_1 u(t)^2\right) dt + \sum_{i=1}^k q_1 x(\tau_i^-) + s_1 x(T^+), \tag{4}$$

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[2]A majority of applications of impulse controls consider fixed costs, see Cadenillas and Zapatero (1999), Berovic and Vinter (2004), Chahim et al. (2012), Bertola et al. (2016), Chahim et al. (2017), Ferrari and Koch (2019), and Aïd et al. (2020).

where the term inside the integral denotes the instantaneous payoff, the second term is the payoff received during the impulse instants, and the third term denotes the terminal payoff. T^+ denotes the time instant just after T. The parameters satisfy $w_1 \in \mathbb{R}$, $R_1 < 0$, $q_1 \in \mathbb{R}\backslash\{0\}$ and $s_1 \in \mathbb{R}$. Player 2 uses a strategy $\tilde{v} \in \mathcal{V}$ to maximize the objective

$$J_2(x_0, u(.), \tilde{v}) = \int_0^T w_2 x(t)dt + \sum_{i=1}^k \left(C + \frac{1}{2} P_2 v_i^2 \right) + s_2 x(T^+), \tag{5}$$

where $C < 0$ denotes the fixed cost of each impulse and $\frac{1}{2}P_2 v_i^2$ the variable cost of the impulse at time instant τ_i, with $P_2 < 0$. Here, $w_2 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ are the instantaneous and terminal payoff parameters respectively. As the objectives of the players are interdependent, (2–5) describes a differential game with impulse controls. Further, as the objectives of the players as well as the dynamics are linear in the state variable, the game described by (2–5) is a linear-state differential game with impulse controls.

Remark 1 *Our main objective is to study the nature of the Nash equilibria when players' strategy spaces are different (piecewise continuous and discrete). The differential game model described by (2–5) is canonical, that is, minimal configuration required to capture the effect of differences in the strategy spaces. For this reason, we consider a two-player game with one player using piecewise continuous controls and the other player using impulse controls. Extension to $n > 2$ player case can be easily formulated with the framework studied in this paper.*

The Nash equilibrium strategies of the players are defined as follows:

Definition 1 *The strategy profile $(u^*(.), \tilde{v}^*)$ is a Nash equilibrium of the differential game (2–5) if the following inequalities are satisfied:*

$$J_1(x_0, (u^*(.), \tilde{v}^*)) \geq J_1(x_0, (u(.), \tilde{v}^*)), \ \forall u(.) \in \mathcal{U}, \tag{6a}$$

$$J_2(x_0, (u^*(.), \tilde{v}^*)) \geq J_2(x_0, (u^*(.), \tilde{v})), \ \forall \tilde{v} \in \mathcal{V}. \tag{6b}$$

In a differential game, the outcome varies with the information that is available to the players, when they take their decisions, also referred to as information structure; see Başar and Olsder (1999). Typically, two information structures are studied in the literature. In the open-loop information structure, players' strategies are functions of time and the initial state x_0, which is a known parameter. In our setting, this implies that Player 1's controls at time $t \in [0, T]$ are given by $u(t) := \gamma(t; x_0) \in \Omega_u$, where $\gamma : [0, T] \times \mathbb{R} \rightarrow \Omega_u$ is a measurable mapping. Similarly, the control action of Player 2 at an impulse instant $\tau_i \in [0, T]$ is given by $v_i := \delta(\tau_i; x_0) \in \Omega_v$, where $\delta : [0, T] \times \mathbb{R} \rightarrow \Omega_v$ is a measurable mapping. In the feedback information structure, the strategies of players are functions of time and the state variable. More precisely, Player 1's controls at time $t \in [0, T]$ are given by $u(t) := \gamma^f(t, x(t)) \in \Omega_u$, where $\gamma^f : [0, T] \times \mathbb{R} \rightarrow \Omega_u$ is a measurable mapping. Similarly, the control action of Player 2 at an impulse instant $\tau_i \in [0, T]$ is given by $v_i := \delta^f(\tau_i, x(\tau_i)) \in \Omega_v$, where $\delta^f : [0, T] \times \mathbb{R} \rightarrow \Omega_v$ is a measurable mapping.

Assumption 1 *We assume that the equilibrium controls of Player 1 and equilibrium impulse levels of Player 2 lie in the interior of Ω_u and Ω_v, respectively.*

In the rest of the paper, we analyze two situations, first by treating the timing of the impulses of Player 2 as a problem parameter (or provided exogenously), and next as a decision variable (or occurs endogenously). In these two situations, we compare the Nash equilibria obtained under the open-loop and feedback information structures. To simplify the notations, we let $\tau_0 = 0$ and $\tau_{k+1} = T$ in the remainder of the paper.

3   Exogenous impulse instants

In this section, we consider the differential game (2–5), where the number of impulse instants k, and the timing of the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are not decision variables of Player 2 but provided exogenously. So, the strategy of Player 2 is the set of control actions $\tilde{v} := \{v_1, v_2, \cdots, v_k\}$ to be taken at the given impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$. We characterize Nash equilibrium strategies for both open-loop and feedback information structures.

3.1   Open-loop Nash equilibrium

Computation of open-loop Nash equilibrium follows from (6a) and (6b). Let $(u^*(.), \tilde{v}^*)$ be the OLNE strategies of the players. From (6a), Player 1 solves an optimal control problem with additional costs, and jumps in the state variable at the impulse instants τ_i, $i = 1, 2, \cdots, k$, which make it a non-standard optimal control problem. The necessary conditions for optimality with jumps in the state variable and additional costs have been studied in the literature; see Geering (1976), Sadana et al. (2019).[3] These conditions differ from those of classical optimal problem in that there is a jump in the co-state variable at the impulse instants. We define the Hamiltonian function of Player 1 as:

$$H_1(x(t), u(t), \lambda_1(t)) := w_1 x(t) + \frac{1}{2} R_1 u(t)^2 + \lambda_1(t)(Ax(t) + Bu(t)),$$

for $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$, where $\lambda_1(t) \in \mathbb{R}$ is the co-state variable at time t. The necessary conditions are then given as follows. For $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$,

$$u^*(t) = \arg \max_{u \in \Omega_u} H_1(x(t), u(t), \lambda_1(t)), \tag{7a}$$

and the state and co-state variables satisfy

$$\dot{x}(t) = H_{1\lambda_1}(x(t), u^*(t), \lambda_1(t)), \ x(0^-) = x_0, \tag{7b}$$

$$\dot{\lambda}_1(t) = -H_{1x}(x(t), u^*(t), \lambda_1(t)), \ \lambda_1(T^+) = s_1. \tag{7c}$$

At the impulse instant τ_i $(i = 1, 2, \cdots, k)$, the jump in the state and co-state variables satisfy

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*, \tag{7d}$$

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + \frac{\partial}{\partial x}(q_1 x)\Big|_{x(\tau_i^-)} = \lambda_1(\tau_i^+) + q_1. \tag{7e}$$

The jump in the co-state equation (7e) is due to the state-dependent payoff accrued by Player 1 at the impulse instant τ_i.

Again from (6b), Player 2 solves an impulse optimal control problem with Player 1's strategies fixed at the Nash equilibrium strategy $u^*(.)$. The necessary conditions associated with an impulse optimal control problem were studied in the literature; see Blaquière (1977a), Chahim et al. (2012). We introduce the Hamiltonian and impulse Hamiltonian functions as:

$$H_2(x(t), u(t), \lambda_2(t)) := w_2 x(t) + \lambda_2(t)(Ax(t) + Bu(t)), \tag{8a}$$

$$H_2^I(x(t), v_i, \lambda_2(t)) := C + \frac{1}{2} P_2 v_i^2 + \lambda_2(t) Q v_i, \tag{8b}$$

where $\lambda_2(t) \in \mathbb{R}$ denotes the co-state variable. The necessary conditions for optimality for Player 2's impulse optimal control problem are stated in the following lemma.

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[3]In Geering (1976), the authors assumed the state variable to be continuous and similar to Sadana et al. (2019), there are additional costs incurred at some exogenous time instants. In Sadana et al. (2019), the state variable is discontinuous, that is, $x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i^*)$, at the corresponding discrete time instants. Due to the state dependent jumps in the state variable and state dependent additional costs, the co-state variables satisfy $\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + \frac{\partial}{\partial x}(q_1 x)\Big|_{x(\tau_i^-)} + \frac{\partial}{\partial x}(g(x, v_i^*))\Big|_{x(\tau_i^-)}$. Here $g(x, v_i^*) = Qv_i^*$ so we have $\frac{\partial}{\partial x}(g(x, v_i^*))\Big|_{x(\tau_i^-)} = 0$.

Lemma 1 *[Theorem 2.2 , Chahim et al. (2012)] Given the equilibrium controls $u^*(t)$ of Player 1 and the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$, let $(x(t), v_1^*, v_2^*, \cdots, v_k^*)$ denote the optimal solution of the impulse control problem of Player 2. Then there exist co-states $\lambda_2(t) \in \mathbb{R}$ such that*

for $t \notin \{\tau_1, \tau_2, \ldots, \tau_k\}$,

$$\dot{x}(t) = Ax(t) + Bu^*(t), \; x(0^-) = x_0, \tag{9a}$$

$$\dot{\lambda}_2(t) = -H_{2x}(x(t), u^*(t), \lambda_2(t)), \; \lambda_2(T^+) = s_2, \tag{9b}$$

for $i = \{1, 2, \cdots, k\}$,

$$v_i^* = \arg \max_{v_i \in \Omega_v} H_2^I(x(\tau_i^-), v_i, \lambda_2(\tau_i^+)), \tag{9c}$$

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*, \tag{9d}$$

$$\lambda_2(\tau_i^-) = \lambda_2(\tau_i^+) + \frac{\partial}{\partial x}(H_2^I(x(t), v_i, \lambda_2(t)))\Big|_{x(\tau_i^-)} = \lambda_2(\tau_i^+). \tag{9e}$$

Using (7) and (9), the next theorem characterizes the OLNE of the differential game described by (2–5).

Theorem 1 (Exogenous OLNE) *Let Assumption 1 hold. If the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are given, then the unique OLNE strategies for $A \neq 0$ are given by*

$$u^*(t) = \frac{B}{R_1}\left(\frac{w_1}{A} - \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{j+1}^- - t)}\right), \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \tag{10a}$$

$$v_i^* = \frac{Q}{P_2}\left(\frac{w_2}{A} - \left(s_2 + \frac{w_2}{A}\right)e^{A(T-\tau_i)}\right), \tag{10b}$$

where $i \in \{1, 2, \cdots, k\}$, $\lambda_1(\tau_{k+1}^+) = s_1$,

$$\lambda_1(t) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{j+1}^- - t)}, \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\},$$
$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1, \tag{10c}$$

so that, at the impulse instants, $\tau_i(i = 1, 2, \cdots, k)$, we have

$$\lambda_1(\tau_i^-) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{i+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1.$$

For $A = 0$, the unique OLNE strategies are given by

$$u^*(t) = \frac{B}{R_1}\left(w_1(t - \tau_{j+1}^-) - \lambda_1(\tau_{j+1}^-)\right), \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \tag{11a}$$

$$v_i^* = \frac{Q}{P_2}\left(w_2(\tau_i - T) - s_2\right), \tag{11b}$$

where $i \in \{1, 2, \cdots, k\}$, $\lambda_1(\tau_{k+1}^+) = s_1$,

$$\lambda_1(t) = w_1(\tau_{j+1}^- - t) + \lambda_1(\tau_{j+1}^-), \forall t \in (\tau_j, \tau_{j+1}), j \in (0, 1, \cdots, k\},$$
$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1. \tag{11c}$$

so that, at the impulse instants $\tau_i(i = 1, 2, \cdots, k)$, we have $\lambda_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + \lambda(\tau_{i+1}^-) + q_1$.

Proof. Under Assumption 1 and from the optimality conditions for Player 1 and Player 2 given in (7a)–(7e), (9a)–(9e), we can write the necessary conditions for OLNE as follows:

for $t \notin \{\tau_1, \tau_2, \ldots, \tau_k\}$,

$$u^*(t) = -\frac{B}{R_1}\lambda_1(t), \tag{12a}$$

$$\dot{x}(t) = Ax(t) - \frac{B^2}{R_1}\lambda_1(t), \ x(0^-) = x_0, \tag{12b}$$

$$\dot{\lambda}_1(t) = -A\lambda_1(t) - w_1, \ \lambda_1(T^+) = s_1, \tag{12c}$$

$$\dot{\lambda}_2(t) = -A\lambda_2(t) - w_2, \ \lambda_2(T^+) = s_2, \tag{12d}$$

for $i = \{1, 2, \cdots, k\}$,

$$v_i^* = -\frac{Q}{P_2}\lambda_2(\tau_i^+), \tag{12e}$$

$$x(\tau_i^+) = x(\tau_i^-) - \frac{Q^2}{P_2}\lambda_2(\tau_i^+), \tag{12f}$$

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1, \tag{12g}$$

$$\lambda_2(\tau_i^-) = \lambda_2(\tau_i^+). \tag{12h}$$

From the above equations, we can obtain the expression for $\lambda_1(t)$ and $\lambda_2(t)$ as follows:

when $A \neq 0$:

$$\lambda_1(t) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{j+1}^- - t)}, \text{ for } t \in (\tau_j, \tau_{j+1}), \ j \in \{0, 1, \cdots, k\}, \tag{13a}$$

$$\lambda_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right)e^{A(T-t)}; \tag{13b}$$

when $A = 0$:

$$\lambda_1(t) = w_1(\tau_{j+1}^- - t) + \lambda_1(\tau_{j+1}^-), \ \forall \, t \in (\tau_j, \tau_{j+1}), \ j \in \{0, 1, \cdots, k\},$$
$$\lambda_2(t) = w_2(T - t) + s_2. \tag{14a}$$

On substituting the expressions of $\lambda_1(t)$ and $\lambda_2(t)$ for $A \neq 0$ and $A = 0$ in (12a) and (12e) respectively, we obtain the equilibrium controls of Player 1 and Player 2 given in (10a) and (10b) for $A \neq 0$, and (11a) and (11b) for $A = 0$. □

3.2   Feedback Nash equilibrium

Feedback Nash equilibrium in the differential game (2)–(5) follows from (6a) and (6b), and can be obtained using dynamic programming. Before proceeding with the characterization of the FNE, we introduce the value function of Player 1, $V_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ and Player 2, $V_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$. From (6a), the value function V_1 is defined as follows:

$$V_1(t, x) = \max_{u(s), \ s \in [t,T]} \left\{ \int_t^T \frac{1}{2}\left(2w_1x(s) + R_1u(s)^2\right)ds + \sum_{i=l}^{k} q_1x(\tau_i^-) + s_1x(T^+)\right\}, \tag{15}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_l, \tau_{l+1}, \cdots, \tau_k\}$, $t \geq \tau_l$, as

$$\dot{x}(s) = Ax(s) + Bu(s), \ x(t) = x,$$

and during a switching instant $\tau_i \ (i = l, l+1, \cdots, k)$ undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*.$$

Similarly, following (6b), we define the value function associated with Player 2's impulse optimal control problem as follows:

$$V_2(t,x) = \max_{\{v_i\}_{i=1}^k} \left\{ \int_t^T w_2 x(s) ds + \sum_{i=l}^k \left(C + \frac{1}{2} P_2 v_i^2 \right) + s_2 x(T^+) \right\}, \tag{16}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_l, \tau_{l+1}, \cdots, \tau_k\}$, $t \geq \tau_l$, as

$$\dot{x}(s) = Ax(s) + Bu^*(s), \ x(t) = x,$$

and at a switching instant τ_i $(i = l, l+1, \cdots, k)$ undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i.$$

Given the linear-state structure of the differential game (2–5), we guess the form of the value function.

Assumption 2 *We assume that the value functions of Player 1 and Player 2 are given by*

$$V_1(t,x) = m_1(t)x + n_1(t), \tag{17a}$$
$$V_2(t,x) = m_2(t)x + n_2(t). \tag{17b}$$

Next, using the dynamic programming principle, the FNE is characterized in the following theorem.

Theorem 2 (Exogenous FNE) *Let Assumption 1 and 2 hold. If the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are given, then the unique FNE for $A \neq 0$ is given by*

$$u^*(t) = \frac{B}{R_1} \left(\frac{w_1}{A} - \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{j+1}^- - t)} \right), \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \tag{18a}$$
$$v_i^* = \frac{Q}{P_2} \left(\frac{w_2}{A} - \left(s_2 + \frac{w_2}{A} \right) e^{A(T - \tau_i)} \right), \tag{18b}$$

where $i \in \{1, 2, \cdots, k\}$, $m_1(\tau_{k+1}^+) = s_1$,

$$m_1(t) = -\frac{w_1}{A} + \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{j+1}^- - t)}, \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\},$$
$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1. \tag{18c}$$

So, at the impulse instants, τ_i $(i \in \{1, 2, \cdots, k\})$, we have

$$m_1(\tau_i^-) = -\frac{w_1}{A} + \left(m_1(\tau_{i+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1.$$

For $A = 0$, the unique FNE strategies are given by

$$u^*(t) = \frac{B}{R_1} \left(w_1(t - \tau_{j+1}^-) - m_1(\tau_{j+1}^-) \right), \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \tag{19a}$$
$$v_i^* = \frac{Q}{P_2} \left(w_2(\tau_i - T) - s_2 \right), \tag{19b}$$

where $i \in \{0, 1, \cdots, k\}$, $m_1(\tau_{k+1}^+) = s_1$,

$$m_1(t) = w_1(\tau_{j+1}^- - t) + m_1(\tau_{j+1}^-), \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\},$$
$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1. \tag{19c}$$

So, at the impulse instants, τ_i, $i \in \{1, 2, \cdots, k\}$, we have $m_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + m_1(\tau_{i+1}^-) + q_1$.

Proof. See Appendix A.   □

In the next theorem, we present the main result of this section that OLNE and FNE coincide in the differential games with impulse controls described by (2–5) when the impulse instants are given.

Theorem 3 *For the differential game described by (2–5), when the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are fixed (or provided exogenously), and Assumption 1 and 2 hold, both OLNE and FNE coincide.*

Proof. Equation (10a) is structurally similar to (18a), and (11a) is structurally similar to (19a) because $\lambda_1(t)$ and $m_1(t)$ have the same dynamics, jump conditions and terminal conditions for $A \neq 0$ (see (10c) and (18c)) and $A = 0$ (see (11c) and (19c)). In particular, on replacing λ_1 with m_1 for $A = 0$ and for $A \neq 0$, we obtain that the OLNE and FNE strategies of Player 1 coincide. The OLNE and FNE strategies of Player 2 coincide because (10b) and (18b) hold true for $A = 0$, and (11b) and (19b) hold true for $A \neq 0$. □

Remark 2 *Since the dynamic programming approach provides the sufficient conditions for Nash equilibria, and the FNE obtained by using the dynamic programming coincides with the OLNE obtained by using the necessary conditions, we have that the candidate OLNE are indeed the Nash equilibria.*

In the next section, we verify if the above result holds when the impulse timing is a decision variable of Player 2.

4   Endogenous impulse instants

In this section, we characterize the OLNE and FNE when the number and timing of impulse instants are part of Player 2's strategies (or occur endogenously). More importantly, we seek to investigate if both these informationally different equilibria also coincide in this case.

4.1   Open-loop Nash equilibrium

Let $(u^*(.), \tilde{v}^*)$ denote the open-loop Nash equilibrium strategy profile of the players. In particular, Player 2's equilibrium strategy is given by $\tilde{v}^* := (\{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_{k^*}^*, v_{k^*}^*)\}, k^*)$, where k^* and τ_i^* $(1 \leq i \leq k)$ denote the number and timing of impulses. From (6a), Player 1 solves an optimal control problem with Player 2's strategies fixed at the open-loop Nash equilibrium strategy $\tilde{v}^*$. This implies that the necessary conditions for optimality associated with Player 1's problem are also given by (7).

Concerning Player 2's impulse optimal control problem (6b), due to the presence of additional decision variables, that is, the number and timing of impulses, the necessary conditions for optimality differ from (9). In particular, additional consistency conditions are required to hold true at equilibrium impulse instants. These conditions follow from Chahim et al. (2012), and are summarized in the next lemma.

Lemma 2 *[Theorem 2.2, Chahim et al. (2012)] Let the optimal solution of the impulse control problem of Player 2 be given by $(\{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_{k^*}^*, v_{k^*}^*)\}, k^*)$. Then there exist absolutely continuous functions $\lambda_2 : [0^-, T^+] \to \mathbb{R}$, with Hamiltonian and impulse Hamiltonian functions defined as (8a) and (8b) respectively, such that the following conditions hold true:*

for $t \notin \{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$,

$$\dot{x}(t) = Ax(t) + Bu^*(t), \ x(0^-) = x_0, \tag{20a}$$

$$\dot{\lambda}_2(t) = -H_{2x}(x(t), u^*(t), \lambda_2(t)), \ \lambda_2(T^+) = s_2, \tag{20b}$$

and for $i = \{1, 2, \cdots, k^*\}$,

$$v_i^* = \arg \max_{v_i \in \Omega_v} H_2^I(x(\tau_i^{*-}), v_i, \lambda_2(\tau_i^+)), \tag{20c}$$

$$x(\tau_i^{*+}) = x(\tau_i^{*-}) + Qv_i^*, \tag{20d}$$

$$\lambda_2(\tau_i^{*-}) = \lambda_2(\tau_i^{*+}) + \frac{\partial}{\partial x}(H_2^I(x(t), v_i, \lambda_2(t)))\Big|_{x(\tau_i^{*-})} = \lambda_2(\tau_i^{*+}), \tag{20e}$$

$$H_2(x(\tau_i^{*+}), u^*(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x(\tau_i^{*-}), u^*(\tau_i^{*-}), \lambda_2(\tau_i^{*-})) \begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T \end{cases}. \tag{20f}$$

Remark 3 *We note that* (20f) *is the additional consistency condition that is required to hold true when the number and timing of impulses are to be determined endogenously. The difference* $H_2(x(\tau_i^{*+}), u(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x(\tau_i^{*-}), u(\tau_i^{*-}), \lambda_2(\tau_i^{*-}))$ *measures the gain made by Player 2 by delaying the impulse by one time instant (see (Léonard and Long, 1992, Chapter 10)).*

Remark 4 *In the characterization of the OLNE, we assume that Player 2 gives a nonzero impulse, that is,* $v_i^* \neq 0$, *at the equilibrium instants,* $\tau_i^*, i \in \{1, 2, \cdots, k^*\}$. *This assumption is justified because in this section, our objective is to show that OLNE and FNE differ when Player 2 decides the number and timing of impulse. Also, we shall see in the feedback case that the equilibrium impulse strategies involve nonzero equilibrium impulse levels.*

Using (7) and (20), we provide a characterization of the candidate OLNE in the next theorem. In the following discussion, to save on notation, we denote by $\delta := \left(\frac{P_2}{R_1}\right)\left(\frac{B}{Q}\right)^2$, then, as $P_2 < 0$ and $R_1 < 0$, we have $\delta > 0$.

Theorem 4 (Endogenous OLNE) *Let Assumption 1 hold, and let* $w_2 \neq \delta q_1$ *when* $A = 0$. *Then, the number of impulse instants for Player 2 is at most three, that is,* $k^* \leq 3$, *in the open-loop equilibrium. Further, when the parameters satisfy* $w_2 \neq \delta q_1$, *and either of the following conditions,*

$$T - \frac{1}{A}\ln\left(\frac{\delta q_1}{As_2 + w_2}\right) > 0, \tag{21a}$$

$$\frac{1}{A}\ln\left(\frac{\delta q_1}{As_2 + w_2}\right) > 0, \tag{21b}$$

then an interior impulse occurs in the time period $(0, T)$. *For* $A = 0$, *there can be no interior impulse.*

An impulse occurs at $\tau_{ol}^1 = 0$ *if*

$$\frac{(As_2 + w_2(e^{AT} - 1))(As_2 + w_2 e^{AT} - \delta q_1)}{A} > 0. \tag{21c}$$

An impulse occurs at $\tau_{ol}^2 = T$ *if*

$$s_2(As_2 - (\delta q_1 - w_2)) < 0. \tag{21d}$$

The equilibrium timing of interior impulse is given by

$$\tau_{ol}^I = T - \frac{1}{A}\ln\left(\delta\frac{q_1}{As_2 + w_2}\right). \tag{22}$$

With $k^* = 1$, the equilibrium control of Player 1 and equilibrium impulse levels of Player 2 are as follows:

For $\tau_{ol}^1 = 0$ and $t \in (0, T]$, we have

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right), & A \neq 0, \\ \frac{B}{R_1}(w_1(t - T) - s_1), & A = 0, \end{cases} \tag{23a}$$

$$v_{ol}^1 = \begin{cases} \frac{Q}{P_2}\left(\frac{w_2}{A} - (s_2 + \frac{w_2}{A})e^{AT}\right), & A \neq 0 \\ -\frac{Q}{P_2}(w_2 T + s_2), & A = 0 \end{cases} \tag{23b}$$

For $\tau_{ol}^I = T - \frac{1}{A}\ln\left(\delta\frac{q_1}{As_2 + w_2}\right)$, $A \neq 0$,

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right) \text{ for } \tau_{ol}^I < t \leq T, \\ \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)} - q_1 e^{A(\tau_{ol}^I - t)}\right) \text{ for } 0 < t < \tau_{ol}^I, \end{cases} \tag{23c}$$

$$v_{ol}^I = \frac{Qw_2}{P_2 A} - \frac{B^2 q_1}{AQR_1}. \tag{23d}$$

For $\tau_{ol}^2 = T$ *and* $t \in [0, T)$, *we have*

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + q_1 + \frac{w_1}{A})e^{A(T-t)}\right) & A \neq 0, \\ \frac{B}{R_1}(w_1(t - T) - s_1 - q_1) & A = 0, \end{cases} \tag{23e}$$

$$v_{ol}^2 = -\frac{Q}{P_2}s_2. \tag{23f}$$

Proof. From (7a) and Assumption 1, the first-order condition gives the equilibrium control of Player 1

$$u^*(t) = -\frac{B}{R_1}\lambda_1(t).$$

When Player 2 solves her optimal control problem (with Player 1's strategy fixed at her OLNE strategy), conditions (20a)–(20f) hold true. From (20c) and Assumption 1, we get the equilibrium impulse level as follows:

$$v_i^* = \frac{Q}{P_2}\lambda_2(\tau_i^+).$$

From (20b) and (20e), the co-state $\lambda_2(t)$ is given by

$$\lambda_2(t) = \begin{cases} -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right)e^{A(T-t)}, & A \neq 0 \\ w_2(T - t) + s_2, & A = 0. \end{cases}$$

Now, we determine the candidates for the equilibrium impulse instant. First, we analyze the situation where the equilibrium impulse instant satisfies $\tau_i^* \in (0, T)$. Following the Hamiltonian continuity condition (20f) at $\tau_i^* \in (0, T)$, we have

$$w_2 x(\tau_i^{*+}) + \lambda_2(\tau_i^{*+})(Ax(\tau_i^{*+}) + Bu(\tau_i^{*+})) = w_2 x(\tau_i^{*-}) + \lambda_2(\tau_i^{*-})(Ax(\tau_i^{*-}) + Bu(\tau_i^{*-})),$$

Substituting $u^*(t)$ in the above equation, and using the conditions, (7e), (20d), (20e), we obtain

$$-\frac{Q^2}{P_2}\left(A\lambda_2(\tau_i^*) + (w_2 - \delta q_1)\right)\lambda_2(\tau_i^*) = 0. \tag{24}$$

Next, we provide a justification for the assumption $w_2 \neq \delta q_1$ when $A = 0$. Assume that $w_2 = \delta q_1$, then the above condition results in $A\lambda_2^2(\tau_i^*) = 0$. If $A = 0$, then (24) holds true at all $\tau^* \in (0, T)$. From the isolated property of the impulse instants (1), this is not possible.

When $A = 0$, and as $(w_2 - \delta q_1) \neq 0$, (24) results in $\lambda_2(\tau_i^*) = 0$, and this contradicts the occurrence of impulse at $\tau_i^* \in (0, T)$. So, there is no interior impulse when $A = 0$ since we have assumed that for admissible equilibrium impulse instants, $v_i \neq 0$.

When $A \neq 0$ and $w_2 = \delta q_1$, we have that $\lambda_2(\tau_i^*) = 0$. This implies that $v_i^* = 0$, which contradicts the idea that impulse occurs at $\tau_i^* \in (0, T)$. So, an impulse does not occur in $(0, T)$ when $A \neq 0$ and $w_2 = \delta q_1$.

When $A \neq 0$, (24) can be written as

$$A \left(\lambda_2(\tau_i^*) - \frac{\delta q_1 - w_2}{A} \right) \lambda_2(\tau_i^*) = 0.$$

This implies that the impulse instant is characterized by $\lambda_2(\tau_i^*) = \frac{\delta q_1 - w_2}{A}$. From (20b), we have $\lambda_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A} \right) e^{A(T-t)}$ for all $t \in [0, T]$. As, the co-state function $\lambda_2 : [0, T] \to \mathbb{R}$ is strictly monotone, we have at most one impulse instant $\tau_i^* \in (0, T)$ that solves the equation

$$\lambda_2(\tau_i^*) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A} \right) e^{A(T-\tau_i^*)} = \frac{\delta q_1 - w_2}{A}.$$

The unique interior equilibrium impulse instant denoted by τ_{ol}^I is given by

$$\begin{aligned}
\tau_{\mathrm{ol}}^I &= T - \frac{1}{A} \ln \left(\left(\frac{B}{Q} \right)^2 \left(\frac{P_2}{R_1} \right) \frac{q_1}{As_2 + w_2} \right) \\
&= T - \frac{1}{A} \ln \left(\frac{\delta q_1}{As_2 + w_2} \right).
\end{aligned} \tag{25}$$

Since $\tau_{\mathrm{ol}}^I \in (0, T)$, we must have (21a)–(21b) which are expressed in terms of problem parameters.

Next, if there is an impulse at the initial time, then from (20f), (7e), (20d), (20e), we have

$$\lambda_2(0) \left(A\lambda_2(0) - (\delta q_1 - w_2) \right) > 0.$$

On substituting $\lambda_2(0) = \frac{As_2 e^{AT} + w_2(e^{AT} - 1)}{A}$, we get inequality (21c) that describes the problem parameters when impulse occurs at the initial time.

Next, if there is an impulse at the final time, then from (20f), (7e), (20d), (20e), we have

$$\lambda_2(T) \left(A\lambda_2(T) - (\delta q_1 - w_2) \right) < 0,$$

On substituting $\lambda_2(T) = s_2$, we find that an impulse occurs at the final time when (21d) holds true.

Using (7c) and (7e), we obtain the co-state variable $\lambda_1(t)$ satisfies (12c) and (12g) at the impulse instants. With $k^* = 1$ and impulses at $t = 0$, $t = \tau_{\mathrm{ol}}^I$, $t = T$, the equilibrium controls of Player 1 and the equilibrium impulse levels of Player 2 are given by (23). $\qquad \square$

Remark 5 *In Theorem 4, we have only provided the equilibrium controls of the players when $k^* = 1$ for brevity. The equilibrium controls of the players for $k^* = 2$ and $k^* = 3$ can be obtained by using the necessary conditions (7) and (20).*

Remark 6 *Since the continuous Hamiltonian of Player 2 is a function of the equilibrium control of Player 1, the impulse timing also depends on the problem parameters of Player 1.*

The parameter values which satisfy the inequalities (21a)–(21b), (21c), (21d) are shown in Figure 1.

4.2   Feedback Nash equilibrium

Next, we characterize the FNE when both the level and timing of the impulse instants are Player 2's decision variables. First, we consider Player 1's optimal control problem assuming that Player 2's equilibrium policy $\tilde{v}^* = \{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_k^*, v_{k^*}^*), k^*\}$ is given. Similar to the analysis done in Section 3.2, let $V_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ denote the value function of Player 1. Then, we have

$$V_1(t, x) = \max_{u(s),\ s \in [t, T]} \left\{ \int_t^T \frac{1}{2} \left(2w_1 x(s) + R_1 u(s)^2 \right) ds + \sum_{i=l}^{k^*} q_1 x(\tau_i^{*-}) + s_1 x(T^+) \right\}, \tag{26}$$

(a) $A = 0$, $q > 0$

(b) $A = 0$, $q < 0$

(c) $A > 0$, $q > 0$

(d) $A > 0$, $q < 0$

(e) $A < 0$, $q > 0$

(f) $A < 0$, $q < 0$
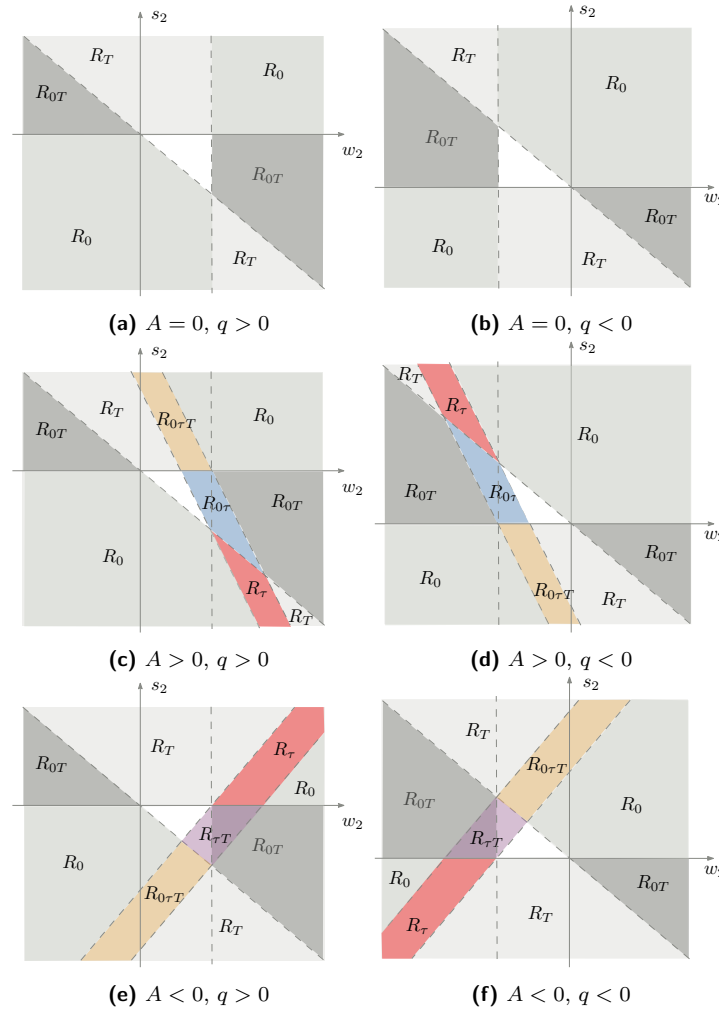
Figure 1: The regions are described as follows: R_0 : **Impulse at** $t = 0$, R_T : **Impulse at** $t = T$, R_{0T} : **Impulse at** $t = 0$ **and** $t = T$, $R_{0\tau}$: **Impulse at** $t = 0$ **and** $t = \tau^I_{ol}$, $R_{\tau T}$: **Impulse at** $t = \tau^I_{ol}$ **and** $t = T$, $R_{0\tau T}$: **Impulse at t=0,** $t = \tau^I_{ol}$ **and** $t = T$.

where the state variable evolves during the non-impulse instants $s \neq \{\tau^*_l, \tau^*_{l+1}, \cdots, \tau^*_{k^*}\}$, $t \geq \tau^*_l$ as

$$\dot{x}(s) = Ax(s) + Bu(s), \ x(t) = x,$$

and during a switching instant $\tau^*_i \ (i = 1, 2, \cdots, k^*)$ undergoes jumps according to

$$x(\tau^{*+}_i) = x(\tau^{*-}_i) + Qv^*_i.$$

In the impulse-free region $[\tau^{*+}_i, \tau^{*-}_{i+1}]$, the following Hamilton-Jacobi-Bellman (HJB) equation holds true:

$$-\frac{\partial V_1(t,x)}{\partial t} = \max_{u \in \Omega_u} \left(w_1 x + \frac{1}{2} R_1 u(t)^2 + \left(\frac{\partial V_1}{\partial x} \right) (Ax + Bu(t)) \right). \tag{27}$$

At the jump instants, $\{\tau^*_1, \tau^*_2, \cdots, \tau^*_{k^*}\}$, the value functions are related as follows:

$$V_1(\tau^{*-}_i, x(\tau^{*-}_i)) = V_1(\tau^{*+}_i, x(\tau^{*+}_i)) + q_1 x(\tau^{*-}_i). \tag{28}$$

Given the equilibrium strategy $u^*(.)$ of Player 1, following (6b), we define the value function associated with Player 2's impulse optimal control problem as follows:

$$V_2(t,x) = \max_{\{(\tau_i, v_i)^k_{i=1}\}} \left\{ \int_t^T w_2 x(s) ds + \sum_{i=1}^k \left(C + \frac{1}{2} P_2 v^2_i \right) + s_2 x(T^+) \right\}, \tag{29}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$, as

$$\dot{x}(s) = Ax(s) + Bu^*(s), \ x(t) = x,$$

and during a switching instant $\tau_i \ (i = 1, 2, \cdots, k)$ undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i.$$

We emphasize that Player 2's problem differs, structurally, in the endogenous case from the exogenous case as the number of impulses k and the timing of the impulses $\tau_i \ (i = 1, 2, \cdots, k)$ are also decision variables to be determined besides the size of the impulses $v_i \ (i = 1, 2, \cdots, k)$. Impulse optimal control problems with endogenous decision variables are closely related to optimal stopping problems, and use tools from quasi-variational inequalities (QVIs); see Bensoussan and Tapiero (1982), Bensoussan and Lions (1982), Bensoussan and Lions (1984) for early works in this area. In the following discussion, we briefly summarize the necessary concepts associated with QVIs before proceeding with the characterization of the FNE.

Assumption 3 *We assume that $V_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable in its arguments.*

Given the value function $V_2(t, x)$ of Player 2, we define the operator $\mathcal{R}$ as follows:

$$\mathcal{R}V_2(t, x) := \max_{v \in \Omega_v} \left(\frac{1}{2} P_2 v^2 + C + V_2(t, x + Qv) \right). \tag{30}$$

We introduce the Hamiltonian function $\mathcal{H}_2 : [0, T] \times \mathbb{R} \times \mathbb{R}$ as follows:[4]

$$\mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) = w_2 x + \frac{\partial V_2}{\partial x} \left(Ax + Bu(t) \right). \tag{31}$$

From Aubin (1982), Bensoussan and Lions (1982) and Bensoussan and Lions (1984), it can be shown that the value function (29) satisfies the following Bensoussan-Lions quasi-variational inequalities, that is,

$$\frac{\partial V_2}{\partial t} + \mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) \leq 0, \qquad \forall(t, x) \in (0, T) \times \mathbb{R}, \tag{32a}$$

$$V_2(t, x) - \mathcal{R}V_2(t, x) \geq 0, \qquad \forall(t, x) \in [0, T] \times \mathbb{R}, \tag{32b}$$

$$\left(\frac{\partial V_2}{\partial t} + \mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) \right) (V_2(t, x) - \mathcal{R}V_2(t, x)) = 0, \qquad \forall(t, x) \in (0, T) \times \mathbb{R}, \tag{32c}$$

$$V_2(T, x) = \max\{\zeta(x), s_2 x\}, \tag{32d}$$

$$\text{where } \zeta(x) = \max_{v \in \Omega_v}\{s_2(x + Qv) + C + \frac{1}{2}P_2 v^2\}. \tag{32e}$$

In the following, we provide a heuristic interpretation of the QVIs (32). When the state is at a given level x at time t, Player 2 can either give an impulse or wait. Suppose that an impulse does not occur in the time interval $[t, t + h]$. Since Player 2 waits, using the dynamic programming principle, we conclude that the value function is bounded from below by the sum of the running profit from t to $t + h$ and the optimal profit from time $t + h$ onwards, that is,

$$V_2(t, x) \geq \int_t^{t+h} w_2 x(s)ds + V_2(t + h, x(t + h)).$$

---

[4]The Hamiltonian associated with the value function of Player 2 is different from the Hamiltonian of Player 2 given in (8a) associated with the co-state of Player 2. The two Hamiltonians are equal when the gradient of the value function is equal to the co-state variable.

From Assumption 3 and using a Taylor series expansion of the above expression, and letting $h \to 0$, we obtain (32a). If it is optimal for Player 2 to give an impulse at time t, then the state jumps from $x(t)$ to $x(t) + Qv$, such that

$$V_2(t, x) \geq \max_{v \in \Omega_v} \left(C + \frac{1}{2} P_2 v^2 + V_2(t, x + Qv) \right) =: \mathcal{R} V_2(t, x).$$

This verifies (32b). Clearly, at any (t, x), Player 2 can either wait, which implies that (32a) holds with equality, or she can give an impulse so that (32b) holds with equality. This implies that the complementarity condition (32c) holds to ensure that either (32a) or (32b) holds with equality. If there is no impulse at the final time, the value function is equal to the salvage value; otherwise, the value function is equal to the maximum value that Player 2 can obtain by giving an impulse at T, and this justifies condition (32d).

Using (30), we define the following two sets. The first is a stopping or intervention set $\mathcal{S}$, which is defined as

$$\mathcal{S} := \left\{ (t, x) \in [0, T] \times \mathbb{R} \mid V_2(t, x) = \mathcal{R} V_2(t, x) \right\}. \tag{33}$$

The stopping set characterizes all the data points $(t, x) \in [0, T] \times \mathbb{R}$ where it is optimal for Player 2 to give an impulse. The second is a continuation set $\mathcal{C}$, defined as

$$\mathcal{C} := \left\{ (t, x) \in [0, T] \times \mathbb{R} \mid V_2(t, x) > \mathcal{R} V_2(t, x) \right\}. \tag{34}$$

Clearly, from the definition of $\mathcal{C}$, it is optimal for Player 2 to not give an impulse at the data point $(t, x) \in \mathcal{C}$. In other words, the continuation set characterizes the impulse-free region.

Due to the linear structure of the game, we have the following assumption on the form of the value functions of the players.

Assumption 4 *We assume that the value function of Player i $(i = 1, 2)$ is given by*

$$V_i(t, x) = \alpha_i(t)x + \beta_i(t). \tag{35}$$

The next theorem characterizes the impulse instants in the FNE when the impulse timing is endogenously determined by Player 2. To save on notation, we introduce $\gamma = \sqrt{\frac{2P_2 C}{Q^2}}$.

Theorem 5 *Let Assumption 1 and 4 hold. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. There can be at most two impulses in the FNE, and they occur at $\tau_{fb}^1 = 0$ and $\tau_{fb}^2 = T$.*

Proof. We substitute the value function of Player 1 given in Assumption 4 in the HJB equation (27) to obtain

$$-\dot{\alpha}_1(t)x - \dot{\beta}_1(t) = \max_{u(t) \in \Omega_u} \{ w_1 x + \frac{1}{2} R_1 u(t)^2 + \alpha_1(t)(Ax + Bu(t)) \}.$$

Following Assumption 1 on the interior solutions, the first-order condition associated with the above maximization problem results in

$$u^*(t) = -\frac{B\alpha_1(t)}{R_1}. \tag{36}$$

Substituting for the above solution in the HJB equation (36), we obtain

$$-\dot{\alpha}_1(t)x - \dot{\beta}_1(t) = w_1 x - \frac{B^2 \alpha_1(t)^2}{2R_1} + A\alpha_1(t)x.$$

Applying the method of undetermined coefficients gives

$$\dot{\alpha}_1(t) = -w_1 - A\alpha_1(t), \ \alpha_1(T^+) = s_1, \tag{37a}$$

$$\dot{\beta}_1(t) = \frac{B^2\alpha_1(t)^2}{2R_1}, \ \beta_1(T^+) = 0. \tag{37b}$$

From (28), at any impulse instant τ_i^*, we have the following relation:

$$\alpha_1(\tau_i^{*-})x(\tau_i^{*-}) + \beta_1(\tau_i^{*-}) = \alpha_1(\tau_i^{*+})(x(\tau_i^{*-}) + Qv_i^*) + \beta_1(\tau_i^{*+}) + q_1 x(\tau_i^{*-}),$$

which implies

$$\alpha_1(\tau_i^{*-}) = \alpha_1(\tau_i^{*+}) + q_1, \tag{38a}$$

$$\beta_1(\tau_i^{*-}) = \beta_1(\tau_i^{*+}) + \alpha_1(\tau_i^{*+})Qv_i^*. \tag{38b}$$

Next, we determine the coefficients of the value function of Player 2 given in Assumption 4. First, we determine the values of $\alpha_2(T)$ and $\beta_2(T)$. Following Assumption 1, we take the partial derivative of the right-hand side of (32e) with respect to v and equate it to 0 to obtain $v_T^* = -\frac{s_2 Q}{P_2}$. Substituting v_T^* in (32e), we obtain

$$\zeta(x) = s_2 x + C - \frac{(s_2 Q)^2}{2P_2}. \tag{39}$$

Clearly, $\zeta(x) \geq s_2 x$ if $\frac{2P_2 C}{Q^2} \leq s_2^2$. From (32d), we obtain that if $\frac{2P_2 C}{Q^2} \leq s_2^2$,

$$\alpha_2(T)x + \beta_2(T) = s_2 x + C - \frac{(s_2 Q)^2}{2P_2},$$

$$\Rightarrow \alpha_2(T) = s_2, \ \beta_2(T) = C - \frac{(s_2 Q)^2}{2P_2}, \tag{40}$$

and if $\frac{2P_2 C}{Q^2} \geq s_2^2$, then

$$\alpha_2(T)x + \beta_2(T) = s_2 x,$$

$$\Rightarrow \alpha_2(T) = s_2, \ \beta_2(T) = 0. \tag{41}$$

In the impulse-free region, (32a) holds with equality. Using (36) in (32a), we obtain

$$w_2 x(t) + \dot{\alpha}_2(t)x + \dot{\beta}_2(t) + \alpha_2(t)\left(Ax - \frac{B^2\alpha_1(t)}{R_1}\right) = 0.$$

Applying the method of undetermined coefficients gives

$$\dot{\alpha}_2(t) = -w_2 - A\alpha_2(t), \alpha_2(T) = s_2, \tag{42a}$$

$$\dot{\beta}_2(t) = \frac{B^2\alpha_1(t)\alpha_2(t)}{R_1}, \tag{42b}$$

where $\beta_2(T)$ is given by (40) if there is an impulse at T, and if it is not optimal to give an impulse then $\beta_2(T)$ is given by (41). Solving for $\alpha_2(t)$, we have

$$\alpha_2(t) = w_2(T - t) + s_2, \ A = 0, \tag{43a}$$

$$\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right), \ A \neq 0. \tag{43b}$$

Under Assumption 4, we compute $\mathcal{R}V_2$ as

$$\mathcal{R}V_2(t, x) = \max_{v \in \Omega_v}\{C + \frac{1}{2}P_2 v^2 + \alpha_2(t)(x + Qv) + \beta_2(t)\}. \tag{44}$$

Following Assumption 1 on the interior solutions, the first-order condition associated with the maximization problem (44) results in

$$v^* = -\frac{Q\alpha_2(t)}{P_2}. \tag{45}$$

Substituting the above solution in (44) yields

$$\mathcal{R}V_2(t,x) = C + \frac{Q^2\alpha^2(t)}{2P_2} + V_2(t,x) \tag{46}$$

$$\Rightarrow V_2(t,x) - \mathcal{R}V_2(t,x) = -C + \frac{Q^2\alpha^2(t)}{2P_2}. \tag{47}$$

Then, the stopping set (33) is given by

$$\mathcal{S} := \left\{ (t,x) \in [0,T] \times \mathbb{R} \ \Big| \ \alpha_2^2(t) = \frac{2P_2C}{Q^2} \right\}, \tag{48}$$

and the continuation set (34) is given by

$$\mathcal{C} := \left\{ (t,x) \in [0,T] \times \mathbb{R} \ \Big| \ \alpha_2^2(t) < \frac{2P_2C}{Q^2} \right\}. \tag{49}$$

When $As_2 + w_2 = 0$, there is an impulse at each instant of time for $s_2 = \gamma$ and for $s_2 = -\gamma$ which means that the impulse instants do not satisfy monotone increasing sequence property given in (1). From (43a) and (43b), we know that for $As_2 + w_2 = 0$, $\alpha_2(t) = s_2$. So, for $As_2 + w_2 = 0$, there is no impulse when $s_2 \neq \gamma$ and $s_2 \neq -\gamma$. Next, we analyze the cases where $As_2 + w_2 \neq 0$.

Clearly, $\alpha_2(t)$ given in (43a) and (43b) is strictly monotone in t for $As_2 + w_2 \neq 0$, so it can take values $\sqrt{\frac{2P_2C}{Q^2}}$ and $-\sqrt{\frac{2P_2C}{Q^2}}$ at most once. This naturally implies from equation (49), that there can be at most two impulses, and they occur at $\tau_{\text{fb}}^1 = 0$ and $\tau_{\text{fb}}^2 = T$. □

Remark 7 *In the linear-state differential games with impulse control, the stopping set given in* (48), *and the continuation set given in* (49) *are independent of the state of the system.*

From (32b), the value function must satisfy $V_2(t,x) \geq \mathcal{R}V_2(t,x) \Rightarrow \alpha_2^2(t) \leq \gamma^2$ for all $(t,x) \in [0,T] \times \mathbb{R}$. As a result, this condition imposes certain restrictions on the parameter region where the linear value function is well-defined. Next theorem characterizes this region.

Theorem 6 *Let Assumption 4 hold true. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. The linear value function* (35) *is well-defined when the parameters satisfy the following conditions.*

   (i) $A = 0$, $w_2 \geq 0$, $Tw_2 + s_2 \leq \gamma$, $s_2 \geq -\gamma$
   (ii) $A = 0$, $w_2 \leq 0$, $Tw_2 + s_2 \geq -\gamma$, $s_2 \leq \gamma$
  (iii) $A > 0$, $As_2 + w_2 > 0$, $s_2 \geq -\gamma$, $As_2e^{AT} + w_2(e^{AT} - 1) - A\gamma \leq 0$
  (iv) $A > 0$, $As_2 + w_2 < 0$, $s_2 \leq \gamma$, $As_2e^{AT} + w_2(e^{AT} - 1) + A\gamma \geq 0$
   (v) $A < 0$, $As_2 + w_2 > 0$, $s_2 \geq -\gamma$, $As_2e^{AT} + w_2(e^{AT} - 1) - A\gamma \geq 0$
  (vi) $A < 0$, $As_2 + w_2 < 0$, $s_2 \leq \gamma$, $As_2e^{AT} + w_2(e^{AT} - 1) + A\gamma \leq 0$

Proof. We recall that the value function $V_2(t,x)$ must satisfy the condition (32b). This implies $\alpha_2^2(t) \leq \gamma^2$ for all $t \in [0,T]$.

With $A = 0$, we get $\alpha_2(t) = w_2(T-t) + s_2$, which is an increasing (decreasing) function of time t when w_2 is negative (positive). Then, we must have $(w_2(T-t) + s_2)^2 \leq \gamma^2$ for all $t \in [0,T]$, and this condition is satisfied when conditions (i)–(ii) hold true.

When $A \neq 0$, we get $\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right)$ is decreasing in t if $As_2 + w_2 > 0$ and is increasing in t if $As_2 + w_2 < 0$. Using a similar analysis as before, for $A > 0$ $(A < 0)$, the value function is defined only in the region where the parameters satisfy $\left(-\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right)\right)^2 \leq \gamma^2$, which is characterized by the conditions (iii)–(iv) and (v)–(vi).   □

The parameter regions where the value function $V_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is well-defined is illustrated in the Figure 2. In particular, the shaded regions in the Figures 2a, 2b and 2c correspond to the regions defined by the conditions (i)–(ii), (iii)–(iv) and (v)–(vi), respectively. We can not comment on the value function for parameter values outside the shaded regions.

The next result characterizes the number and the level of impulses in the FNE.

(a) $A = 0$ **(b)** $A > 0$ **(c)** $A < 0$
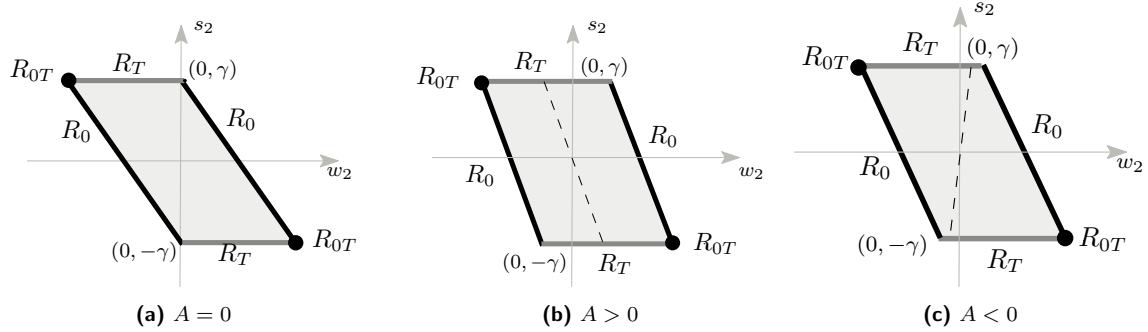
Figure 2: Shaded regions correspond to parameter space in the (w_2, s_2) plane for which the value function is well-defined. An impulse occurs at $t = T$ for parameters corresponding to the upper and lower boundaries of the shaded regions, denoted by R_T. For the left and right boundaries denoted by R_0, there is an impulse at $t = 0$. R_{0T} denotes that impulse occur at $t = 0$ and $t = T$.

Theorem 7 *Let Assumption 1 and 4 hold. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. There can exist at most two impulses in the FNE, that is, $k^* \leq 2$.*

(i) *If the parameters satisfy either of the following conditions, then an impulse occurs at $\tau_{fb}^1 = 0$:*

   (a) *with $A = 0$: either $Tw_2 + s_2 = \gamma$ or $Tw_2 + s_2 = -\gamma$,*

   (b) *with $A \neq 0$: either $As_2 e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$ or $As_2 e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$.*

(ii) *If either $s_2 = \gamma$, or $s_2 = -\gamma$, then an impulse occurs at $\tau_{fb}^2 = T$.*

(iii) *If the parameters satisfy either of the following conditions, then there are exactly two impulses at $\tau_{fb}^1 = 0$ and $\tau_{fb}^2 = T$:*

$$A = 0, \ s_2 = -\gamma, \ Tw_2 = 2\gamma, \tag{50a}$$

$$A = 0, \ s_2 = \gamma, \ Tw_2 = -2\gamma, \tag{50b}$$

$$A \neq 0, \ s_2 = -\gamma, \ w_2 = A\gamma\frac{e^{AT} + 1}{e^{AT} - 1}, \tag{50c}$$

$$A \neq 0, \ s_2 = \gamma, \ w_2 = -A\gamma\frac{e^{AT} + 1}{e^{AT} - 1}. \tag{50d}$$

The equilibrium control of Player 1 when $k^* = 1$, impulse occurs at initial time and $t \in (0, T]$ is

$$u(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right) & A \neq 0 \\ \frac{B}{R_1}\left(w_1(t - T) - s_1\right) & A = 0 \end{cases}, \tag{51a}$$

and when impulse occurs at the final time and $t \in [0, T)$, we have

$$u(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + q_1 + \frac{w_1}{A})e^{A(T-t)}\right) & A \neq 0 \\ \frac{B}{R_1}\left(w_1(t - T) - (s_1 + q_1)\right) & A = 0 \end{cases}. \tag{51b}$$

If either $A = 0$ and $w_2 < 0$, or $A \neq 0$ and $As_2 + w_2 < 0$, then, for $k^* = 1$, the equilibrium impulse levels of Player 2 for impulses at $\tau_{fb}^1 = 0$, $\tau_{fb}^2 = T$ are given by

$$v_{fb}^1 = -\operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}, \ v_{fb}^2 = \operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}. \tag{52a}$$

If either $A = 0$ and $w_2 > 0$, or $A \neq 0$ and $As_2 + w_2 > 0$, then, for $k^* = 1$, the equilibrium impulse levels of Player 2 for impulses at $\tau_{fb}^1 = 0$, $\tau_{fb}^2 = T$ are given by

$$v_{fb}^1 = \operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}, \ v_{fb}^2 = -\operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}. \tag{53a}$$

Proof. In Theorem 5, it is shown that impulses can occur at $\tau_{\mathrm{fb}}^1 = 0$ and $\tau_{\mathrm{fb}}^2 = T$ only. We know from (43a) and (43b) that

$$\alpha_2(t) = \begin{cases} w_2(T - t) + s_2, & A = 0, \\ -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right), & A \neq 0. \end{cases} \tag{54a}$$

From (48), an impulse occurs when $\alpha_2(t)^2 = \gamma^2$. For an impulse to occur at $\tau_{\mathrm{fb}}^1 = 0$, we have either $\alpha_2(0) = \gamma$ or $\alpha_2(0) = -\gamma$. Similarly, an impulse occurs at $\tau_{\mathrm{fb}}^2 = T$ when $\alpha_2(T) = \gamma$ or $\alpha_2(T) = -\gamma$.

Also, $\alpha_2(t)$ is strictly monotone in time. So, two impulses occur at initial and final time if either $\alpha_2(0) = -\gamma$ and $\alpha_2(T) = \gamma$ or $\alpha_2(T) = \gamma$ and $\alpha_2(0) = \gamma$, that is, conditions (50a)–(50d) hold true.

Next, we characterize the FNE of the differential game (2)–(5) when impulses occur at $\tau_{\mathrm{fb}}^1 = 0$ and $\tau_{\mathrm{fb}}^2 = T$. The equilibrium controls of Player 1 given in (51) are obtained by first solving for $\alpha_1(.)$ from (37a) and (38a), and then using $u(t) = -\frac{B}{R_1}\alpha_1(t)$. To obtain the equilibrium impulse levels for Player 2, we insert $\alpha_2(t)$ evaluated at $t = 0$ and $t = T$ from (43a) and (43b) in (45). The impulse levels are given by $v_{\mathrm{fb}}^1 = -\frac{Q}{P_2}\alpha_2(0)$ and $v_{\mathrm{fb}}^2 = -\frac{Q}{P_2}\alpha_2(T)$. When $\alpha_2(t)$ is increasing (decreasing) in time, we have $\alpha_2(0) = -\gamma$ ($\alpha_2(0) = \gamma$) and $\alpha(T) = \gamma$ ($\alpha_2(T) = -\gamma$). Therefore, the impulse levels are given by (52), (53) depending on the problem parameters. $\qquad\square$

Remark 8 *We have the following observations: (i) The level of impulse is a constant and proportional to the ratio of fixed cost C and the coefficient of proportional transaction cost P_2. Note that P_2 can be interpreted as the marginal cost at zero impulse, i.e., $\left.\dfrac{\partial\left(\frac{1}{2}P_2 v_i^2\right)}{\partial v_i}\right|_{v_i=0}$. (ii) The timing of an impulse by Player 2 is independent of Player 1's parameters. Indeed, it depends on Player 2's parameter values and the coefficient entering the state dynamics. Finally, (iii) when there are two impulses, the magnitude of the impulses is the same and they are opposite in sign.*

4.3 Comparison of open-loop and feedback Nash equilibria

From Theorems 4 and 7, it is clear that OLNE and FNE do not coincide when the number and timing of impulse instants are decision variables of Player 2. In the following, we highlight reasons as to why these equilibria differ in the endogenous case.

In the OLNE, the Hamiltonian continuity condition (20f) reduces to an affine function of $\lambda_2(t)$ in $(0, T)$ whereas at $t = 0$ and $t = T$, we obtain an inequality that is quadratic in $\lambda_2(t)$. Since the co-state is strictly monotone, at most three impulses can occur, see Figure 3a.

In the FNE, the continuation set is characterized by the time interval during which the gradient of the value function of Player 2 satisfies $-\gamma < \alpha_2(t) < \gamma$. The stopping set is characterized by the time instants at which $\alpha_2(t)$ takes a value of either γ or $-\gamma$. There is no dependence of stopping set on the equilibrium control of Player 1 while in the OLNE, the Hamiltonian continuity condition, which determines the impulse timing, depends on the equilibrium control of Player 1. From (43), $\alpha_2(t)$ is strictly monotone function of time, and it can achieve a maximum and minimum value of γ and $-\gamma$ at $t = 0$ or $t = T$ for all $x \in \mathbb{R}$; see Figure 3b.
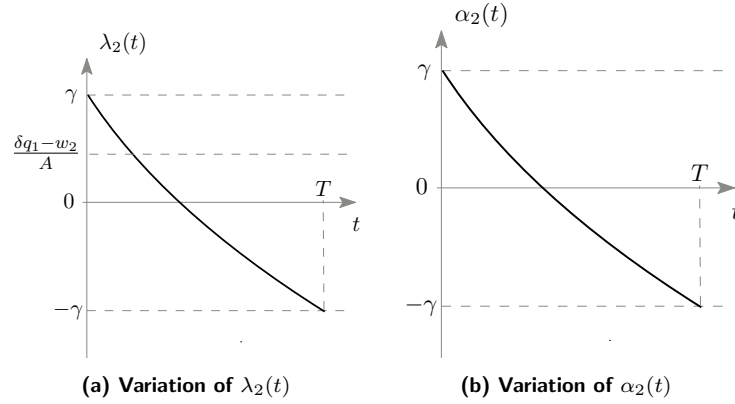
(a) Variation of $\lambda_2(t)$ (b) Variation of $\alpha_2(t)$

Figure 3: Variation of $\lambda_2(t)$ and $\alpha_2(t)$ for three impulses in OLNE whereas there are two impulses in FNE.

Remark 9 *When both the continuous payoff and salvage value of Player 2 either increase in x or decrease in x, i.e., if $w_2 > 0, s_2 > 0$ or $w_2 < 0, s_2 < 0$, then it is clear from Figure 2 that there can be at most one impulse in the FNE while from Figure 1, there can be at most three impulses in the OLNE.*

Now, we study the open-loop and feedback Nash equilibrium solutions for the parameter regions where the value function of Player 2 is well-defined.

(i) Assume that Player 2 incurs a running cost, i.e., $w_2 < 0$ and that the salvage value of Player 2 is decreasing in x, i.e., $s_2 < 0$. Also, assume that $w_2 \neq q_1\delta$ when $A = 0$, and $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$.

With $A = 0$, an impulse can occur at the initial time in the OLNE and FNE when $Tw_2 + s_2 = -\gamma$. However, for other parameter values in the shaded region in Figure 2a, there are no impulses in the FNE, while an impulse can occur at the initial time in the OLNE for all $w_2 < 0, s_2 < 0$; see Figure 1a, Figure 1b.

With $A > 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 < 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 1d). Further, there can be at most three impulses in the OLNE when $q_1 < 0$. FNE has no interior impulses and $\tau_{\text{fb}}^1 = 0$ is an impulse instant when $As_2 e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$ and $\tau_{\text{fb}}^2 = T$ is an impulse instant for $s_2 = -\gamma$. For the other parameter values in the shaded region in Figure 2b, there is no impulse in the FNE.

With $A < 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 1e), or $q_1 < 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 1f). In the OLNE, there can be at most three impulses when $q_1 > 0$. In the FNE, an impulse occurs at $\tau_{\text{fb}}^1 = 0$ when $As_2 e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$, $\tau_{\text{fb}}^2 = T$ is an impulse instant when $s_2 = -\gamma$, and for other parameter values, there is no impulse; see Figure 2c.

(ii) Second, we assume that Player 2 values the state positively so that $w_2 > 0$, and her salvage value is increasing in x, i.e., $s_2 > 0$. Also, assume that $w_2 \neq q_1\delta$ when $A = 0$, and $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$.

With $A = 0$, an impulse can occur at the initial time in the OLNE and FNE when $Tw_2 + s_2 = \gamma$; see Figure 1a, 1b, 2a. There are no impulses in the FNE for any other parameter value while an impulse can occur in the OLNE for all $w_2 > 0, s_2 > 0$.

With $A > 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 1c). There can be at most three impulses in the OLNE when $q_1 > 0$. In the FNE, $\tau_{\text{fb}}^1 = 0$ is an equilibrium impulse instant when $As_2 e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$ and $\tau_{\text{fb}}^2 = T$ is an equilibrium impulse instant when $s_2 = \gamma$. For other parameter values, there is no impulse in the FNE. (see Figure 2b).

With $A < 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 1e) or $q_1 < 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 1f). There can be at most three impulses in the OLNE if $q_1 < 0$. In the FNE, $\tau_{\text{fb}}^1 = 0$ is an impulse instant when $As_2 e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$, $\tau_{\text{fb}}^2 = T$ is an impulse instant when $s_2 = \gamma$, and for other parameter values, there is no impulse; see Figure 2c.

5 Numerical example

In this section, we illustrate our results with a numerical example. Consider a two-player differential game between Player 1 who values the state positively and Player 2 who values the state negatively. For instance, Player 1 can be a firm that aims to increase the security of a system and invests effort in reducing the system vulnerabilities while Player 2 is a hacker that invests effort in reducing the security of a system. Player 2 uses an impulse control that consists of determining the number $k \in \mathbb{N}$ and timing of impulses τ_i, $(i = 1, 2, \cdots, k)$ in addition to the corresponding effort level v_i. We consider that at the impulse times, Player 2 incurs a fixed cost, and a variable cost that is quadratic in the effort level v_i. The fixed cost discourages Player 2 to intervene frequently.

The objective functions of Player 1 and 2 are given by

$$J_1 = \int_0^T [4x(t) - 0.5u(t)^2]dt - \sum_{i=1}^k 0.3x(\tau_i^-) + x(T^+),$$

$$J_2 = -\int_0^T 0.8x(t)dt - \sum_{i=1}^k (0.1v_i^2 + 1) - x(T^+),$$

with the state dynamics given by

$$\dot{x}(t) = -0.1x + 0.6u(t),\ x(0^-) = 5,$$
$$x(\tau_i^+) - x(\tau_i^-) = 0.2v_i,$$

where $T = 5$.

Under the open-loop information structure and using the necessary conditions, the candidate solution for impulse in OLNE is $\tau = 2.4$. The OLNE is given by $(u_{\text{ol}}^*(.), (2.4, -2.6), k^* = 1)$ where equilibrium effort for Player 1 is given by

$$u_{\text{ol}}^*(t) = \begin{cases} 24 - 14.33\,e^{0.1t} & t \in [0, 2.4), \\ 24 - 14.19\,e^{0.1t} & t \in (2.4, T]. \end{cases}$$

The open-loop Nash equilibrium payoff of Player 1 is 167.98 while Player 2 obtains a payoff of -66.42. The FNE is given by $(u_{\text{fb}}^*(t), k^* = 0\})$ where Player 2 does not give any impulse, and the equilibrium effort of Player 1 is given by

$$u_{\text{fb}}^*(t) = 24 - 14.19\,e^{0.1t},\ t \in [0, T].$$

The equilibrium payoff of Player 1 is given by 177.31, and Player 2 obtains a payoff of -66.83.

Next, we consider the following objective for Player 2:

$$J_2 = -\int_0^T 0.34x(t)dt - \sum_{i=1}^k (0.1v_i^2 + 1) - 3x(T^+),$$

while keeping the other parameter values as before.

In this case, the candidate open-loop Nash equilibrium strategy of Player 2 is to give an impulse at the final time T. The OLNE and FNE are given by $(u_{\text{ol}}^*, (T, -3), k^* = 1)$ and $(u_{\text{fb}}^*, (0, -3.16), k^* = 1)$ where

$$u_{\text{ol}}^*(t) = 24 - 14.30\,e^{0.1t},\ t \in [0, T),$$

$$u_{\text{fb}}^*(t) = 24 - 14.19\,\mathrm{e}^{0.1t},\ t \in (0, T].$$

The equilibrium payoff of Player 1 and Player 2 in OLNE is 172.17 and -66.71, respectively while in the FNE, Player 1 and 2 obtain 165.47 and -67.93, respectively.

In both cases, we see that Player 1 uses controls that increases the state while Player 2's equilibrium impulse decreases the state value. When compared with the OLNE, Player 2 obtains a lower payoff in the FNE. Due to the state-dependent costs incurred because of the intervention by Player 2 in $(0, T]$, Player 1's equilibrium strategy is to invest lower effort in OLNE when compared with the FNE.

6   Some extensions

In this section, we consider two extensions of the canonical differential game model described by (2–5). In particular, we show that the conclusions obtained in Sections 3 and 4 remain unaltered, qualitatively, for the following extensions.

6.1   General cost structures

Suppose the piecewise continuous control of Player 1 involves a cost $d(u)$ and the variable cost of impulse for Player 2 is given by $c(v_i)$. We make the following assumption to obtain a unique expression for piecewise continuous control of Player 1 and for the impulse level of Player 2.

Assumption 5 *We assume that the functions $d : \Omega_u \to \mathbb{R}$ and $c : \Omega_v \to \mathbb{R}$ are continuous and twice continuously differentiable. Further, we assume that these functions admit interior maxima, and satisfy $\frac{\partial^2 [d(u)]}{\partial u^2} < 0$ over Ω_u and $\frac{\partial^2 [c(v)]}{\partial v^2} < 0$ over Ω_v.*

Theorem 8 *Let Assumption 1 and 5 hold, and assume that the value functions of both players are linear in state. Then the open-loop and feedback Nash equilibria of the differential game (2–5) coincide when the number and timing of impulses is exogenously given. When the number and timing of impulse instants are decision variables of Player 2, then these two equilibria are different.*

Proof. See Appendix B. $\square$

In the above theorem, we showed that our results hold qualitatively when we consider a general cost structure. Next, we analyze the multi-dimensional extension of our scalar LSDG model.

6.2   Multi-dimensional state

We consider a multi-dimensional extension of the linear-state game described by (2–5), and examine if the conclusions derived in Sections 3 and 4 still hold true. Towards this end, we assume that the state variable is an n-dimensional vector, and the controls satisfy $u(t) \in \mathbb{R}^{m_1}$ and $v \in \mathbb{R}^{m_2}$. The parameters in (2–5) are $w_1, w_2, q_1, s_1, s_2 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_1}$, $Q \in \mathbb{R}^{n \times m_2}$, $R_1 \in \mathbb{R}^{m_1 \times m_1}$, $P_2 \in \mathbb{R}^{m_2 \times m_2}$.

With exogenously given impulse instants, we use the necessary conditions (7a)–(7e), (9a)–(9e) to obtain the equilibrium control $u^*(t)$ of Player 1 and equilibrium impulse level v_i^* of Player 2. Under the feedback information structure, we can use the dynamic programming principle to show that the gradients of the value functions of Player 1 and Player 2 given in Assumption 2 are equal to the co-states of players in the OLNE, and the equilibrium controls are the same for both the players which implies that OLNE and FNE coincide. When the impulse instants are decision variables of Player 2, the equilibrium impulse instants satisfy (20f) where the difference of the Hamiltonian of Player 2 before and after the equilibrium impulse instant is given by

$$\left(q_1' B R_1^{-1} B' - (w_2' + \lambda_2(\tau_i^*)' A) Q P_2^{-1} Q' \right) \lambda_2(\tau_i^*), \tag{55}$$

where $\lambda_2(t) \in \mathbb{R}^n$. In the FNE, value function of the players given in Assumption 4 satisfy (32b) with equality. Therefore, the stopping set $\mathcal{S}$ and the continuation set $\mathcal{C}$ are given by

$$\mathcal{S} = \{(t,x) \mid \alpha_2(t)'QP_2^{-1}Q'\alpha_2(t) = 2C\},$$
$$\mathcal{C} = \{(t,x) \mid \alpha_2(t)'QP_2^{-1}Q'\alpha_2(t) > 2C\}.$$

where $\alpha_2(t) \in \mathbb{R}^n$. Similar to the scalar case, we have that both the stopping set and continuation set are independent of the state of the system. Also, the impulse timing is completely determined by the problem parameters of Player 2 only whereas it is clear from (55) that the impulse instants in OLNE depend on the problem parameters of Player 1.

7   Conclusions

In this paper, we determined open-loop and feedback Nash equilibria in the scalar deterministic finite-horizon two-player nonzero-sum linear-state differential game with impulse controls, in two cases, namely, when the impulse instants are given and when Player 2 endogenously determines the equilibrium timing of the impulses. We showed that open-loop and feedback equilibria coincide when the impulse instants are exogenously given, and that they differ when these instants are endogenously determined.

For future research, it would be interesting to determine the feedback solutions for more general classes of differential games with impulse controls. A natural first candidate is the class of linear-quadratic differential games, which is often used in applications. Clearly, there would be computational challenges since the stopping set condition would involve the state variables that evolve forward in time, while the Ricatti system of Player 1 and Player 2 evolve backwards in time. Another extension of this work could be to consider the case where both players use piecewise continuous as well as impulse controls.

Appendix

A   Proof of Theorem 2

Assuming that the equilibrium strategy of Player 2 is given by $\tilde{v}^*$, Player 1 solves (15). The Hamilton-Jacobi-Bellman (HJB) equation for Player 1 for $t \in (\tau_i^+, \tau_{i+1}^-)$, $i \in \{0, 1, \cdots, k\}$ is given by

$$-\frac{\partial V_1(t,x)}{\partial t} = \max_{u \in \Omega_u} \left(w_1 x + \frac{1}{2} R_1 u(t)^2 + \left(\frac{\partial V_1}{\partial x} \right)(Ax + Bu(t)) \right).$$

Under Assumption 2, we can rewrite the HJB equation as

$$-\dot{m}_1(t)x - \dot{n}_1(t) = \max_{u \in \Omega_u} \left(w_1 x + \frac{1}{2} R_1 u(t)^2 + m_1(t)(Ax + Bu(t)) \right).$$

Since we have assumed that the equilibrium controls lie in the interior of Ω_u (see Assumption 1), the first-order condition gives:

$$u^*(t) = -\frac{Bm_1(t)}{R_1}. \tag{56}$$

Using the equilibrium control in the HJB equation, we obtain

$$-\dot{m}_1(t)x - \dot{n}_1(t) = w_1 x - \frac{B^2 m_1(t)^2}{2R_1} + Am_1(t)x.$$

On comparing the coefficients, we have

$$\dot{m}_1(t) = -w_1 - Am_1(t), \quad m_1(T^+) = s_1, \tag{57a}$$

$$\dot{n}_1(t) = \frac{B^2 m_1(t)^2}{2R_1}, \ n_1(T^+) = 0. \tag{57b}$$

At the impulse instants, the value functions are related as follows:

$$V_1(\tau_i^-, x(\tau_i^-)) = V_1(\tau_i^+, x(\tau_i^-) + Qv_i) + q_1 x(\tau_i^-), \tag{58}$$

where v_i^* denotes the equilibrium impulse level used by Player 2 at the impulse instant τ_i. Using $V_1(t, x) = m_1(t)x + n_1(t)$, we obtain

$$m_1(\tau_i^-)x(\tau_i^-) + n_1(\tau_i^-) = m_1(\tau_i^+)x(\tau_i^-) + m_1(\tau_i^+)Qv_i + n_1(\tau_i^+) + q_1 x(\tau_i^-),$$

which results in the following update equations for $m_1(.)$ and $n_1(.)$:

$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1, \tag{59a}$$

$$n_1(\tau_i^-) = n_1(\tau_i^+) + m_1(\tau_i^+)Qv_i. \tag{59b}$$

Given the equilibrium strategy $u^*(.)$ of Player 1, Player 2 solves (16). For the impulse-free region, we have the following relation:

$$-\frac{\partial V_2(t, x)}{\partial t} = w_2 x + \left(\frac{\partial V_2}{\partial x}\right)(Ax + Bu^*(t)),$$

which, on substituting the equilibrium control $u^*(t)$ of Player 1 and the value function of Player 2, $V_2(t, x) = m_2(t)x + n_2(t)$ (see Assumption 2) simplifies to

$$w_2 x + \dot{m}_2(t)x + \dot{n}_2(t) + m_2(t)(Ax - \frac{B^2 m_1(t)}{R_1}) = 0.$$

On comparing the above coefficients, we get for $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$,

$$\dot{m}_2(t) = -w_2 - Am_2(t), \ m_2(T^+) = s_2, \tag{60a}$$

$$\dot{n}_2(t) = \frac{B^2 m_1(t)m_2(t)}{R_1}, \ n_2(T^+) = 0. \tag{60b}$$

At the impulse instants $\{\tau_2, \tau_2, \cdots, \tau_k\}$, the equilibrium value function of Player 2 satisfies

$$V_2(\tau_i^-, x(\tau_i^-)) = \max_{v_i \in \Omega_v} \left\{ V_2(\tau_i^+, x(\tau_i^-) + Qv_i) + \frac{1}{2}P_2 v_i^2 + C \right\}. \tag{61}$$

The above equation implies that, at the impulse instant, Player 2 selects the equilibrium control to maximize the value-to-go from that instant onwards. From Assumption 1 on interior solution, the equilibrium impulse level is obtained as follows:

$$
\begin{aligned}
v_i^* &= \arg\max_{v_i \in \Omega_v} \left\{ V_2(\tau_i^+, x(\tau_i^-) + Qv_i) + \frac{1}{2}P_2 v_i^2 + C \right\} \\
&= \arg\max_{v_i \in \Omega_v} \left\{ m_2(\tau_i^+)(x(\tau_i^-) + Qv_i) + n_2(\tau_i^+) + \frac{1}{2}P_2 v_i^2 + C \right\} \\
&= -\frac{m_2(\tau_i^+)Q}{P_2}. \tag{62a}
\end{aligned}
$$

Using v_i^* in (61), we obtain

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = m_2(\tau_i^+)x(\tau_i^-) + n_2(\tau_i^+) - \frac{m_2(\tau_i^+)^2 Q^2}{2P_2} + C.$$

The above relation holds for all x. Therefore, we have

$$m_2(\tau_i^-) = m_2(\tau_i^+), \tag{63}$$

$$n_2(\tau_i^-) = n_2(\tau_i^+) - \frac{m_2(\tau_i^+)^2 Q^2}{2P_2} + C. \tag{64}$$

Using (57a), (59a), we obtain that for $A \neq 0$

$$m_1(t) = -\frac{w_1}{A} + \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{j+1}^- - t)}, \, \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\},$$

where $m_1(\tau_{k+1}^+) = s_1$, and

$$m_1(\tau_i^-) = -\frac{w_1}{A} + \left(m_1(\tau_{i+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1,$$

for $i \in \{1, 2, \cdots, k\}$. For $A = 0$, we obtain

$$m_1(t) = w_1(\tau_{j+1}^- - t) + m_1(\tau_{j+1}^-), \, \forall t \in (\tau_j, \tau_{j+1}), \, j \in \{0, 1, \cdots, k\},$$

where $m_1(\tau_{k+1}^+) = s_1$, and

$$m_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + m_1(\tau_{i+1}^-) + q_1,$$

where $i \in \{1, 2, \cdots, k\}$. From (60a), (63), we obtain that for $A \neq 0$,

$$m_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_1}{A} \right) e^{A(T-t)}, \forall t \in [0, T],$$

and for $A = 0$,

$$m_2(t) = w_2(T - t) + s_2, \forall t \in [0, T].$$

From (56) and (62a), the equilibrium controls are given in (18a) and (18b) for $A \neq 0$ and in (19a) and (19b) for $A = 0$.

B Proof of Theorem 8

First, we consider the case when the impulse instants are exogenously given. The OLNE strategies $u^*(t)$ and v_i^* of the players are obtained by solving (7a) and (9c) where the Hamiltonian of Player 1 and Player 2, and the impulse Hamiltonian of Player 2 are respectively given by

$$H_1(x(t), u(t), \lambda_1(t)) = w_1 x(t) + d(u(t)) + \lambda_1(t)(Ax(t) + Bu(t)), \tag{65a}$$

$$H_2(x(t), u(t), \lambda_2(t)) = w_2 x(t) + \lambda_2(t)(Ax(t) + Bu(t)), \tag{65b}$$

$$H_2^I(x(t), v_i, \lambda_2(t)) = C + c(v_i) + \lambda_2(t)Qv_i. \tag{65c}$$

From Assumption 1, the first-order conditions in (7a) and (9c) give

$$H_{1u}(x(t), u^*(t), \lambda_1(t) = 0 \Rightarrow d_u(u^*(t)) + B\lambda_1(t) = 0,$$

$$H_{2v_i}^I(x(\tau_i^-), v_i^*, \lambda_2(\tau_i^+)) = 0 \Rightarrow c_{v_i}(v_i^*) + \lambda_2(\tau_i^+)Q = 0.$$

Following Assumption 5, and from implicit function theorem, there exist continuously differentiable functions $f_1 : \mathbb{R} \to \Omega_u$ and $f_2 : \mathbb{R} \to \Omega_v$ such that

$$u^*(t) = f_1(B\lambda_1(t)), \tag{67a}$$

$$v_i^* = f_2(Q\lambda_2(\tau_i^+)). \tag{67b}$$

From (7b)–(7e) and (9a)–(9e), it follows that $\lambda_1(t)$ and $\lambda_2(t)$ satisfy (12c)–(12d), (12g)–(12h), and the state equations for $i \in \{1, 2, \cdots, k\}$ are given by

$$\dot{x}(t) = Ax(t) + Bf_1(B\lambda_1(t)), \text{ for } t \neq \tau_i, \, x(0^-) = x_0, \tag{68a}$$

$$x(\tau_i^+) = x(\tau_i^-) + Q f_2(Q\lambda_2(\tau_i^+)). \tag{68b}$$

Next, we consider the feedback information structure, and use the dynamic programming principle to obtain the FNE strategies of the players. Since we have considered a linear-state differential game, we assume that the value functions of Player 1 and Player 2 are given by (17a) and (17b). Between the impulse instants, the value function of Player 1 satisfies the HJB equation

$$-\dot{m}_1(t)x - \dot{n}_1(t) = \max_{u \in \Omega_u} \big(w_1 x + d(u(t)) + m_1(t)(Ax(t) + Bu(t)) \big).$$

Following Assumption 1, the first-order condition yields $d_u(u^*(t)) + Bm_1(t) = 0$. From Assumption 5 and from implicit function theorem, there exist continuously differentiable functions $f_1 : \mathbb{R} \to \Omega_u$ such that

$$u^*(t) = f_1(Bm_1(t)). \tag{69}$$

For optimal control $u^*(t)$, the HJB equation is then given by

$$-\dot{m}_1(t)x - \dot{n}_1(t) = w_1 x + d(u^*(t)) + m_1(t)(Ax(t) + Bu^*(t)).$$

On comparing the coefficients, we obtain

$$\dot{m}_1(t) = -w_1 - m_1(t)A, \tag{70a}$$
$$\dot{n}_1(t) = -d(u^*(t)) - m_1(t)Bu^*(t). \tag{70b}$$

The jump in the value function of Player 1 is given by (58)

$$\begin{aligned}
m_1(\tau_i^-)x(\tau_i^-) + n_1(\tau_i^-) &= m_1(\tau_i^+)x(\tau_i^+) + n_1(\tau_i^+) + q_1 x(\tau_i^-) \\
&= m_1(\tau_i^+)(x(\tau_i^-) + Qv_i^*) + n_1(\tau_i^+) + q_1 x(\tau_i^-).
\end{aligned}$$

On comparing the coefficients, we obtain

$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1, \tag{71}$$
$$n_1(\tau_i^-) = n_1(\tau_i^+) + m_1(\tau_i^+)Qv_i^*. \tag{72}$$

Between the impulse instants, the value function of Player 2 (17b) satisfies the HJB equation given by

$$-\dot{m}_2(t)x - \dot{n}_2(t) = w_2 x + m_2(t)(Ax(t) + Bu^*(t)).$$

On comparing the coefficients, we obtain

$$\dot{m}_2(t) = -w_2 - m_2(t)A, \tag{73a}$$
$$\dot{n}_2(t) = -m_2(t)Bu^*(t). \tag{73b}$$

At the impulse instant τ_i, the value function of Player 2 satisfies

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = \max_{v_i \in \Omega_v} \big(m_2(\tau_i^+)(x(\tau_i^-) + Qv_i) + n_2(\tau_i^+) + C + c(v_i) \big). \tag{74}$$

From Assumption 1, the first-order condition yields

$$m_2(\tau_i^+)Q + c_{v_i}(v_i^*) = 0.$$

Following Assumption 5, and from the implicit function theorem, there exist continuously differentiable functions $f_2 : \mathbb{R} \to \Omega_v$ such that

$$v_i^* = f_2(Qm_2(\tau_i^+)). \tag{75}$$

On substituting v_i^* in (74), we obtain

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = m_2(\tau_i^+)(x(\tau_i^-) + Qv_i^*) + n_2(\tau_i^+) + C + c(v_i^*),$$

which on comparing coefficients gives

$$m_2(\tau_i^-) = m_2(\tau_i^+), \tag{76}$$

$$n_2(\tau_i^-) = n_2(\tau_i^+) + m_2(\tau_i^+)Qv_i^* + C + c(v_i^*). \tag{77}$$

The necessary conditions for OLNE require that co-state variables of Player 1 and Player 2 satisfy (12c)–(12d), (12g)–(12h). For the FNE, the gradient of the value function of Player 1 and Player 2 are obtained by solving (70a), (71), (73a), and (76). For both players, $\lambda_1(t) = m_1(t)$, $\lambda_2(t) = m_2(t)$ for all t since $\lambda_1(.)$ and $m_1(.)$, and $\lambda_2(.)$ and $m_2(.)$ have the same dynamics, jump conditions, and terminal conditions. Therefore from (67), (69), (75), we have that OLNE and FNE coincide when the impulse timing is given.

When the impulse instants are decision variables of Player 2, the necessary conditions for OLNE are given in (7) and (20). Using the necessary conditions, and from Assumption 1 on interior solutions, the equilibrium controls are given in (67a)–(67b) where the dynamics and jump equations of co-state variables are given by (12c)–(12d), (12g)–(12h). For an impulse to occur in $[0, T]$, (20f) must hold true which on substituting (65b), (67a), (12h), (68b) simplifies to

$$(w_2 + A\lambda_2(\tau_i^*))Qf_2(Q\lambda_2(\tau_i^*)) + \lambda_2(\tau_i^*)B(f_1(B\lambda_1(\tau_i^{*+})) - f_1(B\lambda_1(\tau_i^{*-}))) \begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T \end{cases}.$$

From the above condition, it is clear that the equilibrium impulse instant in OLNE depends on the problem parameters of Player 1.

Next, we consider the feedback information structure. Given Player 1's equilibrium strategy $u^*(.)$, Player 2 solves (29). We assume linear value function for both players, that is,

$$V_i(t, x) = \alpha_i(t)x + \beta_i(t), \ \forall i = \{1, 2\}.$$

Since Player 2 solves an impulse optimal control problem, the value function of Player 2 satisfies the QVI (32). The stopping set is characterized by the time instant at which (32b) holds with equality, that is,

$$V_2(t, x) = \mathcal{R}V_2(t, x) \Rightarrow \alpha_2(t, x) + \beta_2(t) = \max_{v_i \in \Omega_v} \{\alpha_2(t)(x + Qv_i) + \beta_2(t) + C + c(v_i)\}.$$

From Assumption 1 on interior solutions, the first-order condition gives $\alpha_2(t)Q + c_{v_i}(v_i^*) = 0$. From Assumption 5, we can write

$$v_i^* = f_2(Q\alpha_2(t)). \tag{78}$$

For the equilibrium control v_i^*, we obtain the stopping set condition

$$\alpha_2(t)Qf_2(Q\alpha_2(t)) + c(f_2(Q\alpha_2(t))) + C = 0. \tag{79}$$

Since (32b) must hold for all $(t, x) \in [0, T] \times \mathbb{R}$, the linear value function is well-defined when the following condition holds for all $t \in [0, T]$.

$$\alpha_2(t)Qf_2(Q\alpha_2(t)) + c(f_2(Q\alpha_2(t))) + C \leq 0.$$

Following the proof of Theorem 5, we obtain the gradient of the value function of Player 2 as follows:

$$\alpha_2(t) = w_2(T - t) + s_2, \ A = 0,$$

$$\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)} \left(s_2 + \frac{w_2}{A} \right), \ A \neq 0.$$

The stopping set condition (79) implies that the impulse timing only depends on the problem parameters of Player 2, and is independent of the state of the system. On the other hand, the impulse timing in OLNE involves problem parameters of Player 1. Therefore, OLNE and FNE do not coincide when Player 2 decides the number and timing of impulses.

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