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Spectral properties of threshold graphs

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Abstract: In this paper we study the spectral properties of the threshold graphs. In particular, we give lower and upper bounds for the largest and smallest eigenvalues of a threshold graph. Moreover, we study the spectral properties of the threshold graphs with a few positive eigenvalues.

Keywords: Threshold graphs, spectral radius, smallest eigenvalue, inertia

1 Introduction

Let $G = (V, E)$ denote a graph with vertex set V and edge set E. The *order* $n = |V|$ of G is the number of its vertices, while the *size* $m = |E|$ of G is the number of its edges. The *complement* $\overline{G}$ of $G = (V, E)$ is the graph defined on the vertex set V of G, where an edge uv belongs to $\overline{G}$ if and only if it does not belong to G.

The adjacency matrix $A = (a_{i,j})$ of G is defined by $a_{i,j} = 1$ if vertices v_i and v_j of the graph G are adjacent and 0 otherwise. The eigenvalues of a graph G, denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$ are defined to be the eigenvalues of its adjacency matrix. The largest eigenvalue of G is usually called the *spectral radius* of G and sometimes the *index* of G. The spectral spread of a graph is defined as $\lambda_1 - \lambda_n$. The *positive (negative) inertia* of G, denoted by $n_+(G)$ $(n_-(G))$, is the number of positive (negative) eigenvalues of G. The *nullity* $n_0(G)$ of the graph G is the multiplicity of the eigenvalue zero of G.

The *clique number* $\omega(G)$ of a graph G is the size of a maximum clique in G, that is, the size of largest complete subgraph in G. The independence number $\alpha(G)$ of a graph G is the size of the maximum independent vertex set in G.

A *complete split graph* with parameters n, q $(q \leq n)$, denoted by $CS(n, q)$, is a graph on n vertices consisting of a clique on q vertices and an independent set on the remaining $n - q$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. Obviously if $q = n$, then the complete split graph is isomorphic to the complete graph K_n.

A *threshold graph* is obtained through an iterative process which starts with an isolated vertex, and where, at each step, either a new isolated vertex is added, or a vertex adjacent to all previous vertices (dominating vertex) is added.

For any threshold graph G, we consider a binary sequence $\mathbf{b} = b_1 b_2 \ldots b_n$ such that $b_i = 0$ if the corresponding vertex v_i is isolated and otherwise $b_i = 1$. We called $\mathbf{b}$ the *creation sequence* of G.

The trace T of a threshold graph G is the number of dominating vertices in G. Then the number of the isolated vertices is $n - T$.

From the definition of threshold graphs, we have that the set of all isolated vertices of the threshold graph G is an independent set, that is, the independence number $\alpha(G) \geq n - T$ and the set of all dominating vertices with the first (isolated) vertex of G is a clique in G, that is, the clique number $\omega(G) \geq T + 1$. Then for the threshold graph G, we have

$$\alpha(G) + \omega(G) \geq (n - T) + (T + 1) = n + 1. \tag{1}$$

On the other hand, from [7] we have that for any graph G, $\alpha(G)+\alpha(\overline{G}) \leq n+1$, i.e., $\alpha(G)+\omega(G) \leq n + 1$. This with (1) leads to the following result on threshold graphs:

Lemma 1 *Let G be a threshold graph of order n with the independence number $\alpha(G)$ and the clique number $\omega(G)$. Then*

$$\alpha(G) + \omega(G) = n + 1.$$

Our motivation for considering threshold graphs comes from the spectral graph theory. These graphs arise (within the graphs with fixed order and/or size) as graphs with the largest eigenvalue of the adjacency matrix. Brualdi and Hoffman [3] observed that they admit the stepwise form of the adjacency matrix, while later Hansen (see, for example, [1]) observed that they are split graphs distinguished by a nesting property imposed on vertices in the maximal co-clique, and hence called them the nested split graphs. As far as we know, it was first observed in [13], they are $\{2K_2, P_4, C_4\}$-free graphs, and thus the threshold graphs. In [5, 6] it was observed that they appear in the same role with respect to the signless Laplacian spectrum.

In this paper, we obtain spectral properties of the threshold graphs by giving bounds on its spectral invariants such as the spectral radius, smallest eigenvalue, and spectral spread. Moreover, we compare the energy and Laplacian energy of a threshold graph with only one positive eigenvalue.

The rest of the paper is organized as follows. In Section 2, we give lower bounds and upper bounds respectively for the spectral radius and the smallest eigenvalue of a threshold graph. In Section 3 we obtain some upper bounds on the spectral radius of a threshold graph. In Section 4 we study other spectral properties of a threshold graph with small positive inertia.

2 A lower bound for the spectral radius of a threshold graph

In this section we give lower bounds and upper bounds respectively for the spectral radius and the smallest eigenvalue of a threshold graph. Before this, we need the following lemma which gives a relation between the eigenvalues of a real symmetric matrix and the eigenvalues of its *partitioned matrix.*

Lemma 2 [4] *Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a partition $\{1, 2, \ldots, n\} = V_1 \dot{\cup} V_2 \dot{\cup} \cdots \dot{\cup} V_k$ with $|V_i| = n_i > 0$, consider the corresponding blocking $A = (A_{i,j})$, where $A_{i,j}$ is an $n_i \times n_j$ block and $1 \leq i, j \leq k$. Let $e_{i,j}$ be the sum of the entries in $A_{i,j}$ and set the matrix $B := (e_{i,j}/n_i)$ for $1 \leq i, j \leq k$. Then the eigenvalues of B interlace those of A, i.e. $\lambda_i \geq \rho_i \geq \lambda_{n-k-i}$ for $1 \leq i \leq k$, where ρ_i is the ith largest eigenvalue of B. Moreover, if the block $A_{i,j}$ has constant row sums $b_{i,j}$, then the spectrum of B is contained in the spectrum of A.*

The following results give bounds for the largest and smallest eigenvalues of a graph having a vertex set partitioned in two sets: an independent set and a clique.

Theorem 1 *Let $G = (V, E)$ be a graph on n vertices with the spectral radius λ_1, the smallest eigenvalue λ_n, and the independence number $\alpha(G)$ and let I and $V \backslash I$ be respectively a maximum independent set and a clique in G. If b is the number of edges between I and $V \backslash I$, then*

$$\lambda_1 \geq \frac{1}{2}\left(n - \alpha - 1 + \sqrt{(n-\alpha-1)^2 + \frac{4b^2}{(n-\alpha)\alpha}}\right), \tag{2}$$

and

$$\lambda_n \leq \frac{1}{2}\left(n - \alpha - 1 - \sqrt{(n-\alpha-1)^2 + \frac{4b^2}{(n-\alpha)\alpha}}\right). \tag{3}$$

The quality holds in (2) and (3) if G is a bi-degree graph with the same vertex degrees for the vertices in each partition I and $V \backslash I$.

Proof. Let $V_1 = I$ be a maximum independent set of size α. Then considering the partitions V_1 and $V_2 = V \backslash V_1$ on the vertex set of G with Lemma 2, we arrive at the matrix

$$B = \begin{bmatrix} 0 & b/\alpha \\ b/(n-\alpha) & n-\alpha-1 \end{bmatrix}.$$

The eigenvalues of B are as follows:

$$\lambda_1(B) = \frac{1}{2}\left(n - \alpha - 1 + \sqrt{(n-\alpha-1)^2 + \frac{4b^2}{(n-\alpha)\alpha}}\right),$$

and

$$\lambda_2(B) = \frac{1}{2}\left(n - \alpha - 1 - \sqrt{(n-\alpha-1)^2 + \frac{4b^2}{(n-\alpha)\alpha}}\right).$$

By Lemma 2 we have $\lambda_1(G) \geq \lambda_1(B)$, and $\lambda_n(G) \leq \lambda_2(B)$, which gives the desired result in (2) and (3). The second part of the proof is directly achieved by Lemma 2. $\square$

Let $G = (V, E)$ be a graph with the clique $X = \{v_1, v_2, \ldots, v_k\}$ and the independent set $V \backslash X = \{v_{k+1}, \ldots, v_n\}$. Considering a as the number of edges going between X and $V \backslash X$.

In a threshold graph G, the degree of the common vertex of the maximum clique and the maximum independent set is $k-1$, and then $b = k-1+a$. This fact with Theorem 1, gives the following bounds on the spectral radius and the smallest eigenvalue of a threshold graph.

Corollary 1 *Let G be a threshold graph on n vertices with the spectral radius λ_1, the smallest eigenvalue λ_n, and the clique number k. Let a be the number of edges between the maximum clique C and $V \backslash C$, then*

$$\lambda_1 \geq \frac{1}{2}\left(k-2+\sqrt{(k-2)^2+\frac{4(a+k-1)^2}{(k-1)(n-k+1)}}\right),$$

and

$$\lambda_n \leq \frac{1}{2}\left(k-2-\sqrt{(k-2)^2+\frac{4(a+k-1)^2}{(k-1)(n-k+1)}}\right).$$

For connected threshold graphs, the equality holds in both inequalities if G is the complete split graph $CS(n, k-1)$.

Proof. In a threshold graph G with the clique number k, $|I| = \alpha = n-k+1$ by Lemma 1. Then the first part of the proof is directly achieved by Theorem 1.

By Theorem 1, if the connected threshold graph G is bi-degree with two degrees one on the vertices in the maximum independent set I and other on the vertex set $V \backslash I$, then equality holds in both inequalities. Since the degree of the common vertex of I and the maximum clique in G is $k-1$, then all vertex degrees in I must be $k-1$. Thus G is the complete split graph with a clique of size $k-1$, i.e. $G \cong CS(n, k-1)$. This completes the second part of the proof. $\square$

In the following we give other bounds for λ_1 and λ_n of a threshold graph.

Theorem 2 *Let $G = (V, E)$ be a threshold graph on n vertices with the spectral radius λ_1, the smallest eigenvalue λ_n, and the clique number $\omega(G) = k$ and let $C = \{v_1, v_2, \ldots, v_k\}$ and $V \backslash C$ be respectively the maximum clique and an independent set in G. If a is the number of edges between C and $V \setminus C$, then*

$$\lambda_1 \geq \frac{1}{2}\left(k-1+\sqrt{(k-1)^2+\frac{4a^2}{k(n-k)}}\right),$$

and

$$\lambda_n \leq \frac{1}{2}\left(k-1-\sqrt{(k-1)^2+\frac{4a^2}{k(n-k)}}\right).$$

Proof. Let $V_1 = C$ be the clique of maximum size k. Then considering the partitions V_1 and $V_2 = V \backslash V_1$ on the vertex set of G, with Lemma 2, we arrive at the matrix

$$B = \begin{bmatrix} k-1 & a/k \\ a/(n-k) & 0 \end{bmatrix}.$$

The eigenvalues of B are as follows:

$$\lambda_1(B) = \frac{1}{2}\left(k-1+\sqrt{(k-1)^2+\frac{4a^2}{k(n-k)}}\right), \text{ and } \lambda_2(B) = \frac{1}{2}\left(k-1-\sqrt{(k-1)^2+\frac{4a^2}{k(n-k)}}\right).$$

By Lemma 2 we obtain $\lambda_1 \geq \lambda_1(B)$, and $\lambda_n \leq \lambda_2(B)$, which gives the desired result. □

3 Upper bounds for the spectral radius of a threshold graph

In this section, we give upper bounds for the spectral radius of a threshold graph.

Theorem 3 *Let $G = (V, E)$ be a graph on n vertices with spectral radius λ_1, maximum clique number $\omega(G) = k$ and independence number α, such that $\omega(G) + \alpha(G) = n + 1$. Let $C = \{v_1, v_2, \ldots, v_k\}$ be a maximum clique in G and a be the number of edges between C and $V \setminus C$. Then*

$$\lambda_1 \leq \frac{k-2+\sqrt{k^2+4a}}{2}. \tag{4}$$

The quality holding if and only if $G \cong CS(n, k-1)$, or $G \cong K_k \cup (n-k)K_1$.

Proof. Since $\alpha = n - k + 1$, any maximum independent set in G must contain $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ and one vertex from C. Assume without loss of generality that v_1 is the common vertex.

Let X be the Perron vector of G (a unit non-negative eigenvector belonging to λ_1). The eigenvalue-eigenvector equation corresponding to v_1 is

$$\lambda_1 x_1 = x_2 + x_3 + \cdots + x_k.$$

Therefore

$$\lambda_1 = \frac{x_2 + x_3 + \cdots + x_k}{x_1}. \tag{5}$$

The eigenvalue-eigenvector equation corresponding to v_j, $2 \leq j \leq k$ is

$$\lambda_1 x_j = x_1 + x_2 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k + \sum_{v_p \in N_j \cap (V \setminus C)} x_p,$$

where N_j denotes the set of neighbors of v_j in G.

Adding the k first eigenvalue-eigenvector equations, we get

$$\lambda_1 \sum_{j=1}^{k} x_j = (k-1)\sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n} d_j x_j, \tag{6}$$

where d_j denote the degree of the vertex j in G.

If $x_j = 0$ for all $k+1 \leq j \leq n$ (i.e., all vertices v_j are isolated vertices with degree zero) then we have $\lambda_1 = k - 1$. Otherwise, we consider the eigenvalue-eigenvector equation corresponding to v_j, $k+1 \leq j \leq n$ as follows:

$$\lambda_1 x_j = \sum_{v_p \in N_j} x_p \leq \sum_{p=2}^{k} x_p = \sum_{p=1}^{k} x_p - x_1.$$

Thus, for $k+1 \leq j \leq n$

$$x_j \leq \frac{1}{\lambda_1}\left(\sum_{p=1}^{k} x_p - x_1\right). \tag{7}$$

Combining (6) and (7) and using the fact that $d_{k+1} + d_{k+2} + \cdots + d_n = a$, the number of edges between C and $V \setminus C$, we get

$$\begin{aligned}\lambda_1 \sum_{j=1}^{k} x_j &\le (k-1)\sum_{j=1}^{k} x_j + \frac{1}{\lambda_1}\left(\sum_{j=1}^{k} x_j - x_1\right)\sum_{j=k+1}^{n} d_j \\ &= (k-1)\sum_{j=1}^{k} x_j + \frac{a}{\lambda_1}\sum_{j=1}^{k} x_j - \frac{a x_1}{\lambda_1}.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_1 &\le (k-1) + \frac{a}{\lambda_1} - \frac{a}{\lambda_1}\cdot\frac{x_1}{\sum_{j=1}^{k} x_j} \\ &= (k-1) + \frac{a}{\lambda_1}\cdot\frac{\sum_{j=2}^{k} x_j}{\sum_{j=1}^{k} x_j} \\ &= (k-1) + \frac{a}{\lambda_1}\cdot\left(\frac{x_1 + \sum_{j=2}^{k} x_j}{\sum_{j=2}^{k} x_j}\right)^{-1}.\end{aligned}$$

Using (5), we get

$$\lambda_1 \le k - 1 + \frac{a}{\lambda_1}\left(\lambda_1^{-1} + 1\right)^{-1} = k - 1 + \frac{a}{\lambda_1 + 1}, \tag{8}$$

from which the inequality follows.

To characterize the extremal graphs, we assume that we have equality in (9). This gives all inequalities in the above must be equalities. Then from the proof we have $G \cong K_k \cup (n-k)K_1$, or $N_j = \{2, \ldots, k\}$ for $k+1 \le j \le n$, i.e., $G \cong CS(n, k-1)$ (notice that $N_1 = \{2, \ldots, k\}$).

This completes the second part of the proof. □

Terpai in [16] gave the following Nordhaus–Gaddum type result on the spectral radius of a simple graph G.

Lemma 3 *Let G be a simple graph of order n with the complement $\overline{G}$. Then*

$$\lambda_1(G) + \lambda_1(\overline{G}) \le \frac{4}{3}n - 1.$$

Lemma 3 with Theorem 2 leads to the following result:

Theorem 4 *Let $G = (V, E)$ be a graph on n vertices with spectral radius λ_1, maximum clique number $\omega(G) = k$ and independence number α, such that $\omega(G) + \alpha(G) = n + 1$. Let $C = \{v_1, v_2, \ldots, v_k\}$ be a maximum clique in G and a be the number of edges between C and $V \setminus C$. Then*

$$\lambda_1 \le \frac{4}{3}n - 1 - \frac{1}{2}\left(n - k + \sqrt{(n-k)^2 + \frac{4\overline{a}}{(n-k+1)(k-1)}}\right), \tag{9}$$

where $\overline{a} = (k-1)(n-k) - a$.

Proof. By Lemma 3 we have $\lambda_1(G) + \lambda_1(\overline{G}) \le \frac{4}{3}n - 1$, that is, $\lambda_1(G) \le \frac{4}{3}n - 1 - \lambda_1(\overline{G})$. The maximum independence number of G is $n - k + 1$, that is, the clique number of $\overline{G}$. This with Theorem 2 gives

$$\lambda_1(G) \le \frac{4}{3}n - 1 - \lambda_1(\overline{G}) \le \frac{4}{3}n - 1 - \lambda_1(\overline{G}) \le \frac{4}{3}n - 1 - \frac{1}{2}\left(n - k + \sqrt{(n-k)^2 + \frac{4\overline{a}}{(n-k+1)(k-1)}}\right),$$

where $\overline{a}$ is number of edges between the maximum clique $C(\overline{G})$ of $\overline{G}$ and $V \backslash C(\overline{G})$, that is, $\overline{a} = (k-1)(n-k) - a$.

□

4 Spectral properties of threshold graphs with small positive inertia

In this section, we study more spectral properties of the threshold graphs, in particular, those with positive inertia $n_+(G) = 1, 2$. But before it, we need the following lemmas.

The following result gives the nullity $n_0(G)$, the positive inertia $n_+(G)$ and the negative inertia $n_0(G)$ of a threshold graph G by its creation sequence.

Lemma 4 [2] *In a connected threshold graph G represented with the creation sequence $\mathbf{b}$, $n_-(G)$, $n_0(G)$, and $n_+(G)$ are respectively the number of 1's, the number of strings 00, and the number of strings 01 in $\mathbf{b}$.*

The following result is related to the multiplicity $m_G(-1)$ of the eigenvalue -1 in the spectrum of the threshold graph G.

Lemma 5 [12] *Let G be a connected threshold graph having the creation sequence $\mathbf{b} = 0^{x_1}1^{y_1}\dots 0^{x_s}1^{y_s}$, where the x_i and y_i are positive integers. Then*

$$m_G(-1) = \begin{cases} \sum_{i=1}^{k}(y_i - 1) & \text{if } x_1 > 1 \\ 1 + \sum_{i=1}^{k}(y_i - 1) & \text{if } x_1 = 1. \end{cases}$$

For a connected graph G, it is well-known that $n_+(G) = 1$ if and only if G is a complete k-partite graph. In the following we characterize all connected threshold graphs with a single positive eigenvalue.

Theorem 5 *Let G be a connected threshold graph. Then $n_+(G) = 1$ if and only if G is a complete split graph.*

Proof. Any complete split graph G is a complete partite graph and thus $n_+(G) = 1$. Inversely, let $n_+(G) = 1$. By Lemma 4 we have the number of strings 01 is only one, that is, the creation sequence of G is $\mathbf{b} = 0^{x_1}1^{y_1}$, which implies G is a complete split graph. □

Theorem 6 *The spectrum of a complete split graph G is as follows:*

$$Spec(G) = \left\{\lambda_1, 0^{n-k}, -1^{k-2}, \lambda_n\right\},$$

where $\lambda_1 = \frac{k-2+\sqrt{4nk-3k^2+4k-4n}}{2}$, and $\lambda_n = \frac{k-2-\sqrt{4nk-3k^2+4k-4n}}{2}$. Moreover, λ_1 is an increasing function on the variable k and then has its maximum value at $k = n$, i.e. $G \cong K_n$.

Proof. In the graph G, let V_1 and V_2 be the sets of vertices in G corresponding to zeros and 1's in the creation sequence $\mathbf{b}$, respectively. By Lemma 2 we arrive at

$$B = \begin{bmatrix} 0 & k-1 \\ n-k+1 & k-2 \end{bmatrix},$$

which has two eigenvalues λ_1 and λ_n. By Lemma 2, these eigenvalues are exact eigenvalues of G. The eigenvalues 0 and -1 and their multiplicities are obtained by Lemma 4.

To prove the second part of the theorem, we consider $f(x) = x - 2 + \sqrt{4nx - 3x^2 + 4x - 4n}$, where $1 \le x \le n$. One can easily see that $f'(x)$ is positive on the set $D = \{1, 2, \dots, n\}$ and then $f(x)$ is an increasing function on the domain D. This also implies $\lambda_1(G) = f(k) \le f(n) = n = \lambda_1(K_n)$ for the complete split graph G. □

The *energy* of a graph G with the eigenvalues $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ is defined as [9] $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the degree diagonal matrix of

G. This matrix has nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The *Laplacian energy* of the graph G is defined as [11]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{10}$$

Let S_n^1 be the graph obtained from the star graph $K_{1,n-1}$ by adding an edge between two pendant vertices in $K_{1,n-1}$.

Lemma 6 [8] *Among all connected threshold graph of order $n\,(>4)$, the star graph $K_{1,n-1}$ has the first minimum Laplacian energy. Moreover two graphs K_n and S_n^1 have commonly the second minimum Laplacian enery.*

In [10], it was conjectured that the Laplacian energy is always greater than or equal to the graph energy, The validity of the conjecture was eventually disproved by means of counterexamples [15]. In [14], the authors showed that the conjecture is true for bipartite graphs G, that is, $LE(G) \geq \mathcal{E}(G)$. In the following we give a comparision between these two energies on the complete split graphs.

Theorem 7 *For any complete split graph G, $LE(G) \geq \mathcal{E}(G)$.*

Proof. For the star graph $K_{1,n-1}$, we have $\mathcal{E}(K_{1,n-1}) = 2\sqrt{n-1} \leq LE(K_{1,n-1}) = 2n - 4 + 4/n$ for $n \geq 1$. Now we consider $G \ncong K_{1,n-1}$. By Lemmas 6 and Theorem 6,

$$LE(G) \geq LE(K_n) = \mathcal{E}(K_n) \geq \mathcal{E}(G) = k - 2 + \sqrt{4nk - 3k^2 + 4k - 4n},$$

which completes the proof of the theorem. $\square$

Theorem 8 *Let G be a threshold graph of order n and the clique number k. Then $n^+(G) = 2$ if and only if G has the creation sequence $\mathbf{b} = 0^{x_1}1^{y_1}0^{x_2}1^{y_2}$, where $x_1 + x_2 = n - k + 1$ and $y_1 + y_2 = k - 1$. Moreover, the spectrum of G is as follows:*

$$\{\lambda_1, \lambda_2, 0^{n-k-1}, -1^{k-3}, \lambda_{n-1}, \lambda_n\},$$

where λ_i for $i \in \{1, 2, n-1, n\}$ are roots of the ploynomial

$$p(x) = x^4 + (3-k)x^3 + (-x_1k - x_2y_2 + x_1 - k + 2)x^2 + (x_1 - x_1k - x_2y_2 + x_2y_1y_2)x + x_1x_2y_1y_2.$$

Proof. Suppose that $X_1\,(Y_1)$, and $X_2\,(Y_2)$ be the vertex sets in G corresponding to $x_1\,(y_1)$ and $x_2\,(y_2)$ zeros (ones) in the creation sequence $\mathbf{b}$, respectively. Applying Lemma 2, the matrix B corresponding to the block matrix on partitions X_1, Y_1, $X_2$and Y_2 is as follows: $B = \begin{bmatrix} 0 & y_1 & 0 & y_2 \\ x_1 & y_1 - 1 & 0 & y_2 \\ 0 & 0 & 0 & y_2 \\ x_1 & y_1 & x_2 & y_2 - 1 \end{bmatrix}$.

The characteristic polynomial of the matrix B is

$$\begin{aligned} P(x) = x^4 &+ (-y_1 - y_2 + 2)x^3 + (-x_1y_1 - x_1y_2 - x_2y_2 - y_1 - y_2 + 1)x^2 \\ &+ (x_2y_1y_2 - x_1y_1 - x_1y_2 - x_2y_2)x + x_1x_2y_1y_2. \end{aligned} \tag{11}$$

This with the facts $x_1 + x_2 = n - k + 1$ and $y_1 + y_2 = k - 1$ gives the desired result of the theorem. $\square$

The following corollary is obtained directly by Lemma 2.

Corollary 2 *Let G be a threshold graph. Then $n^+(G) = s$ if and only if G has a creation sequence in the form $\mathbf{b} = 0^{x_1}1^{y_1} \ldots 0^{x_s}1^{y_s}$.*

In [12] Section 4, the authors proved that a threshold graph has no eigenvalues in $(-1, 0)$. In the following we give a shorter proof for this.

Theorem 9 *A threshold graph has no eigenvalues in* $(-1, 0)$.

Proof. The K_k is a clique in G and then by interlacing theorem we have $-1 = \lambda_2(K_k) \geq \lambda_{n-k+2}$, that is, at least $k-1$ eigenvalues of G is located in $(-\infty, -1]$. On the other hand, by Lemma 4, the number of the negative eigenvalues of G is equal to the number of ones in $\mathbf{b}$, that is $k-1$. Then G has no eigenvalues in $(-1, 0)$. □

The following gives a lower bound for the spectral spread $s = \lambda_1 - \lambda_n$ of a threshold graph.

Theorem 10 *Let G Let G be a threshold graph on n vertices with the spectral spread s, and the clique number k. Let a be the number of edges between the maximum clique C and $V \backslash C$, then*

$$s \geq \max\left\{\sqrt{(k-2)^2 + \frac{4(a+k-1)^2}{(k-1)(n-k+1)}},\ \sqrt{(k-1)^2 + \frac{4a^2}{k(n-k)}}\right\}.$$

For connected threshold graphs, the equality holds if G is the complete split graph $CS(n, k-1)$.

Proof. The proof is directly achieved by Corollary 1 and Theorem 2 . □

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