A hybrid optimal control approach to mean field games with applications in optimal execution problems

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Abstract: The paper combines two major contemporary systems and control methodologies to obtain a unique ϵ-Nash equilibrium for optimal execution problems within the stock market, namely Mean Field Game (MFG) theory and Hybrid Optimal Control (HOC) theory. Following standard financial models, the stock market is studied in this paper as a large population non-cooperative game where each trader has stochastic linear dynamics with quadratic costs. We consider the case where there exists one major trader with significant influence on market movements together with a large number of minor traders (within two subpopulations), each with individually asymptotically negligible effect on the market. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. Optimal stopping time strategies together with best response trading policies for all traders are established with respect to their individual cost criteria by an application of LQG HOC theory.

Keywords: Mean field game theory, hybrid optimal control theory, optimal execution problems

Résumé :   Cet article combine deux méthodes contemporaines majeures de systèmes et de contrôle permettant d'obtenir un équilibre unique de ϵ-Nash afin de résoudre les problèmes d'exécution optimale au sein du marché financier, à savoir la "théorie du jeux à champ moyen" (MFG) et la "théorie du contrôle optimal hybride" (HOC). Suivant les modèles financiers classiques, le marché boursier est étudié dans cet article comme un jeu non coopératif à forte population dans lequel chaque commerçant a une dynamique linéaire stochastique avec des coûts quadratiques. Nous considérons le cas où il existe un commerçant majeur ayant une influence significative sur les mouvements du marché ainsi qu'un grand nombre de commerçants mineurs (au sein de deux sous-populations), chacun ayant un effet individuellement asymptotiquement négligeable sur le marché. Les commerçants sont couplés dans leur dynamique et leurs fonctions de coût au taux moyen de commerce du marché (une composante du champ moyen du système) et la fonction hybride par l'indexation de la cessation des transactions par une ou les deux sous-populations de commerçants mineurs par des états discrets. Les stratégies optimales de temps d'arrêt, ainsi que les politiques de commerce avec la meilleure réponse pour tous les commerçants, sont établies en fonction de leurs critères de coût individuels par l'application de la théorie de LQG HOC.

Mots clés :   Théorie du jeux à champ moyen, théorie du contrôle optimal hybride, problèmes d'exécution optimaux

1 Introduction

Mean Field Game (MFG) systems theory establishes the existence of approximate Nash equilibria together with the corresponding individual strategies for stochastic dynamical systems in games involving a large number of agents. The equilibria are termed ϵ-Nash equilibria and are generated by the local, limited information feedback control actions of each agent in the population, where the feedback control actions constitute the best response of each agent with respect to the precomputed behaviour of the mass of agents and where the approximation error converges to zero as the population size goes to infinity.

The analysis of this set of problems originated in Huang et al. (2003, 2006, 2007) (see Caines et al. (2017); Caines (2014)), and independently in Lasry, Lions (2006a,b). In Huang (2010) and Nguyen, Huang (2012) the authors analyse and solve the linear quadratic systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The existence of ϵ-Nash equilibria is established together with the individual agents' control laws that yield the equilibria Nguyen, Huang (2012). The partially observed MFG theory for nonlinear and linear quadratic systems with major and minor agents has been developed in Caines, Kizilkale (2013, 2014, 2017); Şen, Caines (2014, 2015, 2016); Firoozi, Caines (2015).

The notion of Hybrid Systems (HS) is used to describe control systems for which (i) the state has continuous and discrete components and (ii) the continuous component of the state evolves in "continuous time" and the discrete component evolves in "discrete time". While the optimal control of deterministic hybrid systems has been extensively studied in the literature (see e.g. Clarke, Vinter (1989); Bensoussan, Menaldi (1997); Lygeros et al. (1997); Branicky et al. (1998); Sussmann (1999); Xu, Antsaklis (2004); Dharmatti, Ramaswamy (2005); Garavello, Piccoli (2005); Shaikh, Caines (2007); Taringoo, Caines (2013); Pakniyat, Caines (2017c,b)), the optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies Bensoussan, Menaldi (2000); Aghayeva, Abushov (2011); Pakniyat, Caines (2016, 2017a). In Pakniyat, Caines (2016), in particular, first order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle (SHMP). The results are specialized in Pakniyat, Caines (2017a) to a class of Linear Quadratic Gaussian Hybrid Optimal Control Problems (LQG-HOCP) for which the Hamiltonian boundary conditions are path-independent and therefore, the corresponding stochastic Riccati equations are independent from the realization of stochastic diffusion terms.

Optimal execution problems have been addressed in the literature (see e.g. Á. Cartea and S. Jaimungal and J. Penalva (2015); Jaimungal, Kinzebulatov (2014); Almgren, Chriss (2001); Alfonsi et al. (2010); Bayraktar, Ludkovski (2011)) in which an agent must liquidate or acquire a certain amount of shares over a pre-specified time horizon while it balances the price impact and the price uncertainty, and maximizes its final wealth. This problem with the linear models in Á. Cartea and S. Jaimungal and J. Penalva (2015) was formulated as for the nonlinear major minor (MM) MFG model in Huang et al. (2015), and partially observed MM LQG MFG theory was first applied in Firoozi, Caines (2016, 2017) for the case in which traders have partial observations of the market states. In this paper, the stock market consists of an institutional investor, interpreted as the major agent, aims to liquidate a specific amount of shares, and a large population high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate or acquire a certain amount of shares within a specific time horizon. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. This work combines two major contemporary systems and control techniques: MFG theory and Hybrid Optimal Control (HOC) theory to establish optimal stopping time strategies together with best response trading policies for all agents with respect to their individual cost criteria which yield a unique ϵ-Nash equilibria for the market.

We note major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this paper.

The rest of the paper is organized as follows. Section 2 presents the trading dynamics and performance functions in the market. Optimal execution problems in the market are then formulated in the Hybrid MFG framework in Section 3. Finally, concluding remarks are made in Section 4.

2 Trading dynamics of agents in market

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: acquirers $\mathcal{A}_a$ with the population of N_a and liquidators $\mathcal{A}_l$ with the population of N_l, such that $N_a + N_l = N$. All agents trade over the interval $[0, T]$, and minor agents are allowed to stop trading at an optimal time $t_s^i \leq T$. It will be shown in Section 3 that the optimal stopping time policy for each agent is $\mathcal{F}_t$-independent, and depends only on its dynamical parameters. In this paper, for simplicity of exposition the dynamical parameters for all minor traders in their respective type are the same, and hence the stopping times are the same for all agents of each population. Employing the trading model in (**?**), the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the (i) inventories, (ii) trading rates and (iii) prices while the bilinear cash process appears in the quadratic performance function for each agent.

2.1 Inventory dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0, T]$. Hence the major agent's inventory dynamics is given by

$$dq_0(t) = \nu_0(t)dt + \sigma_0^q dw_0^q, \quad 0 \leq t \leq T,$$

where w_0^q is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations; σ_0^q is a positive scalar and we assume that $q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dq_i(t) = \nu_i(t)dt + \sigma_i^q dw_i^q,$$

where for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$. The Wiener process w_i^q models the HFT's information noise, σ_i^q is a positive scalar, $\nu_i(t)$ is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively; $q_i(t)$ is the minor liquidator's remaining shares at time t, or the shares the minor acquirer has bought until time t. However, the initial share stock of the HFTs, $\{q_i(0), 1 \leq i \leq N_a + N_l\}$, are not considered to be large, furthermore they are not motivated to retain shares and are assumed to trade them quickly.

We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathcal{A}_i$, by

$$d\nu_i(t) = u_i(t)dt,$$

where again for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$, and $u_i(t)$ is the trading acceleration of the minor acquirer or liquidator.

2.2 Price dynamics

The trading rate of the major agent and the average trading rate of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

2.2.1 Fundamental asset price

We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + \sigma dw_0^F(t), \quad 0 \le t \le T,$$

where N_t is the number of minor agents trading at time t, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t}\nu_i(t)$ is the average trading rate of the minor agents trading at time t. The Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathcal{A}_0$, $\mathcal{A}_i$, - have no complete or partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, σ denotes the intensity of the market volatility and $\lambda_0, \lambda \ge 0$ denote the strength of the linear permanent impact of the major and minor agents' trading on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent $\mathcal{A}_i$, by

$$dF_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + \sigma dw_i^F(t),$$

where $0 \le t \le t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \le t \le t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t}\nu_i(t)$ is again the average trading rate of the N_t minor agents trading at t, and the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders in the market.

2.2.2 Execution price

The major agent's execution price $S_0(t)$ evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \le t \le T, \tag{1}$$

where $a_0 \ge 0$ is the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to evolve by

$$dS_i(t) = dF_i(t) + a d\nu_i(t), \tag{2}$$

where $0 \le t \le t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \le t \le t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, and a models the temporary impact of a minor agent's trading on its execution price.

2.3 Cash process

The cash processes for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, respectively, are given by

$$dZ_0(t) = -S_0(t)dq_0(t), \quad 0 \le t \le T, \tag{3}$$

$$\begin{cases} dZ_i(t) = -S_i(t)dq_i(t), & \text{for } \mathcal{A}_i \in \mathcal{A}_a, \ 0 \le t \le t_s^a \\ dZ_i(t) = -S_i(t)dq_i(t), & \text{for } \mathcal{A}_i \in \mathcal{A}_l, \ 0 \le t \le t_s^l, \end{cases} \tag{4}$$

where $Z_0(t)$, and $Z_i(t)$ for $\mathcal{A}_i \in \mathcal{A}_l$ are the cash obtained through liquidation of shares, and $Z_i(t)$, for $\mathcal{A}_i \in \mathcal{A}_a$ is the cash paid for acquisition of shares up to time t. We note that the value of $dq_0(t)$ in a stock sale (respectively, buy) is negative (respectively, positive) and hence for positive $S_0(t)$, $Z_0(t)$ increases (respectively, decreases).

2.4 Performance function

2.4.1 Major liquidator

The objective for the major trader is to liquidate $\mathcal{N}_0$ shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time T is $q_0(T)$, it can liquidate it at a lower price than the market asset price reflected at cost function by $q_0(T)(F_0(T) - \alpha q_0(T))$. Further, the major trader's utility in minimizing the inventory over the period $[0, T]$ is modeled by including

the penalty $\phi \int_0^T q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms $\epsilon S_0^2(T)$, $\int_0^T \delta S_0^2(s)ds$, $\beta \nu_0^2(T)$, $\int_0^T \theta \nu_0^2(s)ds$ and $\int_0^T R_0 u_0^2(s)ds$ in the objective function. Therefore, its cost function to be minimized is given by

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[-rZ_0(T) - pq_0(T)\big(F_0(T) - \alpha q_0(T)\big) + \epsilon S_0^2(T) + \beta \nu_0^2(T)$$
$$+ \int_0^T \big(\phi q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s)\big)ds\Big], \quad (5)$$

where r, p, α, ϵ, β, ϕ, δ, θ, and R_0 are positive scalars, and $u_{-0} := (u_1, u_2, ..., u_N)$ are trading strategies of the minor traders. Note that for larger values of ϕ the trader attempts to liquidate its inventory more quickly.

2.4.2 Minor liquidator

In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate $\mathcal{N}_l$ shares over the interval $[0, T]$ with the stopping time $0 \leq t_s^l \leq T$ is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[-r_l Z_i(t_s^l) - p_l q_i(t_s^l)\Big(F_i(t_s^l) - \psi_l q_i(t_s^l)\Big) + \xi_l S_i^2(t_s^l) + \mu_l \nu_i^2(t_s^l)$$
$$+ \int_0^{t_s^l} \Big(\kappa_l q_i^2(s) + \gamma_l S_i^2(s) + \varrho_l \nu_i^2(s) + R_l u_i^2(s)\Big)ds\Big], \text{ for } \mathcal{A}_i \in \mathcal{A}_l \quad (6)$$

where r_l, p_l, ψ_l, ξ_l, μ_l, κ_l, γ_l, ϱ_l and R_l are positive scalars, and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$. Note that $\mathcal{N}_l \ll \mathcal{N}_0$.

2.4.3 Minor acquirer

The objective for a minor acquirer is to buy $\mathcal{N}$ shares during the trading horizon $[0, T]$. Given that it stops trading at $t_s^a \leq T$, it also wishes to minimize the execution cost including the cash $Z_i(t_s^a)$ paid up to time t_s^a, and the cash must be paid at time t_s^a to buy the remaining shares at once at a higher price than the market's asset price, i.e. $\big(\mathcal{N} - q_i(t_s^a)\big)\big(F_i(t_s^a) + \psi_a(\mathcal{N} - q_i(t_s^a))\big)$. It is also intended to avoid high execution prices, large trading intensities and large trading accelerations modeled by including $\xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a) + \int_0^{t_s^a} \big(\gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_A u_i^2(s)\big)ds$ in its objective function

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[p_a(\mathcal{N} - q_i(t_s^a))\big(F_i(t_s^a) + \psi_a(\mathcal{N} - q_i(t_s^a))\big) + r_a Z_i(t_s^a) + \xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a) +$$
$$\int_0^{t_s^a} \big(\kappa_a(\mathcal{N} - q_i(s))^2 + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_a u_i^2(s)\big)ds\Big], \ \mathcal{A}_i \in \mathcal{A}_a, \quad (7)$$

where $\int_0^{t_s^a} \kappa_a(\mathcal{N} - q_i(s))^2 ds$ is to penalize the agent for the remaining shares to be bought up to t_s^a and to expedite the acquisition. The parameters p_a, ψ_a, r_a, ξ_a, μ_a, κ_a, γ_a, ϱ_a, and R_a are positive scalars and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$.

3 Hybrid Mean Field Game formulation of optimal execution problems

In this section we formulate optimal execution problems in the Hybrid MM LQG MFG framework.

3.1 Discrete state association

In order to present the trading dynamics of the stock market in the stochastic hybrid systems framework of (Pakniyat, Caines, 2016, 2017a), the discrete states Q_j, $j = 0, 1, 2$ are introduced, which correspond to the evolution of the market in the intervals $[t_j, t_{j+1})$, where $t_0 = 0$ is the initial time, t_1 and t_2 denote the stopping times of the first population and the second population respectively, and $t_3 = T$ is the terminal time.

We remark that the HS-MFG problems studied in this paper lie within the class of hybrid LQG problems in (Pakniyat, Caines, 2017a) for which optimal switching strategies are $\mathcal{F}_t$-independent, and therefore, optimal stopping strategies depend only on the dynamical parameters of each population.

We associate the discrete state Q_0 to the initial case where both the liquidator and acquirer populations are trading together with the major agent over the interval $[0, t_1)$.

The discrete state Q_1 corresponds to the interval $[t_1, t_2)$ for which two situations can be considered: (i) the liquidator population stops at t_1 while the acquirer population is still trading, in which case $Q_1 = q_{0a}$, and (ii) the acquirer population stops at t_1 while the liquidator population is trading, which corresponds to $Q_1 = q_{0l}$.

The discrete state Q_2 represents the system over the interval $[t_2, T]$ after the second population of HFTs stops at t_2, and hence the major agent is trading in the absence of both populations.

The above discrete state association is summarized in the following table.

Table 1: Discrete State Association

Discrete State		$\mathcal{A}_0$	$\mathcal{A}_a$	$\mathcal{A}_l$
Q_0		✓	✓	✓
Q_1	q_{0a}	✓	✓	×
	q_{0l}	✓	×	✓
Q_2		✓	×	×

3.2 Finite populations

3.2.1 Major agent

The dynamics of the major trader in the market can be modeled as

$$d\nu_0(t) = u_0(t)dt,$$
$$dq_0(t) = \nu_0(t)dt + \sigma_0^q dw_0^q,$$
$$dS_0(t) = \left(\lambda_0 \nu_0(t) + \lambda \nu^{N_t}(t)\right)dt + a_0 u_0(t)dt + \sigma dw_0^F(t).$$

Let the major agent's state be denoted by $x_0 = [\nu_0, q_0, S_0]^T$, then its dynamics can be expressed as

$$dx_0 = A_0 x_0 dt + B_0 u_0 dt + E_0 x^{N_t} dt + D_0 dw_0 \tag{8}$$

with the matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 1 \\ 0 \\ a_0 \end{bmatrix}, \ w_0 = \begin{bmatrix} w_0^q \\ w_0^F \end{bmatrix}, \ E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \ D_0 = \begin{bmatrix} 0 & 0 \\ \sigma_0^q & 0 \\ 0 & \sigma \end{bmatrix}.$$

Note that in (8), N_t takes the following values.

$$N_t = \begin{cases} N_a + N_l & \text{for } Q_0, \\ N_a & \text{for } Q_1 = q_{0a}, \\ N_l & \text{for } Q_1 = q_{0l}, \\ 0 & \text{for } Q_2. \end{cases} \tag{9}$$

The major trader's cost function (5) can also be described in terms of its states with replacing the final cash process by $\mathbb{E}[Z_0(T)] = -\mathbb{E}[\int_0^T S_0(s)\nu_0(s)ds]$, and the fundamental asset price $F_0(T)$ using (1). The Equation (8) together with the cost function (5) form the stochastic LQG problem for the major trader. Note that the major trader is involved with the market's average trading rate in its dynamics while involved with the market's average selling rate in its cost function.

3.2.2 Minor liquidator

Similarly, the stochastic optimal control problem for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, is given by the set of dynamical equations

$$
\begin{aligned}
d\nu_i(t) &= u_i(t)dt, \\
dq_i(t) &= \nu_i(t)dt + \sigma_i^q dw_i^q, \\
dS_i(t) &= \big(\lambda_0 \nu_0(t) + \lambda \nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F.
\end{aligned}
$$

Similar to the major trader, we define a generic minor trader's state vector as $x_i = [\nu_i, q_i, S_i]^T$, and its dynamics can be written as

$$
dx_i = A_l x_i dt + B_l u_i dt + E_l x^{N_t} dt + D_l dw_{l_i} \tag{10}
$$

with

$$
A_l = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad
E_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad
B_l = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix},
$$

$$
G_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad
D_l = \begin{bmatrix} 0 & 0 \\ \sigma_i^q & 0 \\ 0 & \sigma \end{bmatrix}, \quad
w_{l_i} = \begin{bmatrix} w_i^q \\ w_i^F \end{bmatrix}.
$$

The quadratic cost function (6) can also be expressed in terms of the minor agent's state when the final cash process in (6) is replaced by $\mathbb{E}[Z_i(t_s^l)] = -\mathbb{E}[\int_0^{t_s^l} S_i(s)\nu_i(s)ds]$ using (4), and the fundamental asset price $F_i(t_s^l)$ is replaced using (2).

The Equations (10) and (6) form the stochastic LQG problem for a generic minor liquidator. Additionally, they show that a minor liquidator is coupled with the major agent's trading rate and the market's average trading rate in its dynamics while coupled with the market's average selling rate in its cost function.

3.2.3 Minor acquirer agent

The stochastic optimal control problem for a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$, is given by the set of dynamical equations

$$
\begin{aligned}
d\nu_i(t) &= u_i(t)dt, \\
dY_i(t) &= -\nu_i(t)dt + \sigma_i^q dw_i^q, \\
dS_i(t) &= \big(\lambda_0 \nu_0(t) + \lambda \nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F,
\end{aligned}
$$

where $Y_i(t) = \mathcal{N}_a - q_i(t)$ is the remaining shares at t to be acquired until the end of trading horizon. We define a generic minor acquirer's state vector as $x_i = [\nu_i, Y_i, S_i]$, hence its dynamics in compact form would be

$$
dx_i = A_a x_i dt + B_a u_i dt + E_a x^{N_t} dt + D_a dw_{a_i}, \tag{11}
$$

where

$$
A_a = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad
E_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad
B_a = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}
$$

$$
G_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad
D_a = \begin{bmatrix} 0 & 0 \\ \sigma_i^q & 0 \\ 0 & \sigma \end{bmatrix}, \quad
w_{a_i} = \begin{bmatrix} w_i^q \\ w_i^F \end{bmatrix}.
$$

Note that N_t in (11) again takes values as in (9) over the trading horizon. Accordingly, the cost function for acquisition is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[p_a Y_i(t_s^a)\big(S_i(t_s^a) - a\nu_i(t_s^a) + \psi_a Y_i(t_s^a)\big) + \xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a)$$

$$+ \int_0^{t_s^a} \Big(\kappa_a Y_i^2(s) + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) - r_a S_i(s)\nu_i(s) + R_a u_i^2(s)\Big)ds\Big], \text{ for } \mathcal{A}_i \in \mathcal{A}_a. \quad (12)$$

The set of Equations (11)–(12) constitute the standard stochastic LQG problem for a minor acquirer. It can be seen that a generic minor acquirer interacts with the major agent's trading rate as well as the market's average trading rate through it dynamics, and with the market's average buying rate through its cost function.

3.3 Mean field evolution

Following the LQG MFG methodology (Huang, 2010), the mean field, $\bar{x}$, is defined as the L^2 limit, when it exists, of the average of minor agents' states when population size goes to infinity

$$\bar{x}(t) = \lim_{N_t \to \infty} x^{N_t}(t) = \lim_{N \to \infty} \frac{1}{N_t}\sum_{i=1}^{N_t} x_i(t), \ a.s.$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 x_i + L_2 x_0 + \sum_{j\neq i, j=1}^{N_t} L_4 x_j + L_3, \quad 1 \le i \le N_t, \quad (13)$$

then the mean field dynamics can be obtained by substituting (13) in the minor liquidator (respectively, acquirer) agents' dynamics (10) (respectively, (11)), and taking the average and then its L^2 limit as $N \to \infty$.

The set of mean field equations for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{m}dt. \quad (14)$$

For Q_0, $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$ consists of the mean field $\bar{x}_l$ of the liquidator population, and the mean field $\bar{x}_a$ of the acquirer population. The matrices in (14) are defined as

$$\bar{A} = \begin{bmatrix} \bar{A}_a & \bar{A}_{al} \\ \bar{A}_{la} & \bar{A}_l \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_l \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_l \end{bmatrix}, \quad (15)$$

which shall be determined from consistency equations discussed in Section 3.5.

For q_{0a}, $\bar{x} = \bar{x}_a$, and the matrices in (14) are given as

$$\bar{A} = \bar{A}_a, \quad \bar{G} = \bar{G}_a, \quad \bar{m} = \bar{m}_a. \quad (16)$$

For q_{0l}, $\bar{x} = \bar{x}_l$, and the matrices in (14) are given by

$$\bar{A} = \bar{A}_l, \quad \bar{G} = \bar{G}_l, \quad \bar{m} = \bar{m}_l. \quad (17)$$

Finally, for Q_2, $\bar{x} = 0$.

The empirical distribution of the minor traders is denoted by $\pi^N = (\pi_a^N, \pi_l^N)$, $\pi_k^N = \frac{N_k}{N}$, $k \triangleq a, l$. The first assumption is as follows.

Assumption 1 *There exists π such that $lim_{N\to\infty}\pi^N = (\pi_a, \pi_l)$ a.s.*

3.4 Infinite populations

Following the mean field game methodology with a major agent (Huang, 2010; Nourian, Caines, 2013) the hybrid optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced by its infinite population limit, i.e. the

mean field. Then specializing to linear systems (Huang, 2010), the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Then the main results of (Huang, 2010), (Nourian, Caines, 2013) are (i) the existence of infinite population best response strategies which yield the Nash equilibria, and (ii) the infinite population best response strategies applied to the finite population system yield an ϵ-Nash equilibria (see Theorem 1).

3.4.1 Major liquidator agent

The extended dynamics of the major agent in the infinite population, i.e. the dynamic for the x_0^{ex,Q_j} is given by

$$dx_0^{ex,Q_j} = (\mathbb{A}_0^{Q_j} x_0^{ex,Q_j} + \mathbb{M}_0^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j})dt + \mathbb{D}_0^{Q_j} dW_0, \tag{18}$$

$0 \leq j \leq 2$, and the cost function for the extended major agent's system would be

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[\|x_0^{ex,Q_2}(T)\|_{\bar{\mathbb{P}}_0^{Q_2}}^2 + \sum_{j=0}^{2} \int_{t_j}^{t_{j+1}} \big(\|x_0^{ex,Q_j}(s)\|_{\mathbb{P}_0^{Q_j}}^2 + \|u_0^{Q_j}(s)\|_{R_0^{Q_j}}^2\big)ds\Big], \tag{19}$$

where $t_0 = 0, t_3 = T$. Let matrix coefficients $P_0, \bar{P}_0$, respectively, associated with the running and final costs in (5) be given by

$$\bar{P}_0 = \begin{bmatrix} \beta & \frac{1}{2}pa_0 & 0 \\ \frac{1}{2}pa_0 & p\alpha & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \epsilon \end{bmatrix}, \quad P_0 = \begin{bmatrix} \theta & 0 & \frac{1}{2}r \\ 0 & \phi & 0 \\ \frac{1}{2}r & 0 & \delta \end{bmatrix},$$

then over the interval $[t_0, t_1)$, and in the discrete state Q_0, the dynamics of the continuous state $x_0^{ex,Q_0} = [x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$ is determined from (18) with

$$\mathbb{A}_0^{Q_0} = \begin{bmatrix} A_0 & [E_0, E_0] \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0^{Q_0} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix}, \quad \mathbb{B}_0^{Q_0} = \begin{bmatrix} B_0 \\ 0_{6\times 1} \end{bmatrix}, \quad \mathbb{D}_0^{Q_0} = \begin{bmatrix} D_0 & 0_{3\times 6} \\ 0_{6\times 3} & 0_{6\times 6} \end{bmatrix}.$$

and $\mathbb{P}_0^{Q_0}$ in (19) is given by

$$\mathbb{P}_0^{Q_0} = [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}].$$

In case (i) where $Q_1 = q_{0a}$ over the interval $[t_1, t_2)$, the dynamics for $x_0^{ex,q_{0a}} = [x_0^T, \bar{x}_a^T]^T$ is determined from (18) with

$$\mathbb{A}_0^{q_{0a}} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q_{0a}} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q_{0a}} = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{D}_0^{q_{0a}} = \begin{bmatrix} D_0 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

and $\mathbb{P}_0^{q_{0a}}$ is given by

$$\mathbb{P}_0^{q_{0a}} = [I_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}].$$

In this case, the values of the continuous state before and after t_1 are related by the jump map

$$x_0^{ex,q_{0a}}(t_1) = \Psi_{0,a} x_0^{ex,Q_0}(t_1-) \tag{20}$$

where

$$\Psi_{0,a} = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \tag{21}$$

In case (ii) where $Q_1 = q_{0l}$ holds, $x^{ex,q_{0l}} = [x_0^T, \bar{x}_l^T]^T$ and

$$\mathbb{A}_0^{q_{0l}} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_l & \bar{A}_l \end{bmatrix}, \quad \mathbb{B}_0^{q_{0l}} = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{M}_0^{q_{0l}} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m}_l \end{bmatrix}, \quad \mathbb{D}_0^{q_{0l}} = \begin{bmatrix} D_0 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

$$\mathbb{P}_0^{q_{0l}} = [I_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}].$$

In this case, the values of the continuous state of the major trader before and after t_1 are related by the jump map

$$x_0^{ex,q_{0l}}(t_1) = \Psi_{0,l} x_0^{ex,Q_0}(t_1-) \tag{22}$$

where

$$\Psi_{0,l} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix}. \tag{23}$$

For the discrete state Q_2, the continuous state of the major trader is $x_0^{ex,q_{0a}} \equiv x_0$, and

$$\mathbb{A}_0^{Q_2} = A_0, \quad \mathbb{M}_0^{Q_2} = 0_{3\times1}, \quad \mathbb{B}_0^{Q_2} = B_0, \quad \mathbb{D}_0^{Q_2} = D_0$$

$$\bar{\mathbb{P}}_0^{Q_2} = \bar{P}_0, \quad \mathbb{P}_0^{Q_2} = P_0$$

The values continuous state of the major trader before and after t_2 are related by the the jump map

$$x_0^{ex,Q_2}(t_2) = \Psi_{0,2} x_0^{ex,Q_1}(t_2-) \tag{24}$$

where $\Psi_{0,2} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \end{bmatrix}$.

By the definition of the terms $\mathbb{D}_0^{Q_j}$ necessarily satisfy the condition A1 in (Pakniyat, Caines, 2016), which in the LQG takes the following form

$$\mathbb{D}_0^{Q_j} = \Psi_{0,j} \mathbb{D}_0^{Q_{j-1}}, \qquad j = 1, 2. \tag{25}$$

An application of the stochastic hybrid control theory of (Pakniyat, Caines, 2016), specialized to the LQG case in (Pakniyat, Caines, 2017a), yield the infinite population best response hybrid control action as

$$u_0^{Q_j}(t) = -R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j}(t) \, x_0^{ex,Q_j}(t), \tag{26}$$

where $\Pi_0^{Q_j}(t)$ is the solution of

$$-\dot{\Pi}_0^{Q_j} = \Pi_0^{Q_j} \mathbb{A}_0^{Q_j} + \mathbb{A}_{0,Q_j}^T \Pi_0^{Q_j} - \Pi_0^{Q_j} \mathbb{B}_0^{Q_j} R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j} + \mathbb{P}_0, \tag{27}$$

subject to the terminal and boundary conditions

$$\Pi_0^{Q_2}(T) = \bar{\mathbb{P}}_0, \tag{28}$$

$$\Pi_0^{Q_{j-1}}(t_j) = \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j}, \tag{29}$$

$$\mathbb{P}_0^{Q_{j-1}} + \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \mathbb{A}_0^{Q_{j-1}} + \mathbb{A}_{0,Q_{j-1}}^T \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j}$$
$$- \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \mathbb{B}_0^{Q_{j-1}} R_{0,Q_{j-1}}^{-1} \mathbb{B}_{0,Q_{j-1}}^T \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j}$$
$$= \Psi_{0,j}^T \Big(\mathbb{P}_0^{Q_j} + \Pi_0^{Q_j}(t_j) \mathbb{A}_0^{Q_j} + \mathbb{A}_{0,Q_j}^T \Pi_0^{Q_j}(t_j) - \Pi_0^{Q_j}(t_j) \mathbb{B}_0^{Q_j} R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j}(t_j) \Big) \Psi_{0,j}, \text{ for } j = 1, 2. \tag{30}$$

3.4.2 Minor acquirer

A generic minor agent $\mathcal{A}_i$'s extended dynamics in the acquirer population with the extended state x_i^{ex,Q_j} is

$$dx_i^{ex,Q_j} = (\mathbb{A}_a^{Q_j} x_i^{ex,Q_j} + \mathbb{M}_a^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j} + \mathbb{B}_a^{Q_j} u_i^{Q_j})dt + \mathbb{D}_a^{Q_j} dW_i, \tag{31}$$

where for Q_0, $x_i^{ex,Q_0} = [x_i^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$, and

$$\mathbb{A}_a^{Q_0} = \begin{bmatrix} A_a & [G_a, E_a, E_a] \\ 0_{9\times3} & \mathbb{A}_0^{Q_0} - \mathbb{B}_0^{Q_0} R_{0,Q_0}^{-1} \mathbb{B}_{0,Q_0}^T \Pi_0^{Q_0} \end{bmatrix},$$

$$\mathbb{M}_a^{Q_0} = \left[\begin{array}{c} 0_{3\times1,} \\ \mathbb{M}_0 \end{array} \right], \quad \mathbb{B}_a^{Q_0} = \left[\begin{array}{c} B_a \\ 0_{9\times1} \end{array} \right], \quad \mathbb{D}_a^{Q_0} = \left[\begin{array}{cc} D_a & 0_{3\times9} \\ 0_{9\times3} & \mathbb{D}_0^{Q_0} \end{array} \right],$$

and for q_{0a}, $x_i^{ex,q_{0a}} = [x_i^T, x_0^T, \bar{x}_a^T]^T$, and

$$\mathbb{A}_a^{q_{0a}} = \left[\begin{array}{cc} A_a & [G_a, E_a] \\ 0_{6\times3} & \mathbb{A}_0^{q_{0a}} - \mathbb{B}_0^{q_{0a}} R_{0,q_{0a}}^{-1} \mathbb{B}_{0,q_{0a}}^T \Pi_0^{q_{0a}} \end{array} \right],$$

$$\mathbb{M}_a^{q_{0a}} = \left[\begin{array}{c} 0_{3\times1} \\ \mathbb{M}_0 \end{array} \right], \quad \mathbb{B}_a^{q_{0a}} = \left[\begin{array}{c} B_a \\ 0_{6\times1} \end{array} \right], \quad \mathbb{D}_a^{q_{0a}} = \left[\begin{array}{cc} D_a & 0_{3\times6} \\ 0_{6\times3} & \mathbb{D}_0^{q_{0a}} \end{array} \right].$$

In case (i) where the acquirer population is trading over $[t_1, t_2)$, i.e. $Q_1 = q_{0a}$, the total hybrid cost for a minor acquirer is given by

$$J_i^a(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i^{ex,q_{0a}}(t_2)\|_{\bar{\mathbb{P}}_a^{q_{0a}}}^2 + \sum_{j=0}^{1} \int_{t_j}^{t_{j+1}} \big(\|x_i^{ex,Q_j}(s)\|_{\mathbb{P}_a^{Q_j}}^2 + \|u_i^{Q_j}(s)\|_{R_a^{Q_j}}^2 \big) ds \Big], \tag{32}$$

with

$$\bar{\mathbb{P}}_a^{q_{0a}} = [I_{3\times3}, 0_{3\times6}]^T \bar{P}_a [I_{3\times3}, 0_{3\times6}] \tag{33}$$

$$\mathbb{P}_a^{q_{0a}} = [I_{3\times3}, 0_{3\times6}]^T P_a [I_{3\times3}, 0_{3\times6}] \tag{34}$$

$$\mathbb{P}_a^{Q_0} = [I_{3\times3}, 0_{3\times9}]^T P_a [I_{3\times3}, 0_{3\times9}], \tag{35}$$

where $\bar{P}_a, P_a$ are, respectively, associated with the running and final costs in (7) are given by

$$\bar{P}_a = \left[\begin{array}{ccc} \mu_a & -\frac{1}{2}p_a a & 0 \\ -\frac{1}{2}p_a a & p_a \psi_a & \frac{1}{2}p_a \\ 0 & \frac{1}{2}p_a & \xi_a \end{array} \right], \quad P_a = \left[\begin{array}{ccc} \varrho_a & 0 & -\frac{1}{2}r_a \\ 0 & \kappa_a & 0 \\ -\frac{1}{2}r_a & 0 & \gamma_a \end{array} \right]. \tag{36}$$

In this case, the extended state for a generic minor agent in the acquirer population at t_1 satisfies the jump transition map

$$x^{ex,q_{0a}}(t_1) = \Psi_{i,a} x^{ex,Q_0}(t_1-)$$

with

$$\Psi_{i,a} = \left[\begin{array}{cccc} I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \end{array} \right].$$

In case (ii) where $Q_1 = q_{0l}$ holds over the interval $[t_1, t_2)$, the cost for the minor acquirer agent $\mathcal{A}_i$ is given by

$$J_i^a(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i^{ex,Q_0}(t_1)\|_{\bar{\mathbb{P}}_a^{Q_0}}^2 + \int_{t_0}^{t_1} \big(\|x_i^{ex,Q_0}(s)\|_{\mathbb{P}_a^{Q_0}}^2 + \|u_i^{Q_0}(s)\|_{R_a^{Q_0}}^2 \big) ds \Big], \tag{37}$$

with $\bar{\mathbb{P}}^{Q_0} = [I_{3\times3}, 0_{3\times9}]^T \bar{P}[I_{3\times3}, 0_{3\times9}]$.

The optimal stopping problem for a minor acquirer is equivalent to a hybrid optimal control problem in which the dynamics and costs become zero after stopping. By the definition of the terms $\mathbb{D}_a^{Q_j}$ necessarily satisfy the condition A1 in (Pakniyat, Caines, 2016). To be specific, for the case (i) the diffusion coefficients in (31) satisfy

$$\mathbb{D}_a^{q_{0a}} = \Psi_{i,a} \mathbb{D}_a^{Q_0}, \tag{38}$$

$$\mathbb{D}_a^{Q_2} = \Psi_{i,\sigma_{q_{0a},q_{\text{stop}}}} \mathbb{D}_a^{q_{0a}} \equiv 0, \tag{39}$$

where $\sigma_{Q_j,q_{\text{stop}}}$ denotes the stopping event in the discrete state Q_j. Both conditions in (39) are satisfied since $\mathbb{D}_a^{Q_2} = 0$ due to the zero dynamics after stopping and $\Psi_{i,\sigma_{q_{0a},q_{\text{stop}}}} = 0$ due to removal of the minor acquirer trader's state from the market dynamics. For the case (ii) we also have

$$\mathbb{D}_a^{Q_1} = \Psi_{i,\sigma_{Q_0,q_{\text{stop}}}} \mathbb{D}_a^{Q_0} \equiv 0, \tag{40}$$

which holds due to the stopping decision at t_1. The results of (Pakniyat, Caines, 2016, 2017a) yield

$$u_i^{Q_j}(t) = -R_{Q_j}^{-1}\mathbb{B}_{a,Q_j}^T\Pi_a^{Q_j}(t)\,x_i^{ex,Q_j}(t), \tag{41}$$

with

$$-\dot{\Pi}_a^{Q_j} = \Pi_a^{Q_j}\mathbb{A}_a^{Q_j} + \mathbb{A}_{a,Q_j}^T\Pi_a^{Q_j} - \Pi_a^{Q_j}\mathbb{B}_a^{Q_j}R_{a,Q_j}^{-1}\mathbb{B}_{a,Q_j}^T\Pi_a^{Q_j} + \mathbb{P}_a, \tag{42}$$

where for the case (i), in which $Q_1 = q_{0a}$, $\Pi_a^{Q_j}(t)$ is the solution of (42) subject to the terminal conditions

$$\Pi_a^{q_{0a}}(t_2) = \bar{\mathbb{P}}_a^{q_{0a}},$$

$$\left(\mathbb{P}_a^{q_{0a}} + \bar{\mathbb{P}}_a^{q_{0a}}\mathbb{A}_a^{q_{0a}} + \mathbb{A}_{a,q_{0a}}^T\bar{\mathbb{P}}_a^{q_{0a}} - \bar{\mathbb{P}}_a^{q_{0a}}\mathbb{B}_a^{q_{0a}}R_{a,q_{0a}}^{-1}\mathbb{B}_{a,q_{0a}}^T\bar{\mathbb{P}}_a^{q_{0a}}\right)_{t=t_2} = 0,$$

and the boundary conditions

$$\Pi_a^{Q_0}(t_1) = \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}, \tag{43}$$

$$\mathbb{P}_a^{Q_0} + \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}\mathbb{A}_a^{Q_0} + \mathbb{A}_{a,Q_0}^T\Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}$$
$$- \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}\mathbb{B}_a^{Q_0}R_{a,Q_0}^{-1}\mathbb{B}_{a,Q_0}^T\Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}$$
$$= \Psi_{i,a}^T\left(\mathbb{P}_a^{q_{0a}} + \Pi_a^{q_{0a}}(t_1)\mathbb{A}_a^{q_{0a}} + \mathbb{A}_{a,q_{0a}}^T\Pi_a^{q_{0a}}(t_1) - \Pi_a^{q_{0a}}(t_1)\mathbb{B}_a^{q_{0a}}R_{a,q_{0a}}^{-1}\mathbb{B}_{a,q_{0a}}^T\Pi_a^{q_{0a}}(t_1)\right)\Psi_{i,a}, \tag{44}$$

and in case (ii) where $Q_1 = q_{0l}$ holds, $\Pi_a^{Q_0}(t)$ is the solution of (42) subject to the terminal conditions

$$\Pi_a^{Q_0}(t_1) = \bar{\mathbb{P}}_a^{Q_0}, \tag{45}$$

$$\left(\mathbb{P}_a^{Q_0} + \bar{\mathbb{P}}_a^{Q_0}\mathbb{A}_a^{Q_0} + \mathbb{A}_{a,Q_0}^T\bar{\mathbb{P}}_a^{Q_0} - \bar{\mathbb{P}}_a^{Q_0}\mathbb{B}_a^{Q_0}R_{a,Q_0}^{-1}\mathbb{B}_{a,Q_0}^T\bar{\mathbb{P}}_a^{Q_0}\right)_{t=t_1} = 0. \tag{46}$$

3.4.3   Minor liquidator

The hybrid dynamics, jump maps and performance measures for a minor liquidator are presented in a similar form as the minor acquirer, and therefore, due to space limitations, are not presented here. The infinite population best response hybrid control action as

$$u_i^{Q_j}(t) = -R_{Q_j}^{-1}\mathbb{B}_{l,Q_j}^T\Pi_l^{Q_j}(t)\,x_i^{ex,Q_j}(t), \tag{47}$$

with

$$-\dot{\Pi}_l^{Q_j} = \Pi_l^{Q_j}\mathbb{A}_l^{Q_j} + \mathbb{A}_{l,Q_j}^T\Pi_l^{Q_j} - \Pi_l^{Q_j}\mathbb{B}_l^{Q_j}R_{l,Q_j}^{-1}\mathbb{B}_{l,Q_j}^T\Pi_l^{Q_j} + \mathbb{P}_l, \tag{48}$$

where for the case (i), in which $Q_1 = q_{0a}$, $\Pi_l^{Q_j}(t)$ is the solution of (48) subject to the terminal conditions

$$\Pi_l^{Q_0}(t_1) = \bar{\mathbb{P}}_l^{Q_0},$$

$$\left(\mathbb{P}_l^{Q_0} + \bar{\mathbb{P}}_l^{Q_0}\mathbb{A}_l^{Q_0} + \mathbb{A}_{l,Q_0}^T\bar{\mathbb{P}}_l^{Q_0} - \bar{\mathbb{P}}_l^{Q_0}\mathbb{B}_l^{Q_0}R_{l,Q_0}^{-1}\mathbb{B}_{l,Q_0}^T\bar{\mathbb{P}}_l^{Q_0}\right)_{t=t_1} = 0.$$

and in case (ii) where $Q_1 = q_{0l}$ holds, $\Pi_l^{Q_0}(t)$ is the solution of (42) subject to the terminal conditions

$$\Pi_l^{q_{0l}}(t_2) = \bar{\mathbb{P}}_l^{q_{0l}},$$

$$\left(\mathbb{P}_l^{q_{0l}} + \bar{\mathbb{P}}_l^{q_{0l}}\mathbb{A}_l^{q_{0l}} + \mathbb{A}_{l,q_{0l}}^T\bar{\mathbb{P}}_l^{q_{0l}} - \bar{\mathbb{P}}_l^{q_{0l}}\mathbb{B}_l^{q_{0l}}R_{l,q_{0l}}^{-1}\mathbb{B}_{l,q_{0l}}^T\bar{\mathbb{P}}_l^{q_{0l}}\right)_{t=t_2} = 0,$$

and the boundary conditions

$$\Pi_l^{Q_0}(t_1) = \Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l},$$

$$\mathbb{P}_l^{Q_0} + \Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}\mathbb{A}_l^{Q_0} + \mathbb{A}_{l,Q_0}^T\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l} - \Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}\mathbb{B}_l^{Q_0}R_{l,Q_0}^{-1}\mathbb{B}_{l,Q_0}^T\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}$$
$$= \Psi_{i,l}^T\left(\mathbb{P}_l^{q_{0l}} + \Pi_l^{q_{0l}}(t_1)\mathbb{A}_l^{q_{0l}} + \mathbb{A}_{l,q_{0l}}^T\Pi_l^{q_{0l}}(t_1) - \Pi_l^{q_{0l}}(t_1)\mathbb{B}_l^{q_{0l}}R_{l,q_{0l}}^{-1}\mathbb{B}_{l,q_{0l}}^T\Pi_l^{q_{0l}}(t_1)\right)\Psi_{i,l}. \tag{49}$$

The infinite population equilibria is linked to the finite population equilibria by the following theorem.

Theorem 1 (ϵ-Nash equilibria for hybrid MM LQG MFG systems) *Subject to reasonable technical conditions (see (Huang, 2010)), the system equations* (8), (10), (11) *together with the mean field equations* (51) *generate the set of control laws* $\mathcal{U}_{MF}^N \triangleq \{u_i^{Q_j}; 0 \leq i \leq N_t\}$, $1 \leq N_t \leq N < \infty$, *given by* (26), (41), *and* (47) *such that*

(i) *All agent systems* $\mathcal{A}_i$, $0 \leq i \leq N$, *are second order stable.*

(ii) $\{\mathcal{U}_{MF}^N; 1 \leq N < \infty\}$ *yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$;*

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_{i,y}^N} J_i^{s,N}(u_i, u_{-i}^\circ) \leq J_i^{s,N}(u_i^\circ, u_{-i}^\circ). \qquad \square$$

Proof. Applying the approach of (Huang, 2010) backwards from T along the optimal realization of the sequence Q_0, Q_1, Q_2, establishes the existence and uniqueness of the Nash equilibrium and ϵ-Nash equilibrium for the infinite population system and finite population system, respectively. $\qquad \square$

3.5 Mean field consistency equations

The closed loop trading dynamics of a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$ applying (41), or correspondingly a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$ applying (47) is consequently

$$d\nu_i = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt,$$

then the average of the closed loop trading dynamics over the acquirer or liquidator population is obtained as

$$\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} d\nu_i = -\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - \frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt, \qquad (50)$$

where $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$. Then taking the L^2 limit of (50) as the population size $N_{a/l}$ goes to infinity yields the trading rate mean field dynamics

$$d\bar{\nu}_{a/l} = \lim_{N_{a/l}\to\infty} d\nu^{N_{a/l}} = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l} \times \lim_{N_{a/l}\to\infty}\big((x^{N_{a/l}})^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}dt,$$

and hence the consistency equations are given by

$$\begin{aligned}
\bar{A}_{a,11} &= -R_a^{-1}(\Pi_{a,11} + \Pi_{a,17}) - aR_a^{-1}(\Pi_{a,31} + \Pi_{a,37}), \\
\bar{A}_{a,12} &= -R_a^{-1}(\Pi_{a,12} + \Pi_{a,18}) - aR_a^{-1}(\Pi_{a,32} + \Pi_{a,38}), \\
\bar{A}_{a,13} &= -R_a^{-1}(\Pi_{a,13} + \Pi_{a,19}) - aR_a^{-1}(\Pi_{a,33} + \Pi_{a,39}), \\
\bar{A}_{al,11} &= -R_a^{-1}(\Pi_{a,110} + a\Pi_{a,310}), \\
\bar{A}_{al,12} &= -R_a^{-1}(\Pi_{a,111} + a\Pi_{a,311}) \\
\bar{A}_{al,13} &= -R_a^{-1}(\Pi_{a,112} + a\Pi_{a,312}) \\
\bar{A}_{l,11} &= -R_l^{-1}(\Pi_{l,11} + \Pi_{l,110}) - aR_l^{-1}(\Pi_{l,31} + \Pi_{l,310}), \\
\bar{A}_{l,12} &= -R_l^{-1}(\Pi_{l,12} + \Pi_{l,111} - aR_l^{-1}(\Pi_{l,32} + \Pi_{l,311}) \\
\bar{A}_{l,13} &= -R_l^{-1}(\Pi_{l,13} + \Pi_{l,112}) - aR_l^{-1}(\Pi_{l,33} + \Pi_{l,312}) \\
\bar{A}_{la,11} &= -R_l^{-1}\Pi_{l,17} - aR_l^{-1}\Pi_{l,37} \\
\bar{A}_{la,12} &= -R_l^{-1}\Pi_{l,18} - aR_l^{-1}\Pi_{l,38} \\
\bar{A}_{la,13} &= -R_l^{-1}\Pi_{l,19} - aR_l^{-1}\Pi_{l,39} \\
\bar{G}_{a/l,11} &= -R_{a/l}^{-1}(\Pi_{a/l,14} + a\Pi_{a/l,34}), \\
\bar{G}_{a/l,12} &= -R_{a/l}^{-1}(\Pi_{a/l,15} + a\Pi_{a/l,35}), \\
\bar{G}_{a/l,13} &= -R_{a/l}^{-1}(\Pi_{a/l,16} + a\Pi_{a/l,36}), \\
\bar{m}_{a/l,1} &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (51)
\end{aligned}$$

where $\Pi_{a/l,ij} = \Pi_{a/l}(i,j)$ for $i = \{1,3\}$, $j = \{1,2,3,...,12\}$. Hence the matrices in (15) are given as

$$\bar{A}_{a/l} = \begin{bmatrix} \bar{A}_{a/l,11} & \bar{A}_{a/l,12} & \bar{A}_{a/l,13} \\ 1 & 0 & 0 \\ (\pi_{a/l}\lambda + a\bar{A}_{a/l,11}) & a\bar{A}_{a/l,12} & a\bar{A}_{a/l,13} \end{bmatrix}, \quad \bar{A}_{al} = \begin{bmatrix} \bar{A}_{al,11} & \bar{A}_{al,12} & \bar{A}_{al,13} \\ 0 & 0 & 0 \\ \pi_l\lambda + a\bar{A}_{al,11} & a\bar{A}_{al,12} & a\bar{A}_{al,13} \end{bmatrix},$$

$$\bar{m} = \begin{bmatrix} \bar{m}_{a/l,1} \\ 0 \\ a\bar{m}_{a/l,1} \end{bmatrix}, \quad \bar{A}_{la} = \begin{bmatrix} \bar{A}_{la,11} & \bar{A}_{la,12} & \bar{A}_{la,13} \\ 0 & 0 & 0 \\ \pi_a\lambda + \bar{A}_{la,11} & a\bar{A}_{la,12} & a\bar{A}_{la,13} \end{bmatrix},$$

$$\bar{G}_{a/l} = \begin{bmatrix} \bar{G}_{a/l,12} & \bar{G}_{a/l,22} & \bar{G}_{a/l,23} \\ 0 & 0 & 0 \\ (\lambda_0 + a\bar{G}_{a/l,21}) & a\bar{G}_{a/l,22} & a\bar{G}_{a/l,23} \end{bmatrix}.$$

4   Conclusions

Hybrid MFG theory was utilized in a non-cooperative game formulation of the financial market where HFTs (minor agents) may leave the market before the final time. The best response trading and stopping policies for the agents are further shown to yield an ϵ-Nash equilibrium for the the market.

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