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Algorithms and computational complexity on the P_k-hitting set problem

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Abstract: The P_k-hitting set problem consists in removing a minimum number $\psi_k(G)$ of vertices of a given graph G so that the resulting graph does not contain path P_k on k vertices as a subgraph. For instance, the corresponding vertex set for $k = 2$ is a vertex cover. Cubic graphs received more attention for this problem in the last decade, especially for approximation algorithms.

In the present paper, we prove that the P_k-hitting set problem is NP-complete in the class of cubic graphs, for any fixed constant $k \geqslant 4$. Then, we design for cubic graphs a polynomial-time algorithm which is a 1.25-approximation for the P_3-hitting set problem, better than the best-known (1.57-approximation algorithm), and a 2-approximation for the P_4-hitting set problem. Compared to other approximation algorithms, one of its force is its simplicity. We conjecture that for any cubic graph G and any constant $k \geqslant 5$,

$$\frac{|V(G)|}{k} \leqslant \psi_k(G).$$

Moreover, if true, this lower bound is achieved by a family of graphs with arbitrary large value of ψ_k. As a consequence of Bollobas, Robinson and Wormald's result [1, 21, 22], the conjecture is true for almost all cubic graphs. So our algorithm becomes a $\frac{k}{2}$-approximation for the P_k-hitting set problem for almost all cubic graphs, which is better than other constant approximation ratios. Finally, we design, for subcubic graphs, a polynomial-time algorithm which transforms any minimum P_k-hitting set into another one with the following property: itself and its complement are P_k-hitting sets. Besides, there does not exist such algorithm if the graph has at least one vertex with degree $d \geqslant 4$.

Keywords: P_k-hitting set, subcubic graphs, approximation algorithm, computational complexity

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1 Introduction

1.1 Background

In this paper, we consider graphs that are finite, simple, and undirected. Usual concepts and definitions are explained by Diestel [10]. Cubic graphs are graphs with only 3-degree vertices whereas subcubic graphs are subgaphs of cubic graphs. Let $\mathscr{F}$ be a family of graphs. A $\mathscr{F}$*-hitting set* of a graph $G = (V, E)$ is a vertex set X such that the subgraph induced by $V \setminus X$, i.e. $G[V \setminus X]$, does not contain any graphs from $\mathscr{F}$ as a (nonnecessarily induced) subgraph. This contains as special cases the vertex cover and the feedback vertex set, which are obtained by taking $\mathscr{F} = \{P_2\}$ and $\mathscr{F} = \{C_k : k \geqslant 3\}$, respectively, where P_k (respectively C_k) denotes the path (respectively cycle) on k vertices.

We focus on the $\mathscr{F}$-hitting set when $\mathscr{F} = \{P_k\}$ for a fixed constant $k \geqslant 2$. A P_k-hitting set is called *minimum* if its cardinality is minimum and we denote this cardinality by $\psi_k(G)$. Moreover, the corresponding problem are asked, given a graph G and a natural number r, whether $\psi_k(G) \leqslant r$.

In this paper, we first conduct a brief survey and we present then our results on the P_k-hitting set problem in subcubic graphs. Our results are split into three parts. In the first one, we investigate the computational complexity of this problem. Then we establish an approximation algorithm and in the last part, we design an algorithm to find a particular minimum P_k-hitting set.

1.2 State-of-the-art

Some authors [2, 3, 5, 8, 15, 16, 23, 24, 28, 29] investigated the P_k-hitting set problem from different points of view. However, in this paper, we focus only on approximation algorithms.

As it is well known, the vertex cover problem admits a 2-approximation algorithm, and it is widely believed that no better approximation ratio can be achieved unless P = NP (assumption related to the Unique Games Conjecture [17]). For $k = 3$, also a 2-approximation algorithm can be achieved in polynomial time [26, 27] while Camby, Cardinal, Chapelle, Fiorini and Joret [7] designed a primal-dual 3-approximation algorithm for the P_4-hitting set problem.

Jakovac [14] investigated the P_k-hitting set problem in the class of rooted product graphs whereas Zuo Zhang and Zhang [30] focused on product graphs of stars and complete graphs. Moreover, Brešar, Krivoš-Belluš, Semanišin and Šparl [6] studied the weighted version of the problem. Li, Zhang and Huang [19] examined the connected version and presented a k-approximation algorithm if the graph has girth at least k.

Note that for the original problem, a k-approximation can trivially be achieved by taking all vertices in an inclusion-wise maximal packing of vertex-disjoint subgraphs, each isomorphic to P_k, but Funke, Nusser and Storandt [12] designed a $log(|OPT|)$-approximation algorithm for the P_k-hitting set problem, where OPT is an optimal solution. By a straightforward reduction, any α-approximation algorithm for the P_k-hitting set problem can be changed into an α-approximation algorithm for the vertex cover problem. Hence, it is unlikely to obtain such an algorithm with $\alpha < 2$ for some $k \geqslant 2$ in the general class of graphs. However, in the class of cubic or subcubic graphs, algorithms with better approximation ratios can be achieved. Thus, from the basis of our investigations, the following main question arises: is there an $\varepsilon > 0$ such that the P_k-hitting set problem can be approximated to within a constant factor of $(1-\varepsilon)k$ in polynomial time for every fixed $k \geqslant 3$?

1.2.1 Related works on the P_k-hitting set in cubic graphs

Tu and Yang [25] investigated the P_3-hitting set problem in cubic graphs. For cubic graphs, they proved the NP-completness of this problem, proposed a 1.57-approximation algorithm and gave sharp lower and upper bounds on $\psi_3(G)$.

Theorem 1 (Tu, Yang [25]) *Let G be a cubic graph of order n. Then*

$$\frac{2}{5}n \leqslant \psi_3(G) \leqslant \frac{1}{2}n.$$

Moreover, these bounds are sharp by a family of connected graphs with arbitrary large value of $\psi_3(G)$*.*

Notice that this theorem is partly a corollary from result of Brešar et al. [4]: for any cubic graph and $k \in \{3, 4\}$,

$$\frac{5-k}{8-k}n \leqslant \psi_k(G).$$

Li and Tu [18] established the same kind of results for the P_4-hitting set problem. Firstly, they proved roughly that the P_4-hitting set problem is NP-complete for cubic graphs. Secondly, they bounded $\psi_4(G)$ for any cubic graph G by a simple argumentation of counting and by using the well-known decomposition of graphs from Lovasz [20], with the same lower bound found by Brešar et al. [4].

Theorem 2 (Li, Tu [18]) *Let* G *be a cubic graph of order* n*. Then*

$$\frac{n}{4} \leqslant \psi_4(G) \leqslant \frac{n}{2}.$$

Moreover, these bounds are sharp.

Notice that the lower bound is sharp by a family of connected graphs with an arbitrary large value of $\psi_4(G)$ whereas the upper bound is only attained in the paper by two connected graphs on 6 and 8 vertices. Finally, Li and Tu [18] established a method to deduce a 2-approximation algorithm for the P_4-hitting set problem in the class of cubic graphs. However, this method uses also the Lovasz' decomposition [20] and the algorithm is not explicitly given. Recently, Esperet et al. [11] proved that any cubic graph G admits a bipartition (V_1, V_2) of $V(G)$ such that both sets, with the same cardinality, induce a vertex-disjoint union of paths on at most 3 vertices, which implies that both sets are P_4-hitting sets. Furthermore, Devi Mane and Mishra [9] proved that the P_4-hitting set problem is APX-complete for cubic graphs as well as cubic and bipartite graphs and they also designed a greedy based algorithm within a factor of 2 for regular graphs.

2 Results

2.1 Computational complexity of the P_k-hitting set problem for cubic graphs

Theorem 3 *Let* $r \in \mathbb{N}$ *be a natural number and* $k \geqslant 4$*. Given a cubic graph* G*, the problem of deciding whether* $\psi_k(G) \leqslant r$ *is NP-complete.*

Due to space restrictions, proof is omitted and placed in an appendix.

2.2 An approximation algorithm for P_k-hitting set problem in cubic graphs

Given a graph G, an *edge-cut* induced by $X \subseteq V(G)$ is the set of edges between X and $V(G) \setminus X$. We call a *maximal* edge-cut induced by X if its cardinality is greater than that of the edge-cut induced by $X \cup \{v\}$ or $X \setminus \{v\}$, for every $v \in V(G)$. Observe that the edge-cut induced by X is exactly the same than the edge-cut induced by $V \setminus X$ and the condition of maximality does not change in the complement. As mentionned by Esperet et al. [11], the vertex set X from any maximal edge-cut of a cubic graph is a P_3-hitting set. Moreover, the complement of X is also a P_3-hitting set. In fact, this remark is also valid for subcubic graphs, as proved in Theorem 4. The next polynomial-time algorithm build precisely a maximal edge-cut. Recall that $E(X, V \setminus X)$ denotes edges between X and $V \setminus X$. Notice that the last if-condition ensures a low cardinality.

Observe that the while-loop stops when no vertex can be removed or added in the current vertex set by increasing the cardinality of the corresponding edge-cut, i.e. when the corresponding edge-cut is maximal.

Theorem 4 *Let* $k \geqslant 3$ *be a fixed natural number. Given a subcubic graph* G*, Algorithm 1 gives in polynomial-time a* P_k*-hitting set* X *such that* $V(G) \setminus X$ *is also a* P_k*-hitting set.*

Algorithm 1 Algorithm to find a maximal edge-cut.

```
Require: G = (V, E) a graph with V = {v_1, ..., v_n}
Ensure: X a vertex set such that the edge-cut induced by X is maximal
  Let X ← V(G)
  Let Y ← X
  Let Z ← X
  while X ≠ Y do
    X ← Y
    for i = 1, ..., n do
      Z ← Y
      if v_i ∉ Z then
        Z ← Z ∪ {v_i}
      else
        Z ← Z \ {v_i}
      end if
      if |E(Y, V \ Y)| < |E(Z, V \ Z)| then
        Y ← Z
      end if
    end for
  end while
  if |X| < |V \ X| then
    return X
  else
    return V \ X.
  end if
```

Proof. Let X be the vertex set produced by Algorithm 1 and corresponding to a maximal edge-cut. It is sufficient to show that X and $V \setminus X$ are both P_3-hitting sets, since any P_3-hitting set is in particular a P_k-hitting set. On one hand, we assume that X contains a path on 3 vertices x_1, x_2, x_3 with edges x_1x_2, x_2x_3. Since the degree of x_2 is at most 3, the cardinality of the edge-cut induced by $X \setminus \{x_2\}$ is strictly greater than that for X. On the other hand, we suppose that $V \setminus X$ contains a path on 3 vertices x_1, x_2, x_3,with edges x_1x_2, x_2x_3. We deduce the same conclusion with the set $X \cup \{x_2\}$. In both cases, it is a contradiction with the maximality of X.

Observe that Algorithm 1 runs in polynomial-time, exactly in $O(mn)$, where m is the number of edges and n is the number of vertices. Indeed, the while-loop is applied at most m times since at each step, the cardinality of the edge-cut increases strictly. □

From Theorem 4, we deduce the following corollary since either X or $V \setminus X$ has at most $|V|/2$ vertices.

Corollary 1 *Let $G = (V, E)$ be a subcubic graph and $k \geqslant 4$. Every minimum P_k-hitting set has at most $|V|/2$ vertices.*

Besides, Algorithm 1 is actually an approximation algorithm for the P_k-hitting set problem in the class of cubic graphs, when $k = 3$ or 4, as stated in the following theorem.

Theorem 5 *Algorithm 1 is a 1.25-approximation algorithm for the P_3-hitting set problem and a 2-approximation algorithm for the P_4-hitting set problem in cubic graphs.*

Proof. It remains to check the performance ratios. Because of the last if-condition, we know that the size of the solution X given by Algorithm 1 is at most $|V|/2$. Furthermore, Brešar et al. [4] proved that for any cubic graph and $k \in \{3, 4\}$,

$$\frac{5-k}{8-k} n \leqslant \psi_k(G).$$

Thus

$$
\begin{aligned}
|X| &\leqslant \tfrac{|V|}{2} \\
&= \begin{cases} \frac{5}{4}\frac{2}{5}|V| & \text{if } k=3 \\ 2\frac{1}{4}|V| & \text{if } k=4 \end{cases} \\
&\leqslant \begin{cases} \frac{5}{4}\psi_3(G) & \text{if } k=3 \\ 2\psi_4(G) & \text{if } k=4 \end{cases}
\end{aligned}
$$

□

If we find a lower bound on $\psi_k(G)$, when G is a subcubic or cubic graph, then Algorithm 1 could be an approximation algorithm where the performance ratio is depending on the lower bound. However, first Bollobas [1], later Robinson and Wormald [21, 22] proved that almost all regular graphs are Hamiltonian, i.e. in almost all regular graphs, there exists a cycle that visits each vertex exactly once. Since an Hamiltonian cycle can be decomposed in vertex-disjoint paths on k-vertices, almost all regular graphs G satisfy $\frac{|V(G)|}{k} \leqslant \psi_k(G)$. Hence, Algorithm 1 is a $\frac{k}{2}$-approximation algorithm for the P_k-hitting set problem in almost all cubic graphs.

Conjecture 1 *Let G be a cubic graph and $k \geqslant 5$ a natural number. Then*

$$\frac{|V(G)|}{k} \leqslant \psi_k(G).$$

If the conjecture is true, then the lower bound is tight. Indeed, we construct two graphs depending on the parity of k. Let r be a natural number.

Assume k is even, take r vertex-disjoint cycles on k vertices and add a perfect matching in each cycle. In the i-th cycle, choose an edge uv from the perfect matching, remove it and add two edges : one from the $((i-1) modulo\ r)$-th cycle and the vertex v, the other one from the $((i+1) modulo\ r)$-th cycle and the vertex u. The resulting graph $G_{k,r}$ is depicted in Figure 1. Clearly, the set of vertices of type v is a minimum P_k-hitting set. Thus

$$\frac{|V(G_{k,r})|}{k} = \frac{rk}{k} = r = \psi_k(G_{k,r}).$$

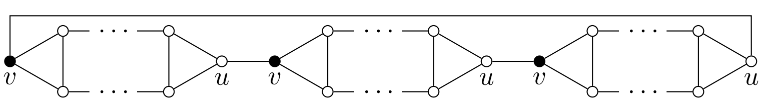

Figure 1: Graph $G_{k,r}$, when k is even and $r = 3$.

Suppose that k is odd. We consider the same structure but for each cycle, we have one vertex x not covered by the perfect matching. To this vertex x, we attach a pendant cycle on k vertices with a perfect matching, as illustrated by Figure 2. Again,

$$\frac{|V(G_{k,r})|}{k} = \frac{2rk}{k} = 2r = \psi_k(G_{k,r}).$$

2.3 A particular minimum P_k-hitting set in subcubic graphs

We have already seen that there always exists a P_k-hitting set X in a subcubic graph G such that $V(G)\setminus X$ is also a P_k-hitting set, by taking a vertex set from a maximal edge-cut. In the following theorem, the algorihm builds a P_k-hitting set holding the same property. The main difference is that the cardinality of the resulting set is exactly $\psi_k(G)$.

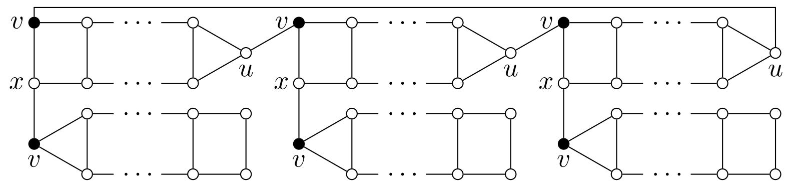

Figure 2: Graph $G_{k,r}$, when k is odd and $r = 3$.

Algorithm 2 Polynomial-time algorithm to find a particular P_k-hitting set.

Require: $G = (V, E)$ a subcubic graph and Y a minimum P_k-hitting set
Ensure: X a minimum P_k-hitting set of G such that $V \setminus X$ is also a P_k-hitting set
  Let $X \leftarrow Y$
  while $G[X]$ contains a subgraph isomorphic to P_k **do**
    Let P be a subgraph of $G[X]$ isomorphic to a k-vertex path
    Let x be a vertex in P with two neighbors in P
    Let P_x be the private path of x in G
    Let u be the neighbor of x in P_x
    $X \leftarrow (X \setminus \{x\}) \cup \{u\}$
  end while
  return X

Theorem 6 *Let $G = (V, E)$ be a subcubic graph and $k \geqslant 3$. Given a minimum P_k-hitting set of G, Algorithm 2 gives in polynomial time a P_k-hitting set X such that $V \setminus X$ is also a P_k-hitting set and*

$$|X| = \psi_k(G).$$

Let us introduce the notion of private path. If X is a P_4-hitting set of the graph G and $x \in X$, a *private path* of x is a (not necessary induced) subgraph P_x of G isomorphic to a 4-vertex path such that $V(P_x) \setminus \{x\} \subseteq V(G) \setminus X$, or in other words, $V(P_x) \cap X = \{x\}$.

Proof. Notice that every vertex v of any minimum P_k-hitting set X has a private path, hence

$$deg_{G[X]}(v) \ \leqslant \ 2.$$

Therefore, every connected component of any minimum P_k-hitting set is either a path or a cycle.

Let P be a subgraph in $G[X]$ isomorphic to a k-vertex path, $x \in V(P)$ be a vertex with two neighbors in P and P_x be the private path of x in G. Since $deg_{G[X]}(x) = 2$, x is an extremity of P_x. Let $u \in V(P_x)$ be the neighbor of x. Let $Z = (X \setminus \{x\}) \cup \{u\}$. Since $N_G(x) \subseteq Z$, i.e. x is an isolated vertex in the subgraph induced by the complement of Z, Z is still a P_k-hitting set with the same cardinality, i.e. a minimum P_k-hitting set.

It remains to check if the while-loop stops. Indeed, in each step, the number of edges between X and $V \setminus X$ is strictly increasing, as shown by Figure 3, which completes the proof.

Besides, this algorithm runs in polynomial time since the number of edges are polynomial in $|V|$. $\square$

Observe that Theorem 6 does not yield anymore if the graph is not subcubic. Indeed, consider the graph G_s by taking $s \geqslant k$ vertex-disjoint paths on $2k-1$ vertices, say $v_1^j, \ldots, v_{2k-1}^j$, for $j = 1, \ldots, s$, and adding edges between vertices $v_k^1, \ldots, v_k^s$, the middle vertices of each path, to obtain an induced path on s vertices. This graph is illustrated by Figure 4. Then the maximum degree of G_s is 4 and $\psi_k(G_s) = s$. Indeed, the set of all middle vertices is the only minimum P_k-hitting set of G_s and its complement is obviously not a P_k-hitting set.

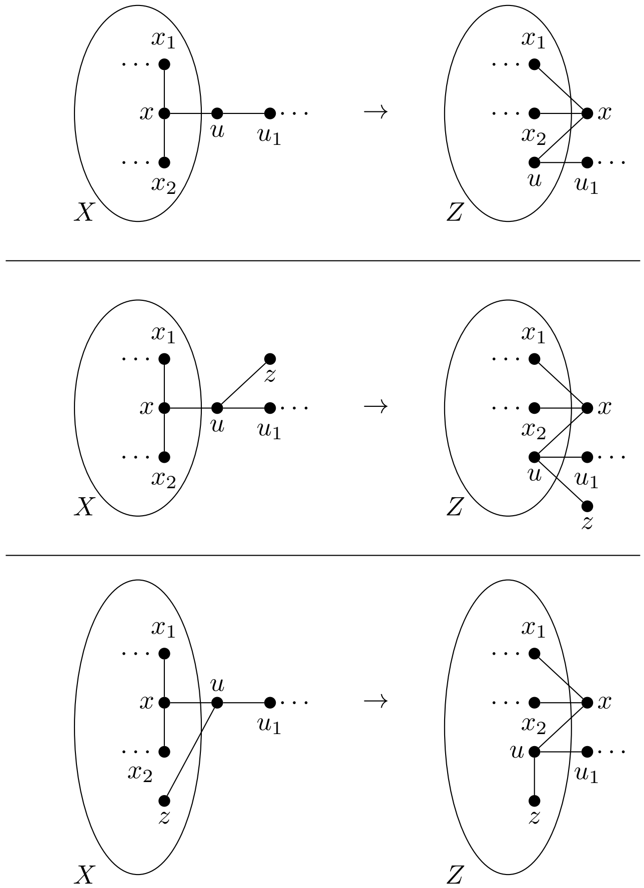

Figure 3: The transformation in each case, depending on the neighborhood of u, increases strictly the number of edges between X and $V \setminus X$.

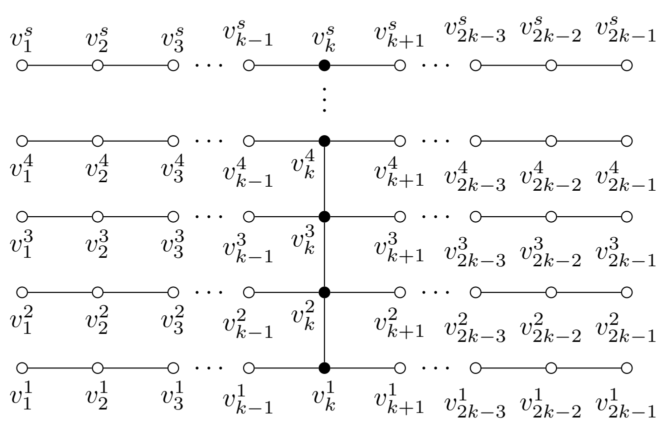

Figure 4: Graph G_s where the set of black vertices is the only minimum P_k-hitting set.

Appendix: Proof of Theorem 3

In this appendix, we present a complete and detailed proof of hardness for P_k-hitting set problem in the class of cubic graphs, where $k \geqslant 4$ is a fixed constant. This proof is inspired from [18].

We indirectly reduce the vertex cover problem for cubic graphs, which is known to be NP-complete [13], to the P_k-hitting set problem for cubic graphs. Firstly, we reduce the vertex cover problem for cubic graphs to the same problem for semi 4-regular graphs. A graph is *semi 4-regular* if the degree of any vertex is either 1 or 4. Secondly, we reduce the vertex cover problem for semi 4-regular graphs to the P_k-hitting set problem for subcubic graphs. The last reduction is from the P_k-hitting set problem for subcubic graphs to the same problem in the class of cubic graphs.

Proof. Clearly, regardless of the class of graphs, the P_k-hitting set problem, resp. the vertex cover problem, is in NP since the recognition of graphs without a path on k vertices, resp. on 2 vertices, as a subgraph yields in polynomial time. Recall that the vertex cover number of a graph G is denoted by $\tau(G)$.

Claim 1 *Let $r \in \mathbb{N}$ be a natural number. Given a semi 4-regular graph G, the problem of deciding whether $\tau(G) \leqslant r$ is NP-complete.*

Let G be a cubic graph. Let G' be a graph obtained from G by adding to each vertex v of $V(G)$ a pendant star S_v isomorphic to $K_{1,3}$ and by attaching the vertex v to the 3-degree vertex x_v of S_v. The situation is illustrated by Figure 5. Clearly, G' is semi 4-regular. On one hand, let X be a vertex cover of G. Then

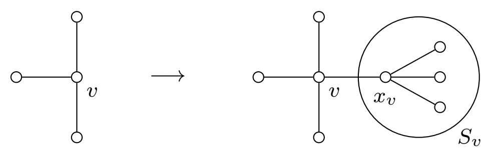

Figure 5: Reduction from a cubic graph to a semi 4-regular graph.

$X' = X \cup \{x_v \mid v \in V(G)\}$ is a vertex cover of G' by construction. On the other hand, let X' be a vertex cover of G'. We can assume without loss of generality that no 1-degree vertex belongs to X'. Accordingly, for every $v \in V(G)$, $X' \cap S_v = \{x_v\}$. Consider $X = X' \setminus \{x_v | v \in V(G)\}$. If an edge $uv \in E(G)$ is not covered by X, then the edge $uv \in E(G')$ is also not covered by X' in G', a contradiction. Thus, X is a vertex cover of G. Observe that the construction yields in polynomial time.

Claim 2 *Let $r \in \mathbb{N}$ be a natural number and $k \geqslant 4$. Given a subcubic graph G, the problem of deciding whether $\psi_k(G) \leqslant r$ is NP-complete.*

Now, we reduce the vertex cover problem for semi 4-regular graphs to the P_k-hitting set problem for subcubic graphs. Given a semi 4-regular graph G, we construct a new graph G' in the following way. Take $s = \lceil \frac{k}{2} \rceil + 1$. First, we state some practical inequalities. Clearly, $k/2 + 1 \leqslant s < k/2 + 2$ and $2s - 2 \geqslant k$. If k is even, $2s = k + 2$, otherwise $2s = k + 3$. Let v be a 4-degree vertex of G and u, w, r, t its neighbors. We split v into two vertices v_1 and v_{2k+1}, where v_1 and v_{2k+1} share the neighborhood of v, for instance $v_1u_1, v_1w_{2k+1}, v_{2k+1}r_1, v_{2k+1}t_{2k+1} \in E(G')$, we add a path on $2k - 1$ new vertices $v_2, \ldots, v_{2k}$ with edges $v_1v_2, \ldots, v_{2k}v_{2k+1} \in E(G')$, and finally, we put two pendant paths on $k - s$ vertices attached to v_s and to v_{2k-s+2}. We call vertices from the pendant paths by $p_1, \ldots, p_{k-s}$ and by $p'_1, \ldots, p'_{k-s}$. The situation is illustrated by Figure 6.

Observe that the reduction is symmetric, in particular path between v_1 and p_{k-s}, resp. between v_{2k+1} and p'_{k-s}, has k vertices. The path between p_{k-s} and v_{2k-s+1}, resp. p'_{k-s} and v_{s+1}, covers $3k - 3s + 2 \geqslant k$ vertices, since $4 \leqslant k$ and $2s \leqslant k+2$ or $k+3$, depending on the parity of k. Hence, the path between p_{k-s} and p'_{k-s} contains at least k vertices. However, the length of the path between v_{s+1} and v_{2k-s+1} is $2k-2s+1 < k$ and it is also the case for paths between p_{k-s} and v_k, or v_{k+2} and p'_{k-s}. Let G'_v be the subgraph induced by $\{v_1, \ldots, v_{2k+1}, p_1, \ldots, p_{k-s}, p'_1, \ldots, p'_{k-s}\}$ in G'.

Let x be a 1-degree vertex of G. We attach to x a pendant path on $s-2$ vertices $x_2, \ldots, x_{s-1}$, as illustrated by Figure 7.

Obviously, the resulting graph G' is subcubic.

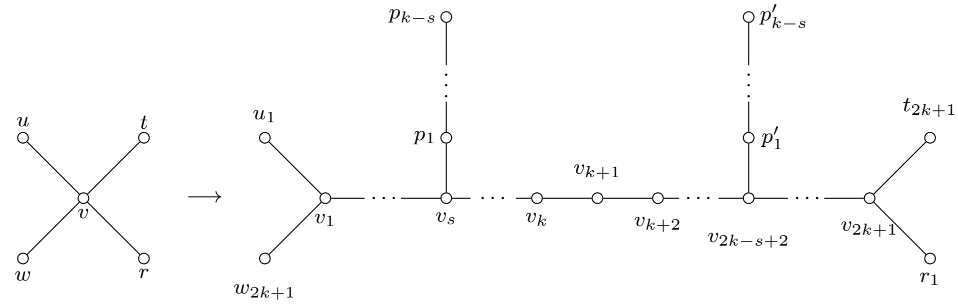

Figure 6: Reduction for a 4-degree vertex from a semi 4-regular graph to a subcubic graph.

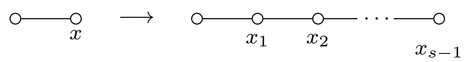

Figure 7: Reduction for a 1-degree vertex from a semi 4-regular graph to a subcubic graph.

On one hand, let X' be a P_k-hitting set of G'. Without loss of generality, we may assume that X' does not contain any vertex in $\{x_2, \ldots, x_{s-1}\}$, coming from x a 1-degree vertex of G. Observe that for each subgraph G'_v coming from a 4-degree vertex $v \in V(G)$, at least two vertices of G'_v belong to X', since G'_v has two vertex-disjoint k-vertex paths. If exactly two vertices of G'_v belong to X', they must be vertices v_s and v_{2k-s+2} because no other pair of vertices from G'_v intersect all k-vertex paths in G'_v.

Then we state that

$$\begin{array}{rcl} X & = & \{v \in V(G) | deg_G(v) = 4, |V(G'_v) \cap X'| \geqslant 3\} \\ & \cup & \{x \in V(G) | deg_G(x) = 1, x_1 \in X'\} \end{array}$$

is a vertex cover of G of size at most $|X'| - 2n_4$, where n_4 is the number of 4-degree vertices in G.

Suppose that in G, an edge e is not covered by X. If both extremities of $e = xy$ have degree 1 then no vertex from each pendant path is in X'. Hence, the $(2s-2)$-vertex path forming by two pendant paths of x and y is not covered by X' in G', a contradiction.

If only one extremity of $e = xv$ has degree 1, say x, then no vertex from the pendant path of x is in X' and $G'_v \cap X' = \{v_s, v_{2k-s+2}\}$. Without loss of generality, we can suppose that x_1 is adjacent to v_1. Hence the $(2s-2)$-vertex path from x_{s-1} to v_{s-1} is not covered by X', a contradiction.

Otherwise both extremeties of $e = uv$ have degree 4. Again, $G'_v \cap X' = \{v_s, v_{2k-s+2}\}$ and $G'_u \cap X' = \{u_s, u_{2k-s+2}\}$. Without loss of generality, we can suppose that v_1 is adjacent to u_1. Thus the $(2s-2)$-vertex path from v_{s-1} to u_{s-1} is not covered by X', a contradiction.

On the other hand, let X be a vertex cover of G. Now we construct a P_k-hitting set X' of G' of size at most $|X| + 2n_4$ as follows. Take

$$\begin{array}{rcl} X' & = & \{v_1, v_{k+1}, v_{2k+1} | v \in X, deg_G(v) = 4\} \\ & \cup & \{v_s, v_{2k-s+2} | v \notin X, deg_G(v) = 4\} \\ & \cup & \{x_1 | x \in X, deg_G(x) = 1\}. \end{array}$$

It remains to prove that X' is a P_k-hitting set of G'. We will describe the connected components of $G'[V(G') \setminus X']$. If $x \in X$ and $deg_G(x) = 1$, then the corresponding $(s-2)$-vertex pendant path in G' is detached by x_1 from the resulting graph. If $v \in X$ and $deg_G(v) = 4$, then $G'[V(G'_v) \setminus \{v_1, v_{k+1}, v_{2k+1}\}]$ is P_k-free. By property of a vertex cover, if a vertex is not in X then all neighbors of this vertex are in X. Accordingly, if $x \notin X$ and $deg_G(x) = 1$ then the connected component of $G'[V(G') \setminus X']$ containing x_1 is isomorphic to a $(s-1)$-vertex path. If $v \notin X$ and $deg_G(v) = 4$ then the connected components of $G'[V(G') \setminus X']$ intersecting G'_v are isomorphic to paths of length at most $2k - 2s + 1 < k$.

Observe that the construction yields again in polynomial time.

Claim 3 *Let $r \in \mathbb{N}$ be a natural number and $k \geqslant 4$. Given a cubic graph G, the problem of deciding whether $\psi_k(G) \leqslant r$ is NP-complete.*

We reduce the P_k-hitting set problem for subcubic graphs to the same problem for cubic graphs. Let G be a subcubic graph. We construct a cubic graph G' depending on the parity of k.

Suppose that k is odd. For a 2-degree vertex v of G, we add a cycle on k vertices $v_1, v_2, \ldots, v_k$ with edges $vv_1, v_1v_2, \ldots, v_kv_1$ and a perfect matching on $\{v_2, \ldots, v_k\}$, as illustrated by Figure 8. We call G'_v the subgraph induced by vertices $v_1, \ldots, v_k$.

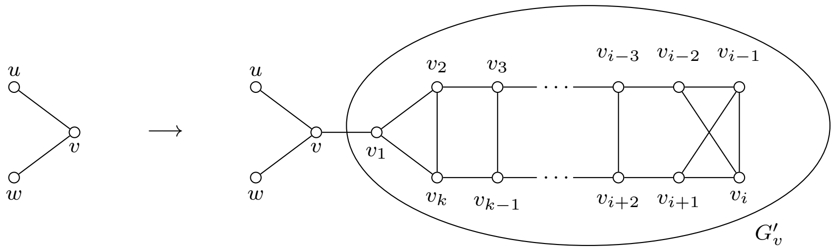

Figure 8: Reduction for a 2-degree vertex from a subcubic graph to a cubic graph, when k is odd.

For a 1-degree vertex v of G, we consider two graphs isomorphic to G'_v on vertices $\{v_1, \ldots, v_k\}$ and $\{v'_1, \ldots, v'_k\}$ and we add edges vv_1 and vv'_1, as illustrated by Figure 9. We call H'_v the subgraph induced by the two graphs isomorphic to G'_v.

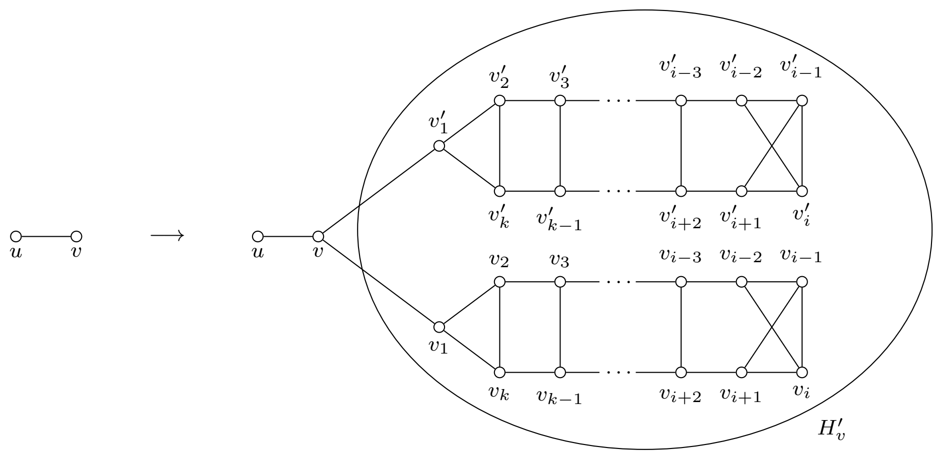

Figure 9: Reduction for a 1-degree vertex from a subcubic graph to a cubic graph, when k is odd.

Let X be a P_k-hitting set of G. Take

$$\begin{aligned} X' \quad = \quad & X \cup \{v_1 \in G'_v | deg_G(v) = 2\} \\ & \cup \{v_1, v'_1 \in H'_v | deg_G(v) = 1\}. \end{aligned}$$

By construction, X' is a P_k-hitting set of G' of size $|X| + n_2 + 2n_1$, where n_i is the number of vertices of degree i in G.

Moreover, let X' be a P_k-hitting set of G'. Observe that for each 2-degree vertex v of G, G'_v has at least one vertex in X' while for each 1-degree vertex v of G, H'_v has at least two vertices in X'. Take

$$\begin{array}{rcl} X & = & (X' \cap V(G)) \\ & \cup & \{v \in V(G) | deg_G(v) = 2, |G'_v \cap X'| \geqslant 2\} \\ & \cup & \{v \in V(G) | deg_G(v) = 1, |H'_v \cap X'| \geqslant 3\}. \end{array}$$

If X is not a P_k-hitting set of G then there is a k-vertex path P in G whose no vertex belongs to X. Then the corresponding path in G' has also no vertex in X', a contradiction.

Now, we assume that k is even. For a 2-degree vertex v of G, we use nearly the same construction from the odd case but the cycle is on $k+1$ vertices instead of k vertices, since k is even if and only if $k+1$ is odd. The situation is illustrated by Figure 10.

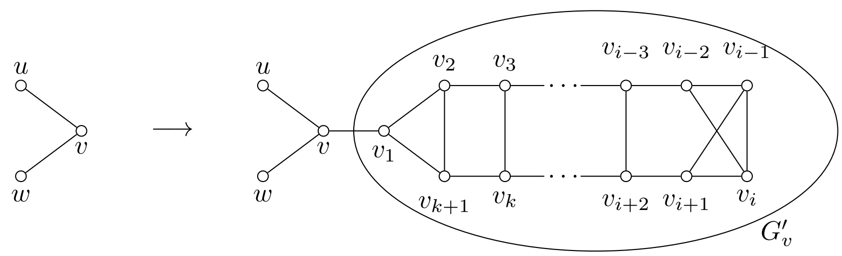

Figure 10: Reduction for a 2-degree vertex from a subcubic graph to a cubic graph, when k is even.

For a 1-degree vertex v of G, we use exactly the reduction when k is odd with the new graph G'_v defined previously, as illustrated by Figure 11.

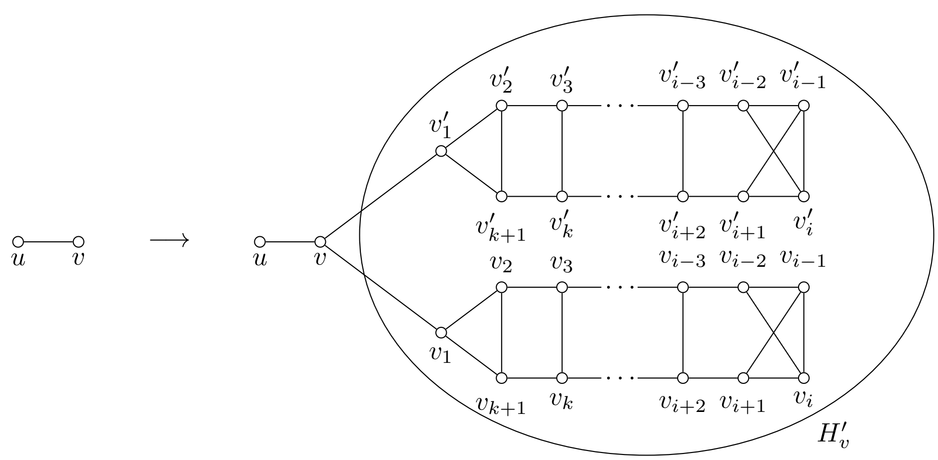

Figure 11: Reduction for a 1-degree vertex from a subcubic graph to a cubic graph, when k is even.

The argumentation of the end of the proof is inspired from the odd case but the main difference is the following : G'_v needs 2 vertices in a P_k-hitting set of G' and H'_v needs 4 vertices. Hence, the relation between sizes of P_k-hitting sets becomes $|X'| = |X| + 2n_2 + 4n_1$. $\square$

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