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On distance Laplacian and distance signless Laplacian eigenvalues of graphs

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Abstract: Let $\mathcal{D}(G)$, $\mathcal{D}^L(G) = \mathcal{D}iag(Tr) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = \mathcal{D}iag(Tr) + \mathcal{D}(G)$ be, respectively, the *distance matrix*, the *distance Laplacian matrix* and the *distance signless Laplacian matrix* of graph G, where $\mathcal{D}iag(Tr)$ denotes the diagonal matrix of the vertex transmissions in G. The eigenvalues of $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ will be denoted by $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_{n-1}^L \geq \partial_n^L = 0$ and $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_{n-1}^Q \geq \partial_n^Q$, respectively. In this paper we study the properties of the distance Laplacian eigenvalues and the distance signless Laplacian eigenvalues of graph G.

Keywords: Distance Laplacian eigenvalues, distance signless Laplacian eigenvalues, domination number, independence number, diameter

Résumé : Soient $\mathcal{D}(G)$, $\mathcal{D}^L(G) = \mathcal{D}iag(Tr) - \mathcal{D}(G)$ et $\mathcal{D}^Q(G) = \mathcal{D}iag(Tr) + \mathcal{D}(G)$, respectivement, la matrice des distances, le laplacien des distances et le laplacien sans signe des distances d'un graph connexe G, où $\mathcal{D}iag(Tr)$ désigne la matrice diagonale des transmissions des sommets de G. Les valeurs propres de $\mathcal{D}^L(G)$ et $\mathcal{D}^Q(G)$ seront notées $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_{n-1}^L \geq \partial_n^L = 0$ et $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_{n-1}^Q \geq \partial_n^Q$, respectivement. Dans cet article, nous étudions les propriétés des valeurs propres du laplacien des distances ainsi que de celles du laplacien sans signe des distances, d'un graphe connexe G.

Mots clés : Valeurs propres du laplacien des distances, valeurs propres du laplacien sans signe des distances, nombre de domination, nombre de stabilité, diamètre

1 Introduction

Throughout this paper we consider simple, undirected and connected graphs. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$, $|E(G)| = m$. Also let d_i be the degree of the vertex $v_i \in V(G)$. For $v_i \in V(G)$, the set of adjacent vertices of the vertex v_i is denoted by $N_G(v_i)$, that is,

$$N_G(v_i) = \{v_j \in V(G) : v_iv_j \in E(G)\}.$$

For $v_i, v_j \in V(G)$, the distance between v_i and v_j, denoted by d_{ij} or $d_G(v_i, v_j)$, is the length of a shortest path connecting them in G. The diameter d (or $d(G)$) of a graph is the maximum distance between any two vertices of G. The complement graph of a graph G is denoted by $\overline{G}$. The transmission $Tr(v_i)$ (or D_i) of a vertex v_i is defined to be the sum of the distances from v_i to all other vertices in G, that is,

$$Tr(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j).$$

Let $\mathcal{D}(G) = (d_{ij})_{n\times n}$ be the distance matrix of a graph G. For research related to the distance matrix, see [8, 9, 10, 11, 12, 14, 15] and especially the survey [4]. Let $\mathcal{D}^L(G) = \mathcal{D}iag(Tr) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = \mathcal{D}iag(Tr) + \mathcal{D}(G)$ be, respectively, the *distance Laplacian matrix* [1] and the *distance signless Laplacian matrix* [1] of the graph G, where $\mathcal{D}iag(Tr)$ denotes the diagonal matrix of the vertex transmissions in G. The eigenvalues of $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ will be denoted by $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_{n-1}^L \geq \partial_n^L = 0$ and $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_{n-1}^Q \geq \partial_n^Q$, respectively. Denote by

$$DLS(G) = (\partial_1^L, \partial_2^L, \dots, \partial_{n-1}^L, \partial_n^L) \text{ and } DSLS(G) = (\partial_1^Q, \partial_2^Q, \dots, \partial_{n-1}^Q, \partial_n^Q),$$

the distance Laplacian spectrum and the distance signless Laplacian spectrum of graph G, respectively. For the mathematical properties of distance Laplacian eigenvalues and distance signless Laplacian eigenvalues, including various lower and upper bounds, see [1, 3, 5, 13, 16, 18] and [1, 2, 7, 21, 22], respectively.

A subset W of vertices in G is called a *dominating set* if every vertex $v \in V(G)$ is either an element of W or is adjacent to an element of W. The *domination number* $\gamma = \gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A subset S of vertices of G is said to be *independent* if its vertices are pairwise nonadjacent. The maximum cardinality of such a subset is called the *independence number* of G and is denoted by $\alpha = \alpha(G)$. As usual, we denote by $K_{1,\,n-1}$ the star, by K_n the complete graph, by P_n the path, by C_n the cycle and by $K_{a,\,b}$ $(a + b = n)$ the complete bipartite graph, each on n vertices. The join $G \vee H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$. Thus, for example, $\overline{K}_p \vee \overline{K}_q = K_{p,\,q}$, the complete bipartite graph.

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present some results on the distance Laplacian eigenvalues with domination number, independence number and diameter of graph G. In Section 4, we obtain some results on the distance signless Laplacian eigenvalues and the independence number of graph G, and also characterize the extremal graphs.

2 Preliminaries

In this section, we shall list some previously known results that will be needed for proving our results in the next two sections.

The first result is known as the interlacing theorem.

Lemma 2.1 [19] *Let A be a $p \times p$ symmetric matrix and let A_k be its leading $k \times k$ submatrix; that is, A_k is the matrix obtained from A by deleting its last $p - k$ rows and columns. Then, for $i = 1, 2, \dots, k$,*

$$\lambda_{p-i+1}(A) \leq \lambda_{k-i+1}(A_k) \leq \lambda_{k-i+1}(A), \tag{1}$$

where $\lambda_i(A)$ is the i-th largest eigenvalue of A.

The next result is a particular case of the well-known *min-max-theorem*, the case corresponding to the largest eigenvalue.

Lemma 2.2 [23] *If M is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then for any $\mathbf{x} \in R^n$ $(\mathbf{x} \neq \mathbf{0})$,*

$$\mathbf{x}^T M\mathbf{x} \leq \lambda_1 \mathbf{x}^T \mathbf{x}. \tag{2}$$

Equality holds if and only if $\mathbf{x}$ is an eigenvector of M corresponding to the largest eigenvalue λ_1.

The Laplacian matrix of graph G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the vertex degrees of graph G and $A(G)$ is the adjacency matrix of graph G. The eigenvalues of $L(G)$ will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. Denote by

$$LS(G) = (\mu_1,\, \mu_2, \ldots,\, \mu_{n-1},\, \mu_n),$$

the Laplacian spectrum of graph G. A lower bound on the Laplacian spectral radius is in the following:

Lemma 2.3 [17] *Let G be a graph on n vertices with at least one edge. Then*

$$\mu_1(G) \geq \Delta + 1. \tag{3}$$

Moreover, if G is connected, then the equality in (3) holds if and only if $\Delta = n - 1$.

The following result is the relation between the Laplacian eigenvalues of G and $\overline{G}$.

Lemma 2.4 [17] *Let G be a graph with Laplacian spectrum $LS(G) = (\mu_1,\, \mu_2, \ldots,\, \mu_n)$. Then the Laplacian spectrum of $\overline{G}$ is $LS(\overline{G}) = (n - \mu_{n-1},\, n - \mu_{n-2}, \ldots,\, n - \mu_2,\, n - \mu_1,\, 0)$, where $\overline{G}$ is the complement of the graph G.*

The next result is about the change in the spectrum, distance Laplacian and distance signless Laplacian, respectively, with respect to the same operation, that is, the edge removal.

Lemma 2.5 [1] *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph $\tilde{G}$ obtained from G by the deletion of an edge.*

i) Let $\left(\partial_1^L,\, \partial_2^L, \ldots,\, \partial_n^L\right)$ and $\left(\tilde{\partial}_1^L,\, \tilde{\partial}_2^L, \ldots,\, \tilde{\partial}_n^L\right)$ denote the distance Laplacian spectrum of G and $\tilde{G}$, respectively. Then $\tilde{\partial}_i^L \geq \partial_i^L$ for all $i = 1,\, 2, \ldots,\, n$.

ii) Let $\left(\partial_1^Q, \partial_2^Q, \ldots \partial_n^Q\right)$ and $\left(\tilde{\partial}_1^Q,\, \tilde{\partial}_2^Q, \ldots,\, \tilde{\partial}_n^Q\right)$ denote the distance signless Laplacian spectrum of G and $\tilde{G}$, respectively. Then $\tilde{\partial}_i^Q \geq \partial_i^Q$ for all $i = 1,\, 2, \ldots,\, n$.

The next result provides a lower bound on the second smallest distance Laplacian eigenvalue ∂_{n-1}^L. It also gives us the information about the connectedness of $\overline{G}$, the complement of G.

Lemma 2.6 [1] *Let G be a connected graph on n vertices. Then $\partial_{n-1}^L \geq n$ with equality holding if and only if $\overline{G}$ is disconnected. Furthermore, the multiplicity of n as an eigenvalue of $\mathcal{D}^L$ is one less than the number of components of $\overline{G}$.*

The following two results on the distance Laplacian and the distance signless Laplacian eigenvalues were proved in [3] and [2], respectively.

Lemma 2.7 [2, 3] *Let G be a graph on n vertices. Also let $S = \{v_1,\, v_2, \ldots,\, v_p\}$ be an independent set of G such that $N(v_i) = N(v_j)$ $(\partial = Tr(v_i) = Tr(v_j))$ for all $i,\, j \in \{1,\, 2, \ldots,\, p\}$. Then $\partial + 2$ and $\partial - 2$ are the eigenvalues with multiplicity at least $p - 1$ of $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$, respectively.*

Lemma 2.8 [2, 3] *Let G be a graph on n vertices. Also let $K = \{v_1,\, v_2, \ldots,\, v_p\}$ is a clique of G such that $N_G(v_i) - K = N_G(v_j) - K$ $(\partial = Tr(v_i) = Tr(v_j))$ for all $i,\, j \in \{1,\, 2, \ldots,\, p\}$. Then $\partial + 1$ and $\partial - 1$ with multiplicity at least $p - 1$ are the eigenvalues of $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$, respectively.*

The relation between Laplacian eigenvalues and distance Laplacian eigenvalues is given in the following result:

Lemma 2.9 [1] *Let G be a connected graph on n vertices with diameter $d \leq 2$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ be the Laplacian eigenvalues of G. Then the distance Laplacian eigenvalues of G are $2n - \mu_{n-1}(G) \geq 2n - \mu_{n-2}(G) \geq \ldots \geq 2n - \mu_1(G) > \partial_n^L(G) = 0$. Moreover, for every $i = 1, 2, \ldots, n-1$, the eigenspaces corresponding to μ_i and to $2n - \mu_i$ are the same.*

3 Distance Laplacian eigenvalues, domination number and independence number of graphs

This section is devoted to the study of relationships between the distance Laplacian eigenvalues and graph parameters, such as diameter, domination number and independence number, etc. First we give a lower bound on the largest distance Laplacian eigenvalue ∂_1^L in terms of the minimum transmission Tr_{min} and diameter d of the graph G. Denote by $K_{\underbrace{b, b, \ldots, b}_{c+1}}$, a complete $(c+1)$-partite graph of order n, where $n = b(c+1)$ and $b \geq 2$, $c \geq 1$. Therefore $K_{\underbrace{b, b, \ldots, b}_{c+1}} = \overline{(c+1)\, K_b}$. Now we want to find out the distance Laplacian spectral radius of graph $K_{\underbrace{b, b, \ldots, b}_{c+1}}$ $(n = b(c+1),\ c \geq 1)$.

Lemma 3.1 *Let $G \cong K_{\underbrace{b, b, \ldots, b}_{c+1}}$ be a graph of order $n = b(c+1)$ $(b \geq 2,\ c \geq 1)$. Then $\partial_1^L(G) = b(c+2)$.*

Proof. Since $G \cong K_{\underbrace{b, b, \ldots, b}_{c+1}}$ $(n = b(c+1))$, we have $\overline{G} \cong (c+1)\, K_b$. One can easily see that $\mu_1(\overline{G}) = b$, that is, $\mu_{n-1}(G) = bc$. Since $d(G) = 2$, by Lemma 2.9, we have $\partial_1^L(G) = bc + 2b$. □

We are now ready to give a lower bound on ∂_1^L.

Theorem 3.2 *Let G be a connected graph on $n \geq 2$ vertices with diameter d and minimum transmission Tr_{min}. Then*

$$\partial_1^L \geq Tr_{min} + d \tag{4}$$

with equality holding if and only if $G \cong K_n$ or $G \cong K_{\underbrace{b, b, \ldots, b}_{c+1}}$, where $n = b(c+1)$ and $b \geq 2$, $c \geq 1$.

Proof. For $d = 1$, we have $G \cong K_n$, $\partial_1^L = n = Tr_{min} + d$ and hence the equality holds in (4). Otherwise, $d \geq 2$. Let v_1 and v_n be two vertices in G such that $d_G(v_1, v_n) = d$ and consider the n-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ defined by

$$x_i = \begin{cases} 1 & \text{if } v_i = v_1, \\ -1 & \text{if } v_i = v_n, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2, we have

$$\partial_1^L \geq \frac{\mathbf{x}^T \mathcal{D}^L \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{Tr(v_1) + Tr(v_n)}{2} + d \geq Tr_{min} + d. \tag{5}$$

Suppose that equality holds in (4). Then all the inequalities in the above must be equalities. We have $Tr(v_1) = Tr(v_n) = Tr_{\min}$. Moreover, $\mathbf{x} = (1, 0, \ldots, 0, -1)^T$ is an eigenvector corresponding to an eigenvalue ∂_1^L of $\mathcal{D}^L$. Let v_2 be a vertex on the diametral path $v_1 v_2 \ldots v_n$ such that $v_1 v_2 \in E(G)$. For $v_2 \in V(G)$, we have $\partial_1^L \cdot 0 = Tr(v_2) \cdot 0 - 1 + d - 1$, that is, $d = 2$. Therefore $Tr_{\min} = Tr(v_1) > n-1$. Let v_s be any maximum degree vertex of degree Δ in G. Then $\Delta \leq n-1$. If $\Delta = n-1$, then

$$Tr_{\min} \leq Tr(v_s) = n-1 < Tr(v_1) = Tr_{\min}, \quad \text{a contradiction.}$$

Otherwise, $\Delta \leq n-2$. Let v_i be the minimum degree vertex of degree d_i in G. We now consider the following two cases:

Case 1: $d_i = n-2$. Then all the vertices are of degree $n-2$ in G as $\Delta \leq n-2$. Then G is $(n-2)$-regular graph and n is even. Hence $G \cong K_{\underbrace{2,\,2,\,\ldots,\,2}_{n/2}}$ (n is even).

Case 2: $d_i \leq n-3$. Since $d = 2$, then there are at least two vertices v_j and v_k, (say), in G such that $d_G(v_i,\, v_j) = 2$ and $d_G(v_i,\, v_k) = 2$. Let $S = \{v_p \in V(G) \,|\, d_G(v_i,\, v_p) = 2\}$. Then $|S| \geq 2$ as $v_j,\, v_k \in S$. Since $\partial_1^L = Tr_{min} + 2$, we must have $Tr(v_p) = Tr_{min}$, $v_p \in S$. If any two vertices in S are adjacent ($v_j v_k \in E(G)$), then we consider the principle submatrix corresponding to three vertices v_i, v_j and v_k of $\mathcal{D}^L(G)$:

$$B = \begin{pmatrix} Tr_{\min} & -2 & -2 \\ -2 & Tr_{\min} & -1 \\ -2 & -1 & Tr_{\min} \end{pmatrix}.$$

The characteristic equation of B is

$$f(x) = 0,$$

where

$$f(x) = x^3 - 3ax^2 + 3a^2x - 9x - a^3 + 9a + 8, \quad a = Tr_{\min}.$$

We have $f(x) \to +\infty$ as $x \to \infty$. Moreover, we have $f(a+2) = -2 < 0$, $f(a) = 8 > 0$ and $f(a-4) = -20 < 0$. Then from the above results, we conclude that $\partial_1^L(B) > a + 2 = Tr_{\min} + 2$.

By Lemma 2.1, we have $\partial_1^L(G) \geq \partial_1^L(B) > Tr_{\min} + 2$, a contradiction. Otherwise, the induced subgraph $G[S \cup \{v_i\}]$ is an empty set, that is, $E(S \cup \{v_i\}) = \emptyset$. From the definition of S, one can easily see that v_i is adjacent to all the vertices in $V(G) \backslash (S \cup \{v_i\})$. If any vertex v_j, (say), in S is not adjacent to vertex v_t, $v_t \in V(G) \backslash (S \cup \{v_i\})$, then again we consider the principle submatrix B corresponding three vertices v_j, v_i and v_t of $\mathcal{D}^L(G)$, that is, $\partial_1^L(G) \geq \partial_1^L(B) > Tr_{\min} + 2$, a contradiction. Otherwise, each vertex in $S \cup \{v_i\}$ is adjacent to all the vertices in $V(G) \backslash (S \cup \{v_i\})$. Therefore $K_{a,\,b}$ is a subgraph of G ($a + b = n$), where $a = |S \cup \{v_i\}|$ and $b = |V(G) \backslash (S \cup \{v_i\})|$. Since v_i is the minimum degree vertex in G and $v_i \in S \cup \{v_i\}$, so we must have $a \geq b$. If all the vertex transmissions are not the same in G, then by (5), $\partial_1^L(G) > Tr_{\min} + 2$, a contradiction. Otherwise, all the vertices have the same vertex transmission in G, that is, $Tr(v_1) = Tr(v_2) = \cdots = Tr(v_n) = a + 2b - 2$ as $a \geq b$. Hence $G \cong \overline{K}_b \vee H$, that is, $\overline{G} \cong K_b \cup \overline{H}$, where H is a $(a-b)$-regular graph of order a ($a(a-b)$ is divisible by 2). Then $\overline{H}$ is a $(b-1)$-regular graph. If any of the connected component in $\overline{H}$ contains at least $b+1$ vertices, then by Lemma 2.3, we get $\mu_1(\overline{H}) > b$, that is, $\mu_1(\overline{G}) > b$, that is, $\mu_{a+b-1}(G) < a$. Since $d(G) = 2$, by Lemma 2.9, $\partial_1^L(G) > a + 2b = Tr_{\min} + d$, a contradiction. Otherwise, each connected component in $\overline{H}$ contains at most b vertices. Since $\overline{H}$ is a $(b-1)$-regular graph, then each connected component contains exactly b vertices and $\overline{H} \cong c\,K_b$ (c is the number of connected components in $\overline{H}$), where $a = bc$, $c \geq 1$. Therefore $H \cong K_{\underbrace{b,\,\ldots,\,b}_{c}}$ and hence $G \cong \overline{K}_b \vee K_{\underbrace{b,\,b,\,\ldots,\,b}_{c}}$ ($a = bc$, $n = a + b$).

Since the minimum degree $d_i \leq n-3$, we have $G \cong K_{\underbrace{b,\,b,\,\ldots,\,b}_{c+1}}$, where $n = b(c+1)$ and $b \geq 3$, $c \geq 1$.

Conversely, let $G \cong K_n$. Then $\partial_1^L(G) = n = Tr_{min} + d$ holds.

Let $G \cong K_{\underbrace{b,\,b,\,\ldots,\,b}_{c+1}}$ ($n = b(c+1)$, $b \geq 2$, $c \geq 1$). Since $d = 2$ and by Lemma 3.1, we have $\partial_1^L(G) = b(c+2) = Tr_{min} + d$ holds. □

The next result consists of a lower bound on the largest distance Laplacian ∂_1^L in terms of the order n, domination number γ and diameter d, and the characterization of the extremal graphs.

Theorem 3.3 *Let G be a connected graph of order $n \geq 2$ with domination number γ and diameter d. Then $\partial_1^L \geq n + \gamma + d - 2$ with equality holding if and only if $G \cong K_n$ or $G \cong K_{\underbrace{2,2,\ldots,2}_{p}}$ $(n = 2p)$.*

Proof. Let v_i be a maximum degree vertex of degree Δ. Consider the vertex set $S = V(G) \setminus N_G(v_i)$. It is easy to see that S is a dominating set in G, and therefore $\gamma \leq n - \Delta$, that is, $\Delta \leq n - \gamma$. Let Tr_{min} denote the minimum transmission in G. It is easy to see that

$$Tr_{min} \geq \Delta + 2(n - \Delta - 1) = 2n - \Delta - 2 \geq 2n - n + \gamma - 2 = n + \gamma - 2.$$

By Theorem 3.2, we have

$$\partial_1^L \geq Tr_{min} + d \geq n + \gamma + d - 2.$$

Therefore the lower bound follows. By Theorem 3.2, the equality holds if and only if $G \cong K_n$ or $G \cong K_{\underbrace{2,2,\ldots,2}_{p}}$ $(n = 2p)$ as $\Delta = n - \gamma = 2n - Tr_{min} - 2$. □

The next result follows immediately from the above theorem and the fact that $d = 1$ if and only if the graph is complete.

Corollary 3.4 *Let G be a connected graph on n vertices with domination number γ. Then $\partial_1^L \geq n + \gamma - 1$ with equality if and only if G is the complete graph K_n.*

The next results shows that if the order n appears as distance Laplacian eigenvalue of a graph G, then its domination number is at most 2.

Proposition 3.5 *Let G be a connected graph of order n with domination number γ. If $\partial_{n-1}^L = n$, then $\gamma \leq 2$.*

Proof. If $\partial_{n-1}^L = n$, then by Lemma 2.6, $\overline{G}$ is disconnected. Consider two vertices u and v in G belonging to different components in $\overline{G}$. Thus $\{u, v\}$ is a dominating set in G, and therefore $\gamma \leq 2$. □

The converse of the above proposition is not true in general. For instance, Figure 1 shows complementary graphs on 5 vertices, both with domination number 2 and none of them has 5 as a distance Laplacian eigenvalue. However, we provide in the next result a sufficient condition in terms of the domination number for n to be a distance Laplacian eigenvalue. Denote by $\mu_G(n)$, the multiplicity of n as an eigenvalue of $\mathcal{D}^L(G)$.

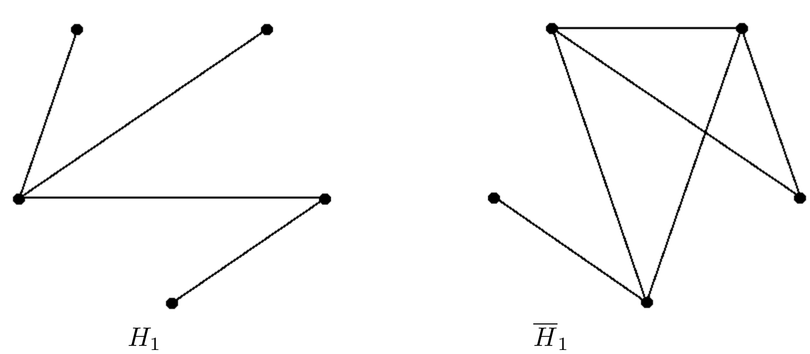


Figure 1: Two complementary graphs H_1 and $\overline{H}_1$ with $\gamma = 2$ for which $n = 5$ is not a distance Laplacian eigenvalue.

Proposition 3.6 *Let G be a connected graph of order n with domination number γ. If $\gamma = 1$, then $\partial^L_{n-1} = n$. Moreover, for any integer p with $1 \leq p \leq n-1$, there exists a connected graph H such that $\mu_H(n) = p$.*

Proof. If $\gamma = 1$, then $\overline{G}$ is disconnected, and thus $\partial^L_{n-1} = n$, by Lemma 2.6. We can construct a connected graph H whose complement contains $p+1$ ($1 \leq p \leq n-1$) connected components among which at least one is an isolated vertex. Since 0 is a Laplacian eigenvalue with multiplicity $p+1$ of $\overline{H}$, by Lemma 2.4, we have that n is a Laplacian eigenvalue with multiplicity p of H. Since $d(H) \leq 2$, by Lemma 2.9 with the above result, we have $\mu_H(n) = p$. □

The next result provides a sharp upper bound on the multiplicity of n in the distance Laplacian spectrum when the domination number of the graph is 2.

Proposition 3.7 *Let G be a connected graph on n vertices with domination number γ and distance Laplacian spectrum $\partial^L_1 \geq \partial^L_2 \geq \cdots \geq \partial^L_n = 0$. If $\gamma = 2$, then $\mu_G(n) \leq \lfloor n/2 \rfloor - 1$. Moreover, for any integer p with $0 \leq p \leq \lfloor n/2 \rfloor - 1$, there exists a connected graph H such that $\mu_H(n) = p$.*

Proof. Since $\gamma = 2$, $\overline{G}$ contains no isolated vertices, and therefore any connected component in $\overline{G}$ must contain at least 2 vertices. Thus $\overline{G}$ contains at most $\lfloor n/2 \rfloor$ components and $\mu_G(n) \leq \lfloor n/2 \rfloor - 1$, by Lemma 2.6. If $0 \leq p \leq \lfloor n/2 \rfloor - 1$, then similarly by the proof of the Proposition 3.6, we can construct a connected graph H, whose complement $\overline{H}$ contains $p+1$ connected components with at least 2 vertices each, satisfies $\mu_H(n) = p$ and $\gamma(H) = 2$. □

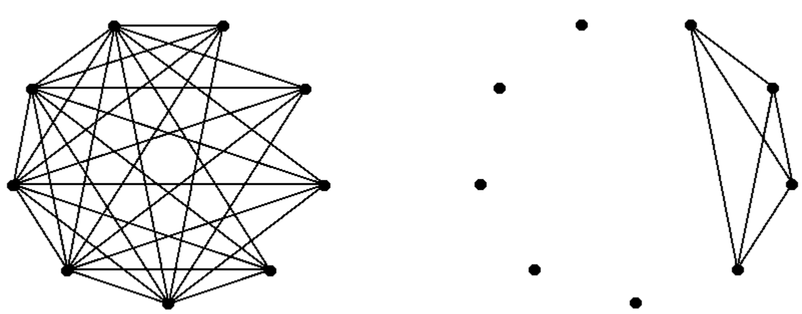
We need the following definition for the rest of the section. For two integers n and α such that $1 \leq \alpha \leq n-1$, *the complete split graph* $CS_{n,n-\alpha}$ is the join graph obtained from a clique $K_{n-\alpha}$ and an empty graph $\overline{K}_\alpha$ (see, Figure 2 for $CS_{9,5}$). Its distance Laplacian spectrum is as follows [3]:

$$DLS(CS_{n,\,n-\alpha}) = \Big(\underbrace{n+\alpha, \ldots, n+\alpha}_{\alpha-1}, \underbrace{n, \ldots, n}_{n-\alpha}, 0\Big). \tag{6}$$

We next give a lower bound on the largest distance Laplacian eigenvalue ∂^L_1 in terms of the order n and the independence number α.

Theorem 3.8 *Let G be a connected graph of order $n \geq 3$ with distance Laplacian spectral radius ∂^L_1 and independence number $\alpha \geq 2$. Then $\partial^L_1(G) \geq n + \alpha$. The bound is best possible as shown by the complete split graph $CS_{n,\,n-\alpha}$.*

Proof. Since G has n vertices and independence number α, we have $G \subseteq CS_{n,\,n-\alpha}$ ("$\subseteq$" denote subgraph). Since $\alpha \geq 2$, by Lemma 2.5, we have $\partial^L_1(G) \geq \partial^L_1(CS_{n,\,n-\alpha}) = n + \alpha$, by (6). Then the result follows. □



$CS_{9,5}$ $\overline{CS}_{9,5}$

Figure 2: Graphs $CS_{9,5}$ and $\overline{CS}_{9,5}$.

Denote by $m_G[0,\, a)$ the number of eigenvalues of $\mathcal{D}^L(G)$, multiplicities included, that belong to the interval $[0,\, a)$. As an immediate consequence of the above theorem, we state the following corollary.

Corollary 3.9 *Let G be a connected graph on $n \geq 3$ with independence number α. Then $\mu_G[0,\, n+\alpha) = n$ if and only if G is the complete graph K_n.*

Proof. If $\alpha = 1$, then $G \cong K_n$ and hence $\partial_1^L = \cdots = \partial_{n-1}^L = n$, $\partial_n^L = 0$. Thus $\mu_G[0, n+\alpha) = n$. Otherwise, $\alpha \geq 2$. By Theorem 3.8, we have $\partial_1^L(G) \geq n+\alpha$ and hence $\mu_G[0,\, n+\alpha) \leq n-1$. This completes the proof of the result. □

The next result characterize of all the graphs when $\mu_G[0,\, n+\alpha) = n-1$.

Corollary 3.10 *Let G ($\ncong K_n$) be a connected graph of order $n \geq 3$ with independence number α. Then $\mu_G[0, n+\alpha) \leq n-1$ with equality if and only if $G \cong K_n - S_p$, where S_p denotes a set of p edges with one common end vertex with $1 \leq p \leq n-1$.*

Proof. Since $G \ncong K_n$, we have $\alpha \geq 2$. First we assume that $G \cong K_n - S_p$ with $1 \leq p \leq n-1$. Thus we have $\alpha(G) = 2$. Since $\overline{G} = K_{1,\,p} \cup (n-p-1)\, K_1$, one can easily see that

$$LS(\overline{G}) = \{p+1,\, \underbrace{1, \ldots,\, 1}_{p-1},\, \underbrace{0, \ldots,\, 0}_{n-p}\}.$$

By Lemma 2.4, we have

$$LS(G) = \{\underbrace{n, \ldots,\, n}_{n-p-1},\, \underbrace{n-1, \ldots,\, n-1}_{p-1},\, n-p-1, 0\}.$$

Since G has diameter 2, by Lemma 2.9, we get

$$DLS(G) = \{n+p+1,\, \underbrace{n+1, \ldots,\, n+1}_{p-1},\, \underbrace{n, \ldots,\, n}_{n-p-1},\, 0\}.$$

Since $\alpha(G) = 2$ and $p \geq 1$, we have $\mu_G[0, n+\alpha) = n-1$ holds.

Next we assume that $G \ncong K_n - S_p$ $(1 \leq p \leq n-1)$. Then either $\alpha \geq 3$ or $\alpha = 2$ and $\overline{G}$ contains at least two disjoint edges.

Case 1: $\alpha \geq 3$. G is a subgraph (not necessarily induced) of the complete split graph $CS_{n,n-\alpha}$ with $n+\alpha$ as a distance Laplacian eigenvalue of multiplicity at least 2. Using Lemma 2.5, G has at least two distance eigenvalues greater than or equal to $n+\alpha$. Therefore $\mu_G[0,\, n+\alpha) \leq n-2$.

Case 2: $\alpha = 2$ and $\overline{G}$ contains at least two disjoint edges, say, e_1 and e_2. In this case G is a subgraph (not necessarily induced) of $H \cong K_n - \{e_1, e_2\}$. By Lemma 2.7, one can easily see that $n+2 = n+\alpha$ is a distance Laplacian eigenvalue of H with multiplicity at least 2. By Lemma 2.5, G contains at least two distance Laplacian eigenvalues greater than or equal to $n+\alpha$. Therefore $\mu_G[0,\, n+\alpha) \leq n-2$. This completes the proof of the result. □

The above two results can be generalized into the following proposition. By Lemmma 2.5 with (6), we get the following result:

Proposition 3.11 *Let G be a connected graph of order $n \geq 3$ with independence number α. Then $\mu_G[0, n+\alpha) \leq n-\alpha+1$. Moreover, the bound is best possible as shown by the complete split graph $CS_{n,n-\alpha}$.*

The following result provides an upper bound on the multiplicity of the order n as a distance Laplacian eigenvalue in terms of the order and the independence number α. The corresponding extremal graph is also characterized.

Theorem 3.12 *Let G be a connected graph of order $n \geq 3$ with independence number α. Then $\mu_G(n) \leq n-\alpha$ with equality if and only if $G \cong CS_{n-\alpha,\,\alpha}$.*

Proof. Since G is a connected graph of order n with independence number α, then G is a subgraph of the complete split graph $CS_{n,n-\alpha}$. By Lemma 2.5 with (6), we get $\mu_G(n) \leq \mu_{CS_{n,n-\alpha}}(n) = n-\alpha$. The first part of the proof is done.

If $G \cong CS_{n,n-\alpha}$, then the equality holds, by (6). Otherwise, $G \ncong CS_{n,n-\alpha}$, that is, $G \subseteq CS_{n,n-\alpha} - e$, where e is any edge in $CS_{n,n-\alpha}$. Let $CS_{n,\,n-\alpha} - e = G_1$ or G_2, where

$$G_1 = \overline{K_\alpha \cup K_2 \cup (n-\alpha-2)\,K_1}, \quad G_2 = \overline{\overline{K_{1,\,\alpha-1} \cup K_1} \cup (n-\alpha-1)\,K_1}.$$

By Lemma 2.4, we have

$$LS(G_1) = \{\underbrace{n, \ldots, n}_{n-\alpha-1},\, n-2,\, \underbrace{n-\alpha, \ldots, n-\alpha}_{\alpha-1},\, 0\}$$

and

$$LS(G_2) = \{\underbrace{n, \ldots, n}_{n-\alpha-1},\, \underbrace{n-\alpha, \ldots, n-\alpha}_{\alpha-1},\, n-\alpha-1,\, 0\}.$$

Both G_1 and G_2 have diameter 2. Then by Lemma 2.9, we have

$$DLS(G_1) = \{\underbrace{n+\alpha, \ldots, n+\alpha}_{\alpha-1},\, n+2,\, \underbrace{n, \ldots, n}_{n-\alpha-1},\, 0\}$$

and

$$DLS(G_2) = \{n+\alpha+1,\, \underbrace{n+\alpha, \ldots, n+\alpha}_{\alpha-1},\, \underbrace{n, \ldots, n}_{n-\alpha-1},\, 0\}.$$

Therefore $\mu_G(n) \leq m_{G_i}(n) = n-\alpha-1$, $i = 1,\, 2$. This completes the proof of the result. □

From the proof of the above theorem, we obtain a lower bound on the α^{th} largest distance Laplacian eigenvalue ∂^L_α as follows:

Proposition 3.13 *Let G be a connected graph of order $n \geq 3$ with independence number α. Then $\partial^L_\alpha \geq n$ with equality if and only if $G \cong CS_{n-\alpha,\,\alpha}$.*

We now present a lower bound on the $(\alpha-1)^{th}$ largest distance Laplacian eigenvalue $\partial^L_{\alpha-1}$.

Proposition 3.14 *Let G be a connected graph on $n \geq 3$ with independence number $\alpha \geq 2$. Then $\partial^L_{\alpha-1} \geq n+\alpha$, and the bound is best possible as shown by the complete split graph $CS_{n-\alpha,\,\alpha}$.*

Proof. Since G is a graph on n vertices with independence number α, then G is a subgraph of the complete split graph $CS_{n,\,n-\alpha}$. Then by Lemma 2.5, we have $\partial^L_i(G) \geq \partial^L_i(CS_{n,n-\alpha}) = n+\alpha$, $i = 1,\, 2, \ldots,\, \alpha-1$. This completes the proof. □

4 On the distance signless Laplacian eigenvalues of graph

In this section, we mainly focus on the study of relationships between the independence number α of a connected graph G and its distance signless Laplacian eigenvalues. Among other properties, we show connections between the frequency of $n-2$ in the distance signless Laplacian spectrum and the independence number α. Recall that (see, [2]) for a connected graph on n vertices, $n-2$ is the smallest possible value that could belong to the distance signless Laplacian spectrum of G.

As for the distance Laplacian spectrum, the family of complete split graphs plays an import role as extremal graphs in our results in the present section. Therefore, we start by recalling distance signless Laplacian spectrum of a complete split graph $CS_{n,\,n-\alpha}$:

$$DSLS(CS_{n,\,n-\alpha}) = \Big(q_1,\, q_2, \underbrace{n+\alpha-4, \ldots,\, n+\alpha-4}_{\alpha-1}, \underbrace{n-2, \ldots,\, n-2}_{n-\alpha-1}\Big), \tag{7}$$

where q_1 and q_2 are given by

$$q_1,\, q_2 = \frac{3n+2\alpha-6 \pm \sqrt{n^2+12\,\alpha^2-4n\alpha+4n-16\alpha+4}}{2}\,.$$

Chung [6] proved that $\alpha \geq \bar{l}$ ($\bar{l}$ denotes the average distance). Using the distance signless Laplacian eigenvalues, the result of Chung reads $\alpha \geq \frac{\overline{\partial}^Q}{n-1}$, where $\overline{\partial}^Q$ denote the average of the distance signless Laplacian eigenvalues. Thus comparing between α and $\frac{\partial_i^Q}{n-1}$, for different values of i, would be interesting. First we compare between α and $\frac{\partial_1^Q}{n-1}$, and then α and $\frac{\partial_n^Q}{n-1}$.

Theorem 4.1 *Let G be a connected graph on $n \geq 3$ with independence number α and distance signless Laplacian spectrum $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q \geq 0$. Then*

i)

$$\alpha - \frac{\partial_1^Q}{n-1} \leq \frac{2n^2-9n+10-\sqrt{9n^2-32n+32}}{2(n-1)} \tag{8}$$

with equality holding if and only if $G \cong K_{1,n-1}$;

ii)

$$\alpha - \frac{\partial_n^Q}{n-1} \geq \frac{1}{n-1} \tag{9}$$

with equality holding if and only if $G \cong K_n$;

iii)

$$\alpha - \frac{\partial_n^Q}{n-1} \leq \frac{2n^2-9n+10+\sqrt{9n^2-32n+32}}{2(n-1)} \tag{10}$$

with equality holding if and only if $G \cong K_{1,n-1}$.

Proof. *i*) Let $\mathbf{x} = (x_1,\, x_2, \ldots,\, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\partial_1^Q(G)$ of $\mathcal{D}^Q(G)$. Then we have

$$\mathcal{D}^Q(G)\mathbf{x} = \partial_1^Q(G)\mathbf{x}. \tag{11}$$

Let S be the independent set of graph G such that $|S| = \alpha$. We can assume that x_i and x_j are the eigencomponents of the eigenvector $\mathbf{x}$ such that

$$x_i = \min_{v_k\in S} x_k\,, \quad \text{and} \quad x_j = \min_{v_k\in V(G)\setminus S} x_k. \tag{12}$$

For $v_i \in S$, using (12), from (11), we have

$$\partial_1^Q(G)\, x_i = D_i\, x_i + \sum_{v_k\in V(G)} d_{ik}\, x_k,$$

that is,

$$\partial_1^Q(G)\, x_i \geq (n+\alpha-2)\, x_i + 2(\alpha-1)x_i + (n-\alpha)x_j,$$

therefore

$$(\partial_1^Q(G) - n - 3\alpha + 4)\, x_i \geq (n-\alpha)\, x_j. \tag{13}$$

For $v_j \in V(G)$, using (12), from (11), we have

$$\partial_1^Q(G)\, x_j = D_j\, x_j + \sum_{v_k \in V(G)} d_{jk}\, x_k\,,$$

that is,

$$\partial_1^Q(G)\, x_j \geq (n-1)\, x_j + \alpha\, x_i + (n-\alpha-1)\, x_j,$$

therefore

$$(\partial_1^Q(G) - 2n + \alpha + 2)\, x_j \geq \alpha\, x_i. \tag{14}$$

From (13) and (14), we get

$$(\partial_1^Q(G) - n - 3\alpha + 4)\,(\partial_1^Q(G) - 2n + \alpha + 2) \geq (n-\alpha)\,\alpha,$$

that is,

$$\partial_1^Q(G)^2 - (3n + 2\alpha - 6)\partial_1^Q(G) + 4n\,\alpha + 2n^2 - 2\alpha^2 - 2\alpha - 10n + 8 \geq 0,$$

that is,

$$\partial_1^Q(G) \geq \frac{3n + 2\alpha - 6 + \sqrt{(3n+2\alpha-6)^2 - 4(4n\alpha + 2n^2 - 2\alpha^2 - 2\alpha - 10n + 8)}}{2}\,,$$

that is,

$$\alpha - \frac{\partial_1^Q(G)}{n-1} \leq \alpha - \frac{3n + 2\alpha - 6 + \sqrt{n^2 + 12\,\alpha^2 - 4n\alpha + 4n - 16\alpha + 4}}{2(n-1)}\,. \tag{15}$$

Let us consider a function

$$f(x) = x - \frac{3n + 2x - 6 + \sqrt{n^2 + 12\,x^2 - 4nx + 4n - 16x + 4}}{2(n-1)} \quad 1 \leq x \leq n-1.$$

Then one can easily see that $f(x)$ is an increasing function on $1 \leq x \leq n-1$. Hence

$$f(x) \leq f(n-1) = n - 1 - \frac{5n - 8 + \sqrt{9n^2 - 32n + 32}}{2(n-1)}.$$

Using the above result with (15), we get the required result in (8). The first part of the proof is done.

Suppose that equality holds in (8). Then all the above inequalities must be equalities. From the equality in (13), we have

$$x_k = x_i \quad \text{for all } v_k \in S \quad \text{and} \quad x_k = x_j \quad \text{for all } v_k \in V(G)\backslash S.$$

Moreover, we have $\alpha = n - 1$. Hence $G \cong K_{1,n-1}$.

Conversely, one can easily see that the equality holds in (8) for $K_{1,n-1}$, by (7) (for $\alpha = n-1$).

ii) Similarly, the proof of Case (i), we get

$$\partial_n^Q(G) \leq \frac{3n + 2\alpha - 6 - \sqrt{(3n+2\alpha-6)^2 - 4(4n\alpha + 2n^2 - 2\alpha^2 - 2\alpha - 10n + 8)}}{2}\,,$$

that is,

$$\alpha - \frac{\partial_n^Q(G)}{n-1} \geq \alpha - \frac{3n + 2\alpha - 6 - \sqrt{n^2 + 12\,\alpha^2 - 4n\alpha + 4n - 16\alpha + 4}}{2(n-1)}\,. \tag{16}$$

Since

$$g(x) = x - \frac{3n + 2x - 6 - \sqrt{n^2 + 12\,x^2 - 4nx + 4n - 16x + 4}}{2(n-1)},$$

is an increasing function on $1 \le x \le n-1$, therefore

$$g(x) \ge g(1) = 1 - \frac{n-2}{n-1} = \frac{1}{n-1}.$$

Hence we get the required result in (9). Moreover, the equality holds in (9) if and only if $G \cong K_n$ $(\alpha = 1)$.

iii) Since G has order n and independence number α, by Lemma 2.5, we have

$$\partial_n^Q(G) \ge \partial_n^Q(CS_{n,\alpha}) = \frac{3n + 2\alpha - 6 - \sqrt{n^2 + 12\,\alpha^2 - 4n\alpha + 4n - 16\alpha + 4}}{2}.$$

Hence

$$\begin{aligned} \alpha - \frac{\partial_n^Q}{n-1} &\le \alpha - \frac{3n + 2\alpha - 6 - \sqrt{n^2 + 12\,\alpha^2 - 4n\alpha + 4n - 16\alpha + 4}}{2(n-1)} \\ &\le n - 1 - \frac{5n - 8 - \sqrt{9n^2 - 32n + 32}}{2(n-1)}. \end{aligned}$$

This gives the required result in (10). Moreover, the equality holds in (10) if and only if $G \cong K_{1,\,n-1}$ $(\alpha = n-1)$. □

To prove our next result, we need the distance signless Laplacian spectrum of the complete bipartite graph $K_{a,\,b}$, given in [1, 2]. It is as follows:

$$DSLS(K_{a,\,b}) = \Big(q_1,\, q_2, \underbrace{2n-b-4, \ldots, 2n-b-4}_{a-1}, \underbrace{2n-a-4, \ldots, 2n-a-4}_{b-1}\Big), \tag{17}$$

where q_1 and q_2 are given by

$$q_1,\, q_2 = \frac{5n - 8 \pm \sqrt{9\,(a-b)^2 + 4ab}}{2}.$$

Now we state a lemma that will be used in the proof of our main result regarding the multiplicity of $n-2$ as a distance signless Laplacian eigenvalue.

Lemma 4.2 *Let $G \cong K_{n-2,\,2} \backslash e$ $(n \ge 4)$, where e is an edge in $K_{n-2,\,2}$(see, Figure 3). Then $\mu_Q(n-2) \le 1$ with equality holding if and only if $G \cong P_4$.*

$DSLS(K_{4,2}) = (15.12, 6.87, 6, 6, 6, 4)$, $\quad DSLS(K_{4,2} - e) = (16.56, 8.37, 6.61, 6, 6, 4.44)$

Figure 3: Two graphs $K_{4,2}$ and $K_{4,2} - e$ with their distance signless Laplacian sepectra.

Proof. For $n \le 6$, by Mathematica [20], one can easily check that $\mu_Q(n-2) \le 1$ with equality holding if and only if $G \cong P_4$. Otherwise, $n \ge 7$. By Lemma 2.7, the distance signless Laplacian eigenvalues of G are

$$\underbrace{2n-6, \ldots, 2n-6}_{n-4}$$

and the remaining four eigenvalues satisfy the following:

$$\phi(x) = 0,$$

where

$$\phi(x) = x^4 - 8nx^3 + 12x^3 + 21n^2x^2 - 52nx^2 - 22n^3x + 64n^2x + 62nx - 192x + 8n^4 - 24n^3 - 72n^2 + 280n - 192.$$

We have $\phi(x) \to +\infty$ as $x \to \infty$. Since $n \geq 7$, we have $\phi(2n) = -8n^3 + 52n^2 - 104n - 192 < 0$, $\phi(n+3) = 5n^2 + 22n - 363 > 0$, $\phi(n-1) = -3n^2 + 14n - 11 < 0$ and $\phi(n-2) = 10n^2 - 68n + 112 > 0$. Then from the above results, we conclude that all the roots of $\phi(x) = 0$ are greater than $n-2$ for $n \geq 7$ and hence $\mu_Q(n-2) < 1$. This completes the proof of the result. □

Denote by $\mu_Q(n-2)$, the multiplicity of $n-2$ as an eigenvalue of $\mathcal{D}^Q(G)$. We now give an upper bound on the multiplicity of the distance signless Laplacian eigenvalue $n-2$ of graph G in terms of order n and independence number α.

Theorem 4.3 *Let G be a connected graph of order $n \geq 4$ with independence number α. For $\alpha = 1$, $\mu_Q(n-2) = n-1$ and for $\alpha = 2$, $\mu_Q(n-2) \leq n-2$ with equality holding if and only if $G \cong CS_{n,\,n-2}$. Otherwise, $\mu_Q(n-2) \leq n-\alpha-1$ with equality holding if and only if $G \cong CS_{n,\,n-\alpha}$ or $G \cong K_{2,\,n-2}$ or $G \cong \overline{P_3 \cup 2K_1}$.*

Proof. For $\alpha = 1$, we have $G \cong K_n$ and hence $\mu_Q(n-2) = n-1$. For $\alpha = n-1$, we have $G \cong K_{1,\,n-1}$ ($= CS_{n,\,1}$) and hence the equality holds, by (7). Otherwise, $2 \leq \alpha \leq n-2$. Since G is a connected graph of order n with independence number α, we have G is a subgraph of $CS_{n,\,n-\alpha}$, that is, $G \subseteq CS_{n,\,n-\alpha}$. For $G \cong CS_{n,\,n-\alpha}$, by (7), one can easily see that $\mu_Q(n-2) = n-2$ ($\alpha = 2$) and $\mu_Q(n-2) = n-\alpha-1$ ($\alpha \geq 3$) and hence the equality holds. Otherwise, $G \subseteq CS_{n,\,n-\alpha} \setminus \{e\}$ and hence by Lemma 2.5, we have $\partial_i^Q(G) \geq \partial_i^Q(CS_{n,\,n-\alpha} \setminus \{e\})$ $(1 \leq i \leq n)$, where $e = v_iv_j$ is an edge in $CS_{n,\,n-\alpha}$. For $n = 4$, one can easily check that the result holds. Thus we have to prove our result for $2 \leq \alpha \leq n-2$ and $n \geq 5$.

Let S be the independent set of vertices of a complete split graph $CS(n,\, n-\alpha)$. Then $|S| = \alpha$. Since $e = v_iv_j$ is an edge in $CS_{n,\,n-\alpha}$, then we consider the following two cases:

Case 1: There is an edge $e = v_iv_j$ with $v_i,\, v_j \in V(CS_{n,\,n-\alpha}) \setminus S$ in $CS(n, n-\alpha)$. By Lemmas 2.7 and 2.8, we conclude that the distance signless Laplacian eigenvalues of $CS_{n,\,n-\alpha} \setminus \{e\}$ are

$$\underbrace{n+\alpha-4, \ldots, n+\alpha-4}_{\alpha-1},\ \underbrace{n-2, \ldots, n-2}_{n-\alpha-2}$$

and the remaining eigenvalues satisfy the following:

$$\begin{aligned}
\partial^Q x_1 &= (n+3\alpha-4)x_1 + 2x_2 + (n-\alpha-2)x_3 \\
\partial^Q x_2 &= \alpha x_1 + (n+2)x_2 + (n-\alpha-2)x_3 \\
\partial^Q x_3 &= \alpha x_1 + 2x_2 + (2n-\alpha-4)x_3.
\end{aligned}$$

Therefore the remaining three distance signless Laplacian eigenvalues of $CS_{n,\,n-\alpha}\setminus\{e\}$ satisfy the following equation:

$$f(x) = 0,$$

where

$$\begin{aligned}
f(x) = x^3 - 4nx^2 - 2\alpha x^2 + 6x^2 + 5n^2x + 6\alpha nx - 16nx - 2\alpha^2x - 2\alpha x \\
+ 4x - 2n^3 - 4\alpha n^2 + 10n^2 + 2\alpha^2 n + 2\alpha n - 4n + 8\alpha - 16.
\end{aligned}$$

Subcase 1.1: $\alpha = n-2$. In this subcase we have $CS_{n,\,n-\alpha} \setminus \{e\} \cong K_{n-2,\,2}$ and hence $\mu_Q(n-2) = n-\alpha-1$, by (17). Moreover, for $G \subseteq K_{n-2,\,2}\setminus e$ (e is any edge in $K_{n-2,\,2}$), by Lemmas 2.5 and 4.2, we have $\mu_Q(n-2) < n-\alpha-1$ as $n \geq 5$.

Subcase 1.2: $2 \leq \alpha \leq n-3$. We have $f(x) \to +\infty$ as $x \to \infty$. Moreover, we have $f(2n-2) = -2\alpha^2 n + 2\alpha n - 4n + 4\alpha^2 + 4\alpha - 8 = -2\left[\alpha\left(\alpha(n-2) - n - 2\right) + 2n + 4\right] \leq -8(n-2) < 0$ as $\alpha \geq 2$, $f(n) = 8(\alpha - 2) \geq 0$ and $f(n-2) = -4\alpha n + 4n + 4\alpha^2 + 4\alpha - 8 = -4(\alpha-1)(n-\alpha-2) < 0$ as $n - \alpha \geq 3$. Let ∂_i^Q $(i = 1, 2, 3)$ be the roots of $f(x) = 0$. Then from the above results with Lemma 2.5, we conclude that $\partial_i^Q(G) \geq \partial_i^Q(CS_{n,\,n-\alpha} \setminus \{e\}) = \partial_i^Q > n-2$, $i = 1, 2, 3$. Hence $\mu_Q(n-2) < n-2$ for $\alpha = 2$ and $\mu_Q(n-2) < n - \alpha - 1$ for $3 \leq \alpha \leq n-3$. The inequality is strictly holds.

Case 2: There is an edge $e = v_i v_j$ with $v_i \in S$, $v_j \in V \setminus S$ in $CS(n, n-\alpha)$. By Lemmas 2.7 and 2.8, we conclude that the distance signless Laplacian eigenvalues of $CS_{n,\,n-\alpha} \setminus \{e\}$ are

$$\underbrace{n+\alpha-4, \ldots, n+\alpha-4}_{\alpha-2},\ \underbrace{n-2, \ldots, n-2}_{n-\alpha-2}$$

and the remaining eigenvalues satisfy the following:

$$\partial^Q x_1 = (n + 3\alpha - 6)x_1 + 2x_2 + (n - \alpha - 1)x_3 + x_4 \tag{18}$$
$$\partial^Q x_2 = (2\alpha - 2)x_1 + (n + \alpha - 1)x_2 + (n - \alpha - 1)x_3 + 2x_4 \tag{19}$$
$$\partial^Q x_3 = (\alpha - 1)x_1 + x_2 + (2n - \alpha - 3)x_3 + x_4 \tag{20}$$
$$\partial^Q x_4 = (\alpha - 1)x_1 + 2x_2 + (n - \alpha - 1)x_3 + nx_4. \tag{21}$$

Thus the remaining four distance signless Laplacian eigenvalues of $CS_{n,\,n-\alpha} \setminus \{e\}$ satisfy the following equation:

$$h(x) = 0,$$

where

$$\begin{aligned} h(x) \quad = \quad & x^4 - 5nx^3 - 3\alpha x^3 + 10x^3 + 9n^2x^2 + 12\alpha n x^2 - 38nx^2 - 17\alpha x^2 + 29x^2 - 7n^3x \\ & -15\alpha n^2 x + 46n^2 x - 2\alpha^2 n x + 47\alpha n x - 75nx + 2\alpha^3 x - 4\alpha^2 x - 22\alpha x + 16x \\ & +2n^4 + 6\alpha n^3 - 18n^3 + 2\alpha^2 n^2 - 30\alpha n^2 + 46n^2 - 2\alpha^3 n + 2\alpha^2 n + 30\alpha n - 22n \\ & +2\alpha^3 - 4\alpha^2 + 2\alpha - 24. \end{aligned}$$

Subcase 2.1: $\alpha = 2$. For $n = 5$, by Mathematica [20], we have $\mu_Q(n-2) \leq n - \alpha - 1$ with equality holding if and only if $G \cong \overline{P_3 \cup 2K_1}$. For $n \geq 6$, one can easily see that the four roots of the equation $h(x) = 0$ are as follows:

$$n-2,\ n-1,\ \text{one is in } (n-0.9,\ n+3),\ \text{and the other one is in } (n+3,\ \infty).$$

Hence $\mu_Q(n-2) < n-2$ and the inequality is strictly holds.

Subcase 2.2: $3 \leq \alpha \leq n-2$. We have $h(x) \to +\infty$ as $x \to \infty$. Moreover, we have $h(n + 2\alpha - 1) = -\alpha^2(6n-26) - 4\alpha^3(\alpha-2) - 10\alpha\, n + 4n - 22\alpha - 20 < 0$ as $\alpha \geq 3$, $h(n+\alpha-3) = 0$, $h(n-1) = -2(\alpha-2)(n-5) < 0$ as $3 \leq \alpha \leq n-2$, and $h(n-2) = 2\alpha\Big[\alpha(n-\alpha) - 3n\Big] + 4n + 4\alpha^2 + 2\alpha - 4 > 0$ (see Appendix). Moreover,

$$h(0) = 2n^4 + 6\alpha n^3 - 18n^3 + 2\alpha^2 n^2 - 30\alpha n^2 + 46n^2 - 2\alpha^3 n + 2\alpha^2 n + 30\alpha n - 22n + 2\alpha^3 - 4\alpha^2 + 2\alpha - 24 \quad > \quad 0.$$

From the above discussion, we conclude that the remaining four distance signless Laplacian eigenvalues of $CS_{n,\,n-\alpha} \setminus \{e\}$ are as follows:

one eigenvalue lies in $(n+2\alpha-1,\ \infty)$, one eigenvalue is exactly $n+\alpha-3$, one eigenvalue lies in $(n-1,\ n-2)$. Since $h(n-2) > 0$ and $h(0) > 0$, then there is one remaining eigenvalue can not be in $(0,\ n-2)$. Hence we conclude that all the four roots of $h(x) = 0$ are strictly greater than $n-2$. Therefore we have $\mu_Q(n-2) < n - \alpha - 1$ and the inequality is strictly holds. This completes the proof of the theorem. □

Appendix

Lemma 4.4 *For $3 \leq x \leq n-2$, we have* $2x\Big[x(n-x)-3n\Big]+4n+4x^2+2x-4>0.$

Proof. Let us consider a function

$$f(x)=2x\Big[x(n-x)-3n\Big]+4n+4x^2+2x-4 \quad \text{for} \quad 3 \leq x \leq n-2.$$

Then $f'(x)=4xn-6n-6x^2+8x+2$. Therefore $f(x)$ is an increasing function on $[3, \frac{n+2+\sqrt{n^2-5n+7}}{3}]$ and decreasing on $[\frac{n+2+\sqrt{n^2-5n+7}}{3}, n-2]$. Hence

$$f(x) \geq \min\{f(3),\, f(n-2)\} > 0$$

as

$$f(3)=4(n-4) \quad \text{and} \quad f(n-2)=2(n^2-7n+12).$$

□

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