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# An optimal execution problem in finance with acquisition and liquidation objectives: An MFG formulation

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**Abstract:** Partially observed major minor LQG mean field game theory is applied to an optimal execution problem in finance; following standard financial models, controlled linear system dynamics are postulated where an institutional investor (interpreted as a major agent) in the market aims to liquidate a specific amount of shares and has partial observations of its own state (which includes its inventory). Furthermore, the market is assumed to have two populations of high frequency traders (interpreted as minor agents) who wish to liquidate or acquire a certain number of shares within a specific time, and each one of them has partial observations of its own state and the major agent's state (which include the corresponding inventories). The objective for each agent is to maximize its own wealth and to avoid the occurrence of large execution prices, large rates of trading and large trading accelerations which are appropriately weighted in the agent's performance function. The existence of  $\epsilon$ -Nash equilibria together with the individual agents' trading strategies yielding the equilibria, were established. A simulation example is provided.

**Keywords:** MFG, finance, optimal execution problem, liquidation, acquisition, LQG

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## 1 Introduction

Partially observed Mean Field Game (PO MFG) theory was introduced and developed in Caines and Kizilkale (2013, 2014, 2016); Şen and Caines (2014, 2015) where it is assumed the major agent's state is partially observed by each minor agent, and the major agent completely observes its own state. Accordingly, each minor agent can recursively estimate the major agent's state, compute the system's mean field and thence generate the feedback control which yields the  $\epsilon$ -Nash property. This PO MFG theory was further extended in the work of Firoozi and Caines (2015) to major-minor LQG systems in which both the major agent and the minor agents partially observe the major agent's state. The existence of  $\epsilon$ -Nash equilibria, together with the individual agents' control laws yielding the equilibria, were established wherein each minor agent recursively generates (i) an estimate of the major agent's state, and (ii) an estimate of the major agent's estimate of its own state (in order to estimate the major agent's control feedback), and hence generates a version of the system's mean field. It is to be noted that the case where each agent has only partial observations on its own state was addressed in the LQG case in Huang et al. (2006) and in the nonlinear case in Şen and Caines (2016a,b).

Optimal execution problems have been addressed in the literature (see e.g. Cartea et al. (2015); Jaimungal and Kinzebulatov (2014); Almgren and Chriss (2001); Alfonsi et al. (2010)) where an agent must liquidate or acquire a certain amount of shares over a pre-specified time horizon at a trading speed to balance the price impact (from trading quickly) and the price uncertainty (from trading slowly), while it maximizes its final wealth. Further, in Bayraktar and Ludkovski (2011) the partially observed setting where the market liquidity variable is not observed was studied. This problem with the linear models in Cartea et al. (2015) was formulated as for the nonlinear major minor (MM) MFG model in Jaimungal and Nourian (2015). The PO MM LQG MFG theory was first applied to an optimal execution problem with linear models of Cartea et al. (2015) in Firoozi and Caines (2016) where an institutional investor, interpreted as a major agent, aims to liquidate a specific amount of shares and it has only partial observations of its own state (which includes its inventory). Furthermore, there is a large population of high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate their shares, and each of them has partial observations of its own state and the major agent's state (which include the corresponding inventories). This work is improved in the formulation of market dynamics in the MFG framework, and also is extended to consider two populations of HFTs with liquidation or acquisition objectives who wish to, respectively, liquidate or acquire a certain number of shares within a specific duration of time in the current paper. The theory is then utilized to establish the existence of  $\epsilon$ -Nash equilibria together with the best response trading strategies such that each agent attempts to maximize its own wealth and avoid the occurrence of large execution prices, and large trading accelerations which are appropriately weighted in the agent's performance function. A simulation example is provided at the end.

We note that the terms major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this paper.

The paper is organized as follows. Section 2 is devoted to the description of trading dynamics in the market and the execution problem. In Section 3 the optimal execution problem is formulated in the mean field game framework. Full observation and partial observation optimal execution problems are addressed in Sections 4 and 5, respectively. Section 6 presents the simulation results.

## 2 Trading dynamics of market agents

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: liquidators and acquirers. Employing the trading model in Cartea et al. (2015), the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the inventories, trading rates and prices while the bilinear cash process appears in the quadratic performance function for each agent.

## 2.1 Inventory dynamics

It is assumed that the institutional investor liquidates its inventory of shares,  $Q_0(t)$ , by trading at a rate  $\nu_0(t)$  during the trading period  $[0, T]$ . Hence the major agent's inventory dynamics is given by

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q(t), \quad 0 \leq t \leq T,$$

where  $w_0^Q(t)$  is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations;  $\sigma_0^Q$  is a positive scalar and we assume that  $Q_0(0) \gg 1$ . The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q(t), \quad 1 \leq i \leq N_a + N_l, \quad 0 \leq t \leq T$$

where  $N_a$  and  $N_l$  are respectively liquidator and acquirer populations of  $N$  minor traders, i.e.  $N = N_a + N_l$ ,  $w_i^Q$  is a Wiener process that models the HFT's information noise,  $\sigma_i^Q$  is a positive scalar,  $\nu_i(t)$  is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively;  $Q_i(t)$  is the minor liquidator agent's remaining shares at time  $t$ , or the shares the minor acquirer agent has bought until time  $t$ . However, the initial shares of the HFTs,  $\{Q_i(0), 1 \leq i \leq N_a + N_l\}$ , are not considered to be large, furthermore they are not motivated to retain shares and are assumed to trade them quickly.

We assume that the trading rate of the major agent is controlled via  $u_0(t)$  as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy  $u_0(t)$  can be seen to be the trading acceleration of the major trader. Correspondingly,  $u_i(t)$  controls the trading rate of minor agent,  $\mathcal{A}_i$ , by

$$d\nu_i(t) = u_i(t)dt, \quad 1 \leq i \leq N_a + N_l, \quad 0 \leq t \leq T.$$

## 2.2 Price dynamics

The trading rate of the major agent and the average trading rates of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

### 2.2.1 Fundamental asset price:

We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = (\lambda_0 \nu_0(t) + \frac{\lambda}{N} \sum_{i=1}^N \nu_i(t))dt + \sigma dw_0^F(t), \quad 0 \leq t \leq T,$$

where the Wiener process  $w_0^F(t)$  models the aggregate effect of all traders in the market which - unlike the major and minor agents  $\mathcal{A}_0, \mathcal{A}_i$ , - have no partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further,  $\sigma$  denotes the intensity of the market volatility and  $\lambda_0, \lambda \geq 0$  denote the strengths of the linear permanent impact of the major and minor agents' tradings on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent  $\mathcal{A}_i$ , by

$$dF_i(t) = (\lambda_0 \nu_0(t) + \frac{\lambda}{N} \sum_{i=1}^N \nu_i(t))dt + \sigma dw_i^F(t), \quad 0 \leq t \leq T,$$

where  $1 \leq i \leq N_a + N_l$ , and the Wiener process,  $w_i^F(t)$ , represents the mass effect of all uninformed traders in the market. The time differences between agents in getting data from fast changing limit order book make the Wiener processes,  $w_i^F$ ,  $0 \leq i \leq N_a + N_l$  independent.

### 2.2.2 Execution price:

The major agent's execution price  $S_0(t)$  evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \leq t \leq T, \quad (1)$$

where  $a_0 \geq 0$  is the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price,  $S_i(t)$ , is assumed to evolve by

$$dS_i(t) = dF_i(t) + a d\nu_i(t), \quad 1 \leq i \leq N_a + N_l, \quad 0 \leq t \leq T, \quad (2)$$

where  $a$  models the temporary impact of a minor agent's trading on its execution price.

### 2.3 Cash process

The cash process for the major agent and a generic minor agent,  $Z_0(t)$ ,  $Z_i(t)$ , are given by

$$dZ_0(t) = -S_0(t)dQ_0(t), \quad 0 \leq t \leq T, \quad (3)$$

$$dZ_i(t) = -S_i(t)dQ_i(t), \quad 1 \leq i \leq N_a + N_l, \quad 0 \leq t \leq T, \quad (4)$$

where  $Z_0(t)$ ,  $Z_i(t)$ ,  $1 \leq i \leq N_l$  are the cash obtained through liquidation of shares, and  $Z_i(t)$ ,  $0 \leq i \leq N_a$  is the cash paid for acquisition of shares up to time  $t$ . We note that the value of  $dQ_0(t)$  in a stock sale is negative and hence for positive  $S_0(t)$ ,  $Z_0(t)$  increases.

### 2.4 Cost function

#### 2.4.1 Major liquidator trader:

The objective for the major trader is to liquidate  $\mathcal{N}_0$  shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize  $Z_0(T)$ , and if the remaining inventory at the final time  $T$  is  $Q_0(T)$ , it can liquidate it at lower price than the market asset price reflected at cost function by  $Q_0(T)(F_0(T) - \alpha Q_0(T))$ . Further, the major trader's utility in minimizing the inventory over the period  $[0, T]$  is modeled by including the penalty  $\phi \int_0^T Q_0^2(s)ds$  in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms  $\epsilon S_0^2(T)$ ,  $\int_0^T \delta S_0^2(s)ds$ ,  $\beta \nu_0^2(T)$ ,  $\int_0^T \theta \nu_0^2(s)ds$  and  $\int_0^T R_0 u_0^2(s)ds$  in the objective function. Therefore, its cost function to be minimized is given by

$$J_0(u_0, u_{-0}) = \mathbb{E} \left[ -rZ_0(T) - pQ_0(T)(F_0(T) - \alpha Q_0(T)) \right. \\ \left. + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right. \\ \left. + \int_0^T (\phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s))ds \right], \quad (5)$$

where  $r$ ,  $p$ ,  $\alpha$ ,  $\epsilon$ ,  $\beta$ ,  $\phi$ ,  $\delta$ ,  $\theta$ , and  $R_0$  are positive scalars, and  $u_{-0} := (u_1, \dots, u_{N_a+N_l})$  are trading strategies of the minor traders. Note that for larger values of  $\phi$  the trader attempts to liquidate its inventory more quickly.

#### 2.4.2 Minor liquidator trader:

In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate  $\mathcal{N}_l$  shares during interval  $[0, T]$  is given by

$$J_i(u_i, u_{-i}) = \mathbb{E} \left[ -r_l Z_i(T) - p_l Q_i(T)(F_i(T) - \psi_l Q_i(T)) \right. \\ \left. + \xi_l S_i^2(T) + \mu_l \nu_i^2(T) + \int_0^T (\kappa_l Q_i^2(s) + \gamma_l S_i^2(s) \right. \\ \left. + \varrho_l \nu_i^2(s) + R_l u_i^2(s))ds \right], \quad 1 \leq i \leq N_l, \quad (6)$$

where  $r_l$ ,  $p_l$ ,  $\psi_l$ ,  $\xi_l$ ,  $\mu_l$ ,  $\kappa_l$ ,  $\gamma_l$ ,  $\varrho_l$  and  $R_l$  are positive scalars, and  $u_{-i} := (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{N_a+N_l})$ . Note that  $\mathcal{N}_l \ll \mathcal{N}_0$ .

### 2.4.3 Minor acquirer trader:

The objective for a minor acquirer trader is to buy  $\mathcal{N}$  shares during trading horizon  $[0, T]$ , while it minimizes the execution cost including the cash  $Z_i(T)$  paid up to time  $T$ , and the cash must be paid at time  $T$  to buy the remaining shares at once at a higher price than the market's asset price, i.e.  $(\mathcal{N} - Q_i(T))(F_i(T) + \psi_a(\mathcal{N} - Q_i(T)))$ . It is also intended to avoid high execution prices, large trading intensities and large trading accelerations modeled by including  $\xi_a S_i^2(T) + \mu_a \nu_i^2(T) + \int_0^T (\gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_a u_i^2(s)) ds$  in its objective function

$$J_i(u_i, u_{-i}) = \mathbb{E} \left[ p_a (\mathcal{N} - Q_i(T)) (F_i(T) + \psi_a (\mathcal{N} - Q_i(T))) \right. \\ \left. + r_a Z_i(T) + \xi_a S_i^2(T) + \mu_a \nu_i^2(T) + \int_0^T (\kappa_a (\mathcal{N} - Q_i(s))^2 \right. \\ \left. + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_a u_i^2(s)) ds \right], \quad 1 \leq i \leq N_a, \quad (7)$$

where  $\int_0^T \kappa_a (\mathcal{N} - Q_i(s))^2 ds$  is to penalize the agent for the remaining shares to be bought up to  $T$  and to expedite the acquisition. The parameters  $p_a, \psi_a, r_a, \xi_a, \mu_a, \kappa_a, \gamma_a, \varrho_a$  and  $R_a$  are positive scalars, and  $u_{-i} := (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{N_a+N_i})$ .

## 3 MFG formulation of the optimal execution problem

In this section we formulate the optimal execution problem in the MM LQG MFG framework.

### 3.1 Finite populations

#### 3.1.1 Major agent:

The stochastic optimal control problem for major trader is modeled

$$d\nu_0 = u_0 dt, \quad (8)$$

$$dQ_0 = \nu_0 dt + \sigma_0^Q dw_0^Q, \quad (9)$$

$$dS_0 = (\lambda_0 \nu_0 + \frac{\lambda}{N} \sum_{i=1}^N \nu_i) dt + a_0 u_0 dt + \sigma dw_0^F, \quad (10)$$

with the cost function

$$J_0(u_0, u_{-0}) = \mathbb{E} \left[ -Q_0(T) (S_0(T) - a_0 \nu_0(T) - \alpha Q_0(T)) \right. \\ \left. + \epsilon S_0(T)^2 + \beta \nu_0^2(T) + \int_0^T (\phi Q_0^2(s) + S_0(s) \nu_0(s) \right. \\ \left. + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s)) ds \right], \quad (11)$$

wherein the final cash process in (5) was replaced by  $\mathbb{E}[Z_0(T)] = -\mathbb{E}[\int_0^T S_0(s) \nu_0(s) ds]$ , and the fundamental asset price  $F_0(T)$  was replaced using (1).

As can be seen, the major agent is coupled with the minor agents by the average term  $\frac{\lambda}{N} \sum_{i=1}^N \nu_i$  in the execution price dynamics (10).

Now let the major agent's state be denoted by  $x_0 = [\nu_0, Q_0, S_0]^T$ . Subsequently, the major agent's cost function will be written in the standard quadratic form

$$J_0(u_0, u_{-0}) = \mathbb{E} \left[ \|x_0(T)\|_{M_0}^2 + \int_0^T (\|x_0(s)\|_{P_0}^2 + \|u_0(s)\|_{R_0}^2) ds \right], \quad (12)$$

with

$$M_0 = \begin{bmatrix} \beta & \frac{1}{2}pa_0 & 0 \\ \frac{1}{2}pa_0 & p\alpha & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \epsilon \end{bmatrix}, \quad P_0 = \begin{bmatrix} \theta & 0 & \frac{1}{2}r \\ 0 & \phi & 0 \\ \frac{1}{2}r & 0 & \delta \end{bmatrix}, \quad R_0 > 0.$$

The Equations (8)–(10) together with the cost function (12) form the standard stochastic LQG problem for the major agent. It should be remarked that for  $M_0, P_0$  to be positive semi-definite matrices, the conditions  $\beta\alpha \geq \frac{1}{4}a_0^2p$ ,  $\beta(\alpha\epsilon - \frac{1}{4}p) \geq \frac{1}{4}a_0^2p\epsilon$  and  $\theta\delta \geq \frac{1}{4}r^2$  must hold, respectively, and this will be assumed throughout this paper.

### 3.1.2 Minor liquidator agent:

Similarly, the stochastic optimal control problem for a minor trader  $\mathcal{A}_i$ ,  $1 \leq i \leq N_l$ , is given by the set of dynamical equations

$$d\nu_i = u_i dt, \quad (13)$$

$$dQ_i = \nu_i dt + \sigma_i^Q dw_i^Q, \quad (14)$$

$$dS_i = (\lambda_0\nu_0 + \frac{\lambda}{N} \sum_{i=1}^N \nu_i) dt + au_i dt + \sigma dw_i^F, \quad (15)$$

The equations above show that a minor agent is coupled with the major agent and other minor agents through the fundamental asset price dynamics (19).

Similar to the major trader, we define a generic minor trader's state vector as  $x_i = [\nu_i, Q_i, S_i]^T$ , and its quadratic cost function where the final cash process in (6) has been replaced using (4) by  $\mathbb{E}[Z_i(T)] = -\mathbb{E}[\int_0^T S_i(s)\nu_i(s)ds]$ , and the fundamental asset price  $F_i(T)$  were replaced using (2) is given by

$$J_i(u_i, u_{-i}) = \mathbb{E} \left[ \|x_i(T)\|_{M_l}^2 + \int_0^T (\|x_i(s)\|_{P_l}^2 + \|u_i(s)\|_{R_l}^2) ds \right], \quad (16)$$

where

$$M_l = \begin{bmatrix} \mu_l & \frac{1}{2}p_l a & 0 \\ \frac{1}{2}p_l a & p_l \psi_l & -\frac{1}{2}p_l \\ 0 & -\frac{1}{2}p_l & \xi_l \end{bmatrix}, \quad P_l = \begin{bmatrix} \varrho_l & 0 & \frac{1}{2}r_l \\ 0 & \kappa_l & 0 \\ \frac{1}{2}r_l & 0 & \gamma_l \end{bmatrix}, \quad R_l > 0.$$

The set of Equations (13)–(15) and the cost function (16) constitute the standard stochastic LQG problem for a minor liquidator trader. Again, for the matrices  $M_l, P_l$  to be positive semi definite,  $\mu_l \psi_l > \frac{1}{4}a^2 p_l$ ,  $\mu_l(\psi_l \xi_l - \frac{1}{4}p_l) \geq \frac{1}{4}\xi_l p_l a^2$  and  $\gamma_l \varrho_l > \frac{1}{4}r_l^2$  must be, respectively, satisfied and this is adopted as an assumption.

### 3.1.3 Minor acquirer agent:

The stochastic optimal control problem for a minor acquirer trader  $\mathcal{A}_i$ ,  $1 \leq i \leq N_a$ , is given by the set of dynamical equations

$$d\nu_i = u_i dt, \quad (17)$$

$$dY_i = -\nu_i dt + \sigma_i^Q dw_i^Q, \quad (18)$$

$$dS_i = (\lambda_0\nu_0 + \frac{\lambda}{N} \sum_{i=1}^N \nu_i) dt + au_i dt + \sigma dw_i^F, \quad (19)$$

where  $Y_i(t) = \mathcal{N}_a - Q_i(t)$  is the remaining shares at  $t$  to be acquired until the end of trading horizon. Accordingly, the cost function for acquisition is given by

$$J_i(u_i, u_{-i}) = \mathbb{E} \left[ Z_i(T) + Y_i(T)(F_i(T) + \psi_a Y_i(T)) \right. \\ \left. + \xi_a S_i^2(T) + \mu_a \nu_i^2(T) + \int_0^T (\kappa_a Y_i(s)^2 + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_a u_i^2(s)) ds \right], \quad 1 \leq i \leq N_a. \quad (20)$$

We define a generic minor acquirer trader's state vector as  $x_i = [\nu_i, Y_i, S_i]^T$  and its quadratic cost function is given by

$$J_i(u_i, u_{-i}) = \mathbb{E} \left[ \|x_i(T)\|_{M_a}^2 + \int_0^T (\|x_i(s)\|_{P_a}^2 + \|u_i(s)\|_{R_a}^2) ds \right], \quad (21)$$

where

$$M_a = \begin{bmatrix} \mu_a & -\frac{1}{2}p_a a & 0 \\ -\frac{1}{2}p_a a & p_a \psi_a & \frac{1}{2}p_a \\ 0 & \frac{1}{2}p_a & \xi_a \end{bmatrix}, P_a = \begin{bmatrix} \varrho_a & 0 & -\frac{1}{2}r_a \\ 0 & \kappa_a & 0 \\ -\frac{1}{2}r_a & 0 & \gamma_a \end{bmatrix}, R_a > 0.$$

The set of Equations (17)–(19) and the cost function (21) constitute the standard stochastic LQG problem for a minor liquidator trader. Again, for the matrices  $M_l$ ,  $P_l$  to be positive semi definite,  $\mu_a \psi_a > \frac{1}{4}a^2 p_a$ ,  $\mu_a(\psi_a \xi_a - \frac{1}{4}p_a) \geq \frac{1}{4}\xi_a p_a a^2$  and  $\gamma_a \varrho_a > \frac{1}{4}r_a^2$  must be, respectively, satisfied and this is adopted as an assumption.

### 3.2 Mean field evolution

Following the LQG MFG methodology (Huang, 2010), the mean field,  $\bar{x}$ , is defined as the  $L^2$  limit, when it exists, of the average of minor agents' states when the population size goes to infinity

$$\bar{x}(t) = \lim_{N \rightarrow \infty} x^N(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N x_i(t), \text{ a.s.}$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 x_i + L_2 x_0 + \sum_{j \neq i, j=1}^N L_4 x_j + L_3, \quad 1 \leq i \leq N, \quad (22)$$

then mean field dynamics can be obtained by substituting (22) in the minor agents' dynamics (17)–(19) and taking the average and then its  $L^2$  limit as  $N \rightarrow \infty$ .

The dynamics of the mean field,  $\bar{x} = [\bar{\nu}, \bar{Q}, \bar{F}]^T$ , for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0dt + \bar{m}dt, \quad (23)$$

with the matrices in the above equation defined as

$$\bar{A} = \begin{bmatrix} \bar{L}_{1,1} & \bar{L}_{1,2} & \bar{L}_{1,3} \\ 1 & 0 & 0 \\ (\lambda + a\bar{L}_{1,1}) & a\bar{L}_{1,2} & a\bar{L}_{1,3} \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{L}_3 \\ 0 \\ a\bar{L}_3 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{L}_{2,1} & \bar{L}_{2,2} & \bar{L}_{2,3} \\ 0 & 0 & 0 \\ (\lambda_0 + a\bar{L}_{2,1}) & a\bar{L}_{2,2} & a\bar{L}_{2,3} \end{bmatrix},$$

where  $\{\bar{L}_{i,j}, i = 1, 2, j = 1, 2, 3\}$  are scalars that can be determined from consistency equations.

### 3.3 Infinite populations

The stochastic optimal control problem for each agent in the infinite population case where the finite population average term is replaced with its infinite  $L^2$  limit, i.e. the mean field, is given below.

#### 3.3.1 Major liquidator agent:

Major trader's stochastic optimal control problem in the infinite population case is given by

$$dx_0 = A_0 x_0 dt + B_0 u_0 dt + E_0 \bar{x} dt + D_0 dw_0, \quad (24)$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \\ a_0 \end{bmatrix}, \quad w_0 = \begin{bmatrix} w_0^Q \\ w_0^F \end{bmatrix}$$

$$E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & 0 \\ \sigma_0^Q & 0 \\ 0 & \sigma \end{bmatrix},$$

together with the cost function (12).

### 3.3.2 Minor liquidator agent:

The stochastic optimal control problem for a minor liquidator agent in the infinite population case is given by

$$dx_i = A_l x_i dt + E_l \bar{x} dt + B_l u_i dt + G_l x_0 dt + D_l dw_i, \quad (25)$$

$1 \leq i \leq N_l$ , with the matrices

$$A_l = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B_l = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

$$G_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad D_l = \begin{bmatrix} 0 & 0 \\ \sigma_i^Q & 0 \\ 0 & \sigma \end{bmatrix}, \quad w_i = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix},$$

together with the cost function (16).

### 3.3.3 Minor acquirer agent:

The stochastic optimal control problem for an acquirer agent in the infinite population case is given by

$$dx_i = A_a x_i dt + E_a \bar{x} dt + B_a u_i dt + G_a x_0 dt + D_a dw_i, \quad (26)$$

$1 \leq i \leq N_a$ , where

$$A_a = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B_a = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

$$G_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad D_a = \begin{bmatrix} 0 & 0 \\ \sigma_i^Q & 0 \\ 0 & \sigma \end{bmatrix}, \quad w_i = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix},$$

together with the cost function (21).

## 4 Completely observed optimal execution problem

Following the mean field game methodology with a major agent (Nourian and Caines, 2013), the optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced with its infinite population limit, i.e. the mean field. Then specializing to MFG linear systems (Huang, 2010), the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Finally the infinite population best response strategies are applied to the finite population system which yields an  $\epsilon$ -Nash equilibria (see Theorem 5.1).

### 4.1 Major liquidator agent

The dynamics for the major trader's extended state  $x_0^{ex} = [x_0^T, \bar{x}^T]^T$  in the infinite population case is given by

$$dx_0^{ex} = \mathbb{A}_0 x_0^{ex} dt + \mathbb{M}_0 dt + \mathbb{B}_0 u_0 dt + \mathbb{D}_0 dW_0, \quad (27)$$

with the matrices in above equation defined as

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & E_0 \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0_{3 \times 1} \\ \bar{m} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{3 \times 1} \end{bmatrix}, \quad \mathbb{D}_0 = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_0 = \begin{bmatrix} w_0 \\ 0 \end{bmatrix}.$$

Following Huang (2010); Nguyen and Huang (2012), the infinite population best response control is given by

$$u_0^\circ(t) = -R_0^{-1} \mathbb{B}_0^T \Pi_0 (x_0^T, \bar{x}^T)^T. \quad (28)$$

Let us define  $\bar{N}_0 = [I_{3 \times 3}, 0_{3 \times 3}]^T N_0 [I_{3 \times 3}, 0_{3 \times 3}]$  and  $\bar{M}_0 = [I_{3 \times 3}, 0_{3 \times 3}]^T M_0 [I_{3 \times 3}, 0_{3 \times 3}]$ , then  $\Pi_0$  is calculated by the following Riccati equation as

$$-\dot{\Pi}_0 = \Pi_0 \mathbb{A}_0 + \mathbb{A}_0^T \Pi_0 - \Pi_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 + \bar{N}_0,$$

with  $\Pi_0(T) = \bar{M}_0$ .

## 4.2 Minor acquirer/liquidator agent

For brevity, the notation  $(\cdot)_{a/l}$  is used in the rest of this paper to denote the matrices and parameters corresponding to a generic acquirer or a liquidator agent, respectively. Accordingly, a generic minor (acquirer/liquidator) agent's extended dynamics is

$$\begin{aligned} \begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \end{bmatrix} &= \begin{bmatrix} A_{a/l} & [G_{a/l} & E_{a/l}] \\ 0_{6 \times 3} & \mathbb{A}_0 \end{bmatrix} \begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} 0_{3 \times 1} \\ \mathbb{M}_0 \end{bmatrix} dt \\ &+ \begin{bmatrix} 0_{3 \times 1} \\ \mathbb{B}_0 \end{bmatrix} u_0 dt + \begin{bmatrix} B_{a/l} \\ 0_{6 \times 1} \end{bmatrix} u_i dt + \begin{bmatrix} D_{a/l} & 0_{3 \times 6} \\ 0_{6 \times 3} & \mathbb{D}_0 \end{bmatrix} \begin{bmatrix} dw_i \\ dw_0 \\ 0 \end{bmatrix}. \end{aligned}$$

Substituting the major agent's control action (28) into (29) yields

$$dx_i^{ex} = \mathbb{A}_{a/l} x_i^{ex} dt + \mathbb{M}_{a/l} dt + \mathbb{B}_{a/l} dt + \mathbb{D}_{a/l} dW_i \quad (29)$$

where

$$\begin{aligned} \mathbb{A}_{a/l} &= \begin{bmatrix} A_{a/l} & [G_{a/l} & E_{a/l}] \\ 0_{6 \times 3} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 \end{bmatrix}, \quad \mathbb{M}_{a/l} = \begin{bmatrix} 0_{3 \times 1} \\ \mathbb{M}_0 \end{bmatrix}, \\ \mathbb{B}_{a/l} &= \begin{bmatrix} B_{a/l} \\ 0_{6 \times 1} \end{bmatrix}, \quad \mathbb{D}_{a/l} = \begin{bmatrix} D_{a/l} & 0_{3 \times 6} \\ 0_{6 \times 3} & \mathbb{D}_0 \end{bmatrix}, \quad W_i = \begin{bmatrix} w_i \\ w_0 \\ 0 \end{bmatrix}. \end{aligned}$$

Then the best response control for a generic minor agent is

$$u_i^\circ(t) = -R_{a/l}^{-1} \mathbb{B}_{a/l}^T \Pi_{a/l} (x_i^T, x_0^T, \bar{x}^T)^T, \quad (30)$$

where  $\Pi_{a/l}$  is calculated from

$$-\dot{\Pi}_{a/l} = \Pi_{a/l} \mathbb{A}_{a/l} + \mathbb{A}_{a/l}^T \Pi_{a/l} - \Pi_{a/l} \mathbb{B}_{a/l} R_{a/l}^{-1} \mathbb{B}_{a/l}^T \Pi_{a/l} + \bar{P}_{a/l}, \quad (31)$$

with  $\Pi_{a/l}(T) = \bar{M}_{a/l}$ , and the matrices in (31) are

$$\begin{aligned} \bar{P}_{a/l} &= [I_{3 \times 3}, 0_{3 \times 6}]^T P_{a/l} [I_{3 \times 3}, 0_{3 \times 6}] \\ \bar{M}_{a/l} &= [I_{3 \times 3}, 0_{3 \times 6}]^T M_{a/l} [I_{3 \times 3}, 0_{3 \times 6}]. \end{aligned}$$

## 4.3 Consistency condition

The closed loop trading dynamics of a generic minor agent  $\mathcal{A}_i$ ,  $1 \leq i \leq N = N_a + N_l$ , applying (30) is consequently

$$dv_i = -R_{a/l}^{-1} \mathbb{B}_{a/l}^T \Pi_{a/l} (x_i^T, x_0^T, \bar{x}^T)^T dt,$$

and so we obtain the mean field  $\bar{\nu}$  process as follows.

$$\sum_{i=1}^N d\nu_i = - \sum_{i=1}^{N_a} R_a^{-1} \mathbb{B}_a^T \Pi_a (x_i^T, x_0^T, \bar{x}^T)^T dt - \sum_{i=1}^{N_l} R_l^{-1} \mathbb{B}_l^T \Pi_l (x_i^T, x_0^T, \bar{x}^T)^T dt$$

which yields

$$\lim_{N \rightarrow \infty} d\nu^N = -R_a^{-1} \mathbb{B}_a^T \Pi_a \lim_{N_a \rightarrow \infty} ((x^N)^T, x_0^T, \bar{x}^T)^T dt - R_l^{-1} \mathbb{B}_l^T \Pi_l \lim_{N_l \rightarrow \infty} ((x^N)^T, x_0^T, \bar{x}^T)^T dt,$$

and hence the consistency equations become

$$\begin{aligned} \bar{L}_{1,1} &= -R_a^{-1}(\Pi_{a_{1,1}} + \Pi_{a_{1,7}}) - aR_a^{-1}(\Pi_{a_{3,1}} + \Pi_{a_{3,7}}) - R_l^{-1}(\Pi_{l_{1,1}} + \Pi_{l_{1,7}}) - aR_l^{-1}(\Pi_{l_{3,1}} + \Pi_{l_{3,7}}), \\ \bar{L}_{1,2} &= -R_a^{-1}(\Pi_{a_{1,2}} + \Pi_{a_{1,8}}) - aR_a^{-1}(\Pi_{a_{3,2}} + \Pi_{a_{3,8}}) - R_l^{-1}(\Pi_{l_{1,2}} + \Pi_{l_{1,8}}) - aR_l^{-1}(\Pi_{l_{3,2}} + \Pi_{l_{3,8}}) \\ \bar{L}_{1,3} &= -R_a^{-1}(\Pi_{a_{1,3}} + \Pi_{a_{1,9}}) - aR_a^{-1}(\Pi_{a_{3,3}} + \Pi_{a_{3,9}}) - R_l^{-1}(\Pi_{l_{1,3}} + \Pi_{l_{1,9}}) - aR_l^{-1}(\Pi_{l_{3,3}} + \Pi_{l_{3,9}}), \\ \bar{L}_{2,1} &= -R_a^{-1}(\Pi_{a_{1,4}} + a\Pi_{a_{3,4}}) - R_l^{-1}(\Pi_{l_{1,4}} + a\Pi_{l_{3,4}}), \\ \bar{L}_{2,2} &= -R_a^{-1}(\Pi_{a_{1,5}} + a\Pi_{a_{3,5}}) - R_l^{-1}(\Pi_{l_{1,5}} + a\Pi_{l_{3,5}}), \\ \bar{L}_{2,3} &= -R_a^{-1}(\Pi_{a_{1,6}} + a\Pi_{a_{3,6}}) - R_l^{-1}(\Pi_{l_{1,6}} + a\Pi_{l_{3,6}}), \\ \bar{L}_3 &= 0 \end{aligned} \tag{32}$$

where the  $\Pi_{a/l_{i,j}} = \Pi_{a/l}(i, j)$  and the scalars  $\bar{L}_{(\cdot)}$  were defined in (23).

## 5 Partially observed optimal execution problem

We now follow the general development in Firoozi and Caines (2015) for PO MM LQG MFG systems where a generic minor agent has partial observations of its own states as well as the major agent's states, and the major agent has only partial observations on its own states.

### 5.1 Major liquidator agent

Let the major agent's observation process be

$$dy_0 = \mathbb{H}_0[x_0^T, \bar{x}^T]^T dt + \sigma_{v_0} dv_0 \tag{33}$$

where  $\mathbb{H}_0$  is a constant matrix with appropriate dimensions. Then the Kalman filter equation generating the estimates of the major agent's states is given by

$$d\hat{x}_0^{ex}|_{\mathcal{F}_0^y} = \mathbb{A}_0 \hat{x}_0^{ex}|_{\mathcal{F}_0^y} dt + \mathbb{M}_0 dt + \mathbb{B}_0 \hat{u}_0|_{\mathcal{F}_0^y} dt + K_0(t)[dy_0 - \mathbb{H}_0 \hat{x}_0^{ex}|_{\mathcal{F}_0^y} dt], \tag{34}$$

where the filter gain is

$$K_0(t) = V_0(t) \mathbb{H}_0^T R_{v_0}^{-1}, \tag{35}$$

with  $R_{v_0} = \sigma_{v_0} \sigma_{v_0}^T$ . The associated Riccati equation is

$$\dot{V}_0(t) = \mathbb{A}_0 V_0(t) + V_0(t) \mathbb{A}_0^T - K_0(t) R_{v_0} K_0(t)^T + Q_{w_0}. \tag{36}$$

Following the procedure in Firoozi and Caines (2015) the cost function (12) can be decomposed into

$$\begin{aligned} J_0 = \mathbb{E} \left[ \|\hat{x}_0|_{\mathcal{F}_0^y}(T)\|_{M_0}^2 + \int_0^T (\|\hat{x}_0|_{\mathcal{F}_0^y}(s)\|_{P_0}^2 + \|u_0(s)\|_{R_0}^2) ds \right. \\ \left. + \|x_0(T) - \hat{x}_0|_{\mathcal{F}_0^y}(T)\|_{M_0}^2 + \int_0^T (\|x_0(s) - \hat{x}_0|_{\mathcal{F}_0^y}(s)\|_{P_0}^2) ds \right], \end{aligned}$$

and hence by the Separation Principle the corresponding infinite population best response control action is given by

$$\hat{u}_0^\circ = -R_0^{-1} \mathbb{B}_0^T \Pi_0 (\hat{x}_0^T|_{\mathcal{F}_0^y}, \hat{x}_1^T|_{\mathcal{F}_0^y})^T. \tag{37}$$

## 5.2 Minor (acquirer/liquidator) agent

Following Firoozi and Caines (2015) the extended state for a generic minor (acquirer/liquidator) agent shall be

$$X_i = [x_i^T, x_0^T, \bar{x}^T, \hat{x}_{0|\mathcal{F}_0^y}^T, \hat{x}_{|\mathcal{F}_0^y}^T]^T. \quad (38)$$

Correspondingly, the extended dynamics of a minor agent is given by

$$\begin{aligned} \begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \\ d\hat{x}_{0|\mathcal{F}_0^y} \\ d\hat{x}_{|\mathcal{F}_0^y} \end{bmatrix} &= \begin{bmatrix} A_{a/l} & [G_{a/l}, E_{a/l}] & & & \\ 0_{6 \times 3} & A_0 & 0_{3 \times 6} & & \\ 0_{6 \times 3} & K_0 \mathbb{H}_0 & -\mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 & & \\ & & A_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 - K_0 \mathbb{H}_0 & & \end{bmatrix} \begin{bmatrix} x_i \\ x_0 \\ \bar{x} \\ \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{x}_{|\mathcal{F}_0^y} \end{bmatrix} dt \\ &+ \begin{bmatrix} 0_{3 \times 1} \\ \mathbb{M}_0 \\ \mathbb{M}_0 \end{bmatrix} dt + \begin{bmatrix} \mathbb{B}_{a/l} \\ 0_{6 \times 1} \end{bmatrix} u_i(t) dt + \begin{bmatrix} \mathbb{D}_{a/l} & 0 \\ 0 & K_0 \end{bmatrix} \begin{bmatrix} dW_i \\ dv_0 \end{bmatrix}, \quad (39) \end{aligned}$$

or equivalently

$$dX_i = \mathcal{A}_{a/l} X_i dt + \mathcal{M}_{a/l} dt + \mathcal{B}_{a/l} u_i dt + \Sigma_{a/l} [dW_i^T, dv_0]^T \quad (40)$$

Let the minor agent's observation process be given by

$$dy_i(t) = \mathbb{H}_{a/l} [x_i^T, x_0^T, \bar{x}^T, \hat{x}_{0|\mathcal{F}_0^y}^T, \hat{x}_{|\mathcal{F}_0^y}^T]^T dt + \sigma_{v_i} dv_i \quad (41)$$

with the constant matrix  $\mathbb{H}_{a/l}$ . The Kalman filter which generates the estimates of the minor (liquidator/acquirer) agent's states is

$$d\hat{X}_{i|\mathcal{F}_i^y} = \mathcal{A}_{a/l} \hat{X}_{i|\mathcal{F}_i^y} dt + \mathcal{M}_{a/l} dt + \mathcal{B}_{a/l} \hat{u}_{i|\mathcal{F}_i^y} dt + K_{a/l}(t) [dy_i - \mathbb{H}_{a/l} \hat{X}_{i|\mathcal{F}_i^y} dt], \quad (42)$$

where the filter gain is given as

$$K_{a/l}(t) = V_{a/l}(t) \mathbb{H}_{a/l}^T R_{v_i}^{-1}, \quad (43)$$

with  $R_{v_i} = \sigma_{v_i} \sigma_{v_i}^T$ .

The corresponding Riccati equation is

$$\dot{V}_{a/l}(t) = \mathcal{A}_{a/l} V_{a/l}(t) + V_{a/l}(t) \mathcal{A}_{a/l}^T - K_{a/l}(t) R_{v_i} K_{a/l}(t)^T + Q_w. \quad (44)$$

Again employing the methodology in Firoozi and Caines (2015), the cost function (21) is decomposed to

$$\begin{aligned} J_i &= \mathbb{E} \left[ \|\hat{x}_{i|\mathcal{F}_i^y}(T)\|_{M_{a/l}}^2 + \int_0^T (\|\hat{x}_{i|\mathcal{F}_i^y}(s)\|_{P_{a/l}}^2 + \|u_i(s)\|_{R_{a/l}}^2) ds \right. \\ &\quad \left. + \|x_i(T) - \hat{x}_{i|\mathcal{F}_i^y}(T)\|_{M_{a/l}}^2 + \int_0^T \|x_i(s) - \hat{x}_{i|\mathcal{F}_i^y}(s)\|_{P_{a/l}}^2 ds \right], \end{aligned}$$

hence by the Separation Principle the corresponding infinite population best response control is given by

$$\hat{u}_i^\circ(t) = -R_{a/l}^{-1} \mathbb{B}_{a/l}^T \Pi_{a/l} (\hat{x}_{i|\mathcal{F}_i^y}^T, \hat{x}_{0|\mathcal{F}_i^y}^T, \hat{x}_{|\mathcal{F}_i^y}^T)^T. \quad (45)$$

The infinite population best response control laws applied to a finite population system yields an  $\epsilon$ -Nash equilibrium.

**Theorem 1**  *$\epsilon$ -Nash Equilibria for PO MM LQG MFG Systems: Subject to reasonable technical assumptions (see Firoozi and Caines (2015)), the KF-MF state estimation scheme (34)–(36) and (42)–(44) together with the MM-MFG equation scheme (32) generate the set of control laws  $\hat{\mathcal{U}}_{MF}^N \triangleq \{\hat{u}_i^\circ; 0 \leq i \leq N\}$ ,  $1 \leq N < \infty$ , given by*

$$\begin{aligned} \hat{u}_0^\circ &= -R_0^{-1} \mathbb{B}_0^T \Pi_0 (\hat{x}_{0|\mathcal{F}_0^y}^T, \hat{x}_{|\mathcal{F}_0^y}^T)^T, \\ \hat{u}_i^\circ &= -R^{-1} \mathbb{B}^T \Pi (\hat{x}_{i|\mathcal{F}_i^y}^T, \hat{x}_{0|\mathcal{F}_i^y}^T, \hat{x}_{|\mathcal{F}_i^y}^T)^T, \quad 1 \leq i \leq N \end{aligned}$$

such that

- (i) All agent systems  $\mathcal{A}_i$ ,  $0 \leq i \leq N$ , are second order stable.
- (ii)  $\{\hat{\mathcal{U}}_{MF}^N; 1 \leq N < \infty\}$  yields an  $\epsilon$ -Nash equilibrium for all  $\epsilon$ , i.e. for all  $\epsilon > 0$ , there exists  $N(\epsilon)$  such that for all  $N \geq N(\epsilon)$ ;

$$J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_{i,y}^N} J_i^{s,N}(u_i, \hat{u}_{-i}^\circ) \leq J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ).$$

## 6 Simulations

In numerical experiments, it is assumed that the trading action takes place within  $T = 1000$  seconds. The strength of the temporary market impact of the major agent's trading and of a generic minor agent's trading are both given by  $a_0 = a = 5.43 \times 10^{-6}$ , while their permanent impact strengths are both taken to be  $\lambda_0 = \lambda = 2 \times 10^{-8}$ . The diffusion coefficients in the trading dynamics are selected to be  $\sigma_0^Q = 0.05$ , and  $\sigma_i^Q = 0.02$ , respectively. The weights in the cost function for the major trader are  $\alpha = 5a_0 \times 10^5$ ,  $\phi = 10^{-6}a_0$ ,  $\delta = 1/(2a_0)$ ,  $\epsilon = 1/(2\alpha)$ ,  $\theta = 1/(2\delta)$ ,  $\beta = 10$ ,  $p = 100$ , and  $r = 1$ ; and those of a generic minor (liquidator/acquirer) trader are  $\psi_l = \psi_a = 5a \times 10^5$ ,  $\kappa_l = \kappa_a = 10^{-1}a$ ,  $\xi_l = \xi_a = 1/(2\psi)$ ,  $\gamma_l = \gamma_a = 1/(2a)$ ,  $\varrho_l = \varrho_a = 1/(2\gamma)$ ,  $\mu_l = \mu_a = 10$ ,  $p_l = p_a = 1000$ , and  $r_l = r_a = 1$ . Furthermore, the market volatility is assumed to be  $\sigma = 0.6565$ , the initial asset price is taken to be  $F_0(0) = F_i(0) = \$35$ , and the initial inventory stock of the major trader to be liquidated is set to  $Q_0(0) = 5 \times 10^6$ , while the minor liquidator HFT aims to sell  $Q_i(0) = 5000$  shares and the acquirer HFT wants to buy  $Q_i(0) = 5000$  shares. In the estimation part the measurement noise standard deviation for the major trader is  $\sigma_{\nu_0} = 0.05$ , and for the HFT is  $\sigma_{\nu_i} = 0.5$ .

The resulting  $\epsilon$ -Nash equilibrium trajectories of the major agent and the generic acquirer and liquidator HFTs in the complete observation case are displayed in Figures 1–3, and those in the partial observation case are depicted in Figures 4–6.

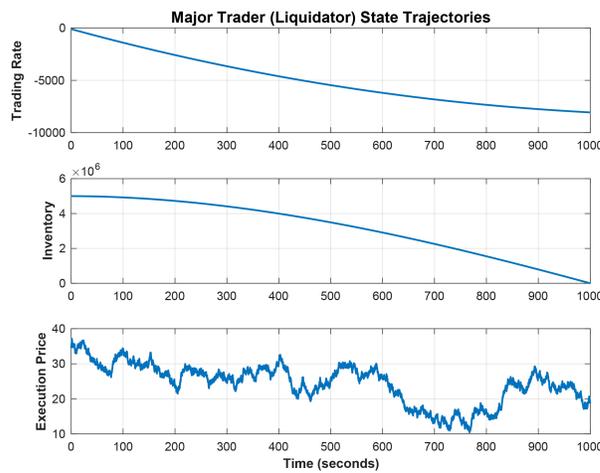


Figure 1: Major liquidator's state trajectories with complete observations.

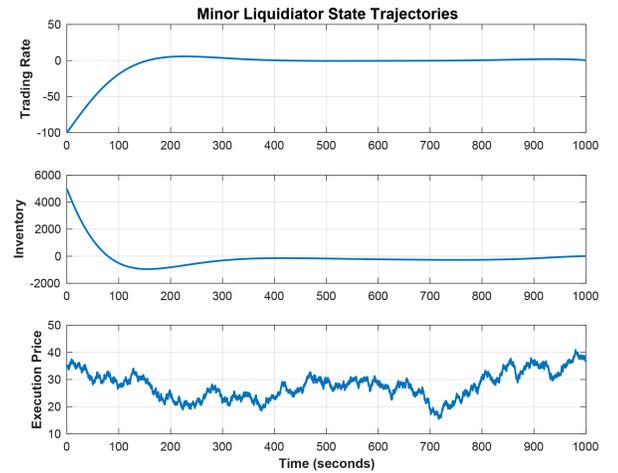


Figure 2: A minor liquidator's state trajectories with complete observations.

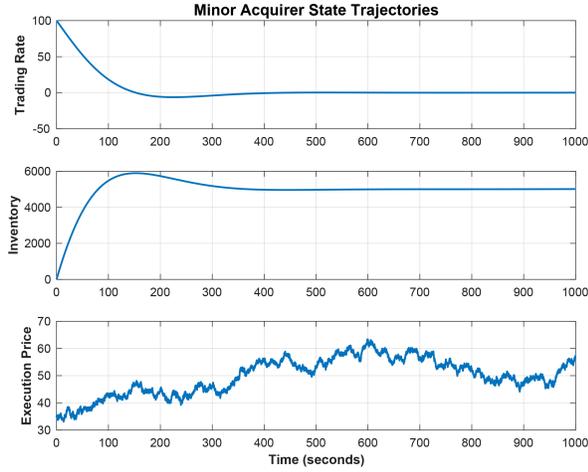


Figure 3: A minor acquirer's state trajectories with complete observations.

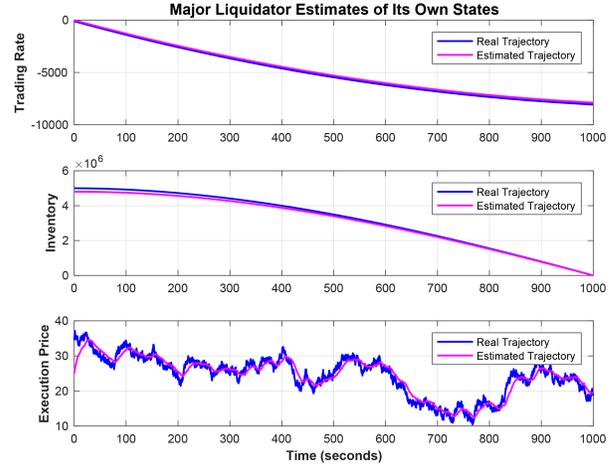


Figure 4: Major agent's state trajectories and major agent's estimates of its own states in the partial observation case.

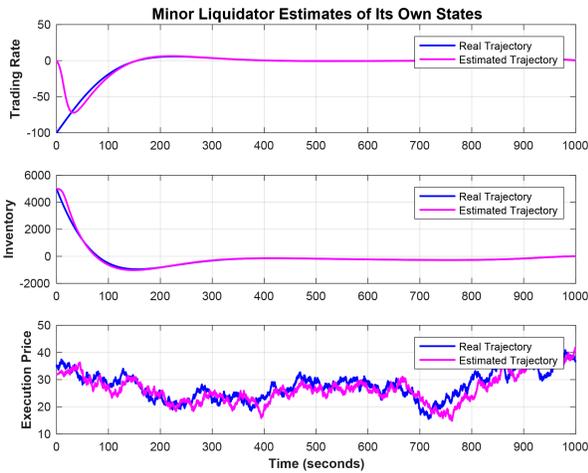


Figure 5: A generic minor liquidator agent's state trajectories and minor agent's estimates of its own states in the partial observation case.

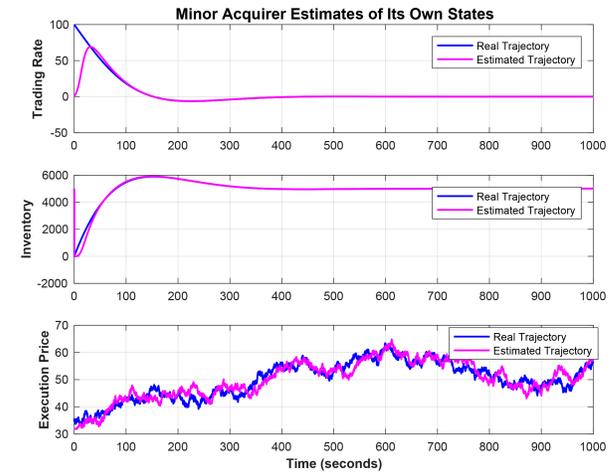


Figure 6: A generic minor acquirer agent's state trajectories and minor agent's estimates of its own states in the partial observation case.

## References

- Alfonsi, A., A. Fruth, and A. Schied. 2010. Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance* 10:143–157.
- Almgren, R., and N. Chriss. 2001. Optimal execution of portfolio transactions. *Journal of Risk* pp. 5–39.
- Bayraktar, E., and M. Ludkovski. 2011. OPTIMAL TRADE EXECUTION IN ILLIQUID MARKETS. *Mathematical Finance* 21:681–701.
- Caines, P. E., and A. C. Kizilkale. 2013. Recursive estimation of common partially observed disturbances in MFG systems with application to large scale power markets. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 2505–2512.
- Caines, P. E., and A. C. Kizilkale. 2014. Mean Field Estimation for Partially Observed LQG Systems with Major and Minor Agents. In *Proceedings of the 19th World Congress of the International Federation of Automatic Control (IFAC)*, pp. 8705–8709.
- Caines, P. E., and A. C. Kizilkale. 2016.  $\epsilon$ -Nash Equilibria for Partially Observed LQG Mean Field Games with Major Player. *IEEE Transaction on Automatic Control* (online) .
- Cartea, A., S. Jaimungal, and J. Penalva. 2015. *Algorithmic and High-Frequency Trading*. United Kingdom : Cambridge University Press.
- Şen, N., and P. E. Caines. 2014. Mean field games with partially observed major player and stochastic mean field. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pp. 2709–2715.
- Şen, N., and P. E. Caines. 2015.  $\epsilon$ -Nash equilibria for a partially observed mean field game with major player. In *Proceedings of the 2015 American Control Conference (ACC)*, pp. 4791–4797.
- Şen, N., and P. E. Caines. 2016a. Mean field game theory for agents with individual-state partial observations. In *Proceedings of the 55th IEEE Conference on Decision and Control*, pp. 6105–6110.
- Şen, N., and P. E. Caines. 2016b. On Mean Field Games and nonlinear filtering for agents with individual-state partial observations. In *Proceedings of the 2016 American Control Conference (ACC)*, pp. 4681–4686.
- Firoozi, D., and P. E. Caines. 2015.  $\epsilon$ -Nash equilibria for partially observed LQG mean field games with major agent: Partial observations by all agents. In *Proceedings of the 54th IEEE Conference on Decision and Control (CDC)*, pp. 4430–4437.
- Firoozi, D., and P. E. Caines. 2016. Mean Field Game  $\epsilon$ -Nash Equilibria for Partially Observed Optimal Execution Problems in Finance. In *Proceedings of the 55th IEEE Conference on Decision and Control*.
- Huang, M. 2010. Large-Population LQG Games Involving a Major Player: The Nash Certainty Equivalence Principle. *SIAM Journal on Control and Optimization* 48:3318–3353.
- Huang, M., P. E. Caines, R. P. Malhamé, et al. 2006. Distributed multi-agent decision-making with partial observations: asymptotic Nash equilibria. In *Proceedings of the 17th Internat. Symposium on Math. Theory on Networks and Systems (MTNS'06)*, Kyoto, Japan, pp. 2725–2730.
- Jaimungal, S., and D. Kinzebulatov. 2014. Optimal Execution with a Price Limiter. *Journal of Risk*.
- Jaimungal, S., and M. Nourian. 2015. Mean-Field Game Strategies for a Major-Minor Agent Optimal Execution Problem. Available at SSRN.
- Nguyen, S. L., and M. Huang. 2012. Linear-Quadratic-Gaussian Mixed Games with Continuum-Parametrized Minor Players. *SIAM Journal on Control and Optimization* 50:2907–2937.
- Nourian, M., and P. E. Caines. 2013.  $\epsilon$ -Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM Journal on Control and Optimization* 51:3302–3331.