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A structural model for valuing exchangeable bonds

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Abstract: An exchangeable bond is a debt that is convertible into shares of a firm's equity other than the bond's issuer. We evaluate an exchangeable bond within a two-dimensional structural model, where the assets' value of the bond's issuer and the underlying equity value are the state variables. Our model, based on dynamic programming, finite elements, and parallel computing, accommodates arbitrary debt portfolio including an exchangeable bond, several seniority classes, bankruptcy costs and tax benefits. We conduct a numerical investigation that highlights the main characteristics of exchangeable bonds and their distinction from a straight bond.

Keywords: Credit risk, exchangeable bond, dynamic programming, finite elements, paralell computing

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1 Introduction

The main aim of this paper is to value exchangeable bonds. Unlike a convertible debt whose the payoff is associated with the performance of the issuer's stock, the payoff of an exchangeable debt depends on the stock of a different firm. Specifically, a firm that issues an exchangeable debt gives bondholders the option to exchange their bonds for shares of another firm's equity. The exercise decision is also closely related to the issuer's financial situation, in particular its credit risk default.

Exchangeable debt has been offered by firms since the early 1970s. The Association of Convertible Bonds Management reported that, in 2001, about one third of the European convertible bond market was made of exchangeable bonds. According to Grimwood and Hodges (2002), this proportion represents 14% of the total bond market in the US.

Assume a company holds equity shares of another public company and makes the decision to divest of this intercorporate holding because of negative expectations regarding its future prospects. Divesting strategies include block sales, secondary distributions or issuing exchangeable debt. As documented in corporate finance (Barber, 1993), the latter is preferred over the other two alternatives. In fact, announcing a secondary distribution (Mikkelson and Partch, 1985) or block sales (Holthausen et al., 1987) can provoke a negative price reaction, which can be avoided by issuing an exchangeable debt. Jones and Mason (1986) discuss also tax advantages as motivation for exchangeable debt issues.

Many articles address the valuation of ordinary convertible bonds, see, e.g. Ingersoll (1977) and Brennan and Schwartz (1977), to cite a few. Despite the relevance of exchangeable debts, less attention has been given to their theoretical valuation. Realdon (2004) proposes a structural valuation model for these bonds and uses the Hopscotch finite difference method to solve the problem. He considers the case of an exchangeable bond when the issuer owns the underlying shares and when the issuer does not own these shares. He explains however that the latter case is not realistic. He also discusses some features and distinctions of the exchangeable bond. Moreover, Guo and Ren (2009) present a pricing model for exchangeable debt under the least-squares regression approach proposed by Longstaff and Schwartz (2001).

In this paper, we extend Realdon (2004) by presenting a two-factor structural model and incorporating the exchangeable debt as part of the debt portfolio of the firm under a setting comparable to Ayadi et al. (2016). Our model accounts for tax benefits, bankruptcy costs and an arbitrary debt portfolio, allowing our model to be flexible and able to accommodate any financial structural. The two factors are the value of the issuer's assets and the value of the equity shares against which the bond can be exchanged. Our methodology is based on a two-dimensional dynamic program coupled with bilinear interpolations and parallel computing. We suppose that the issuer owns the shares of the underlying equity, which are pledged to the bondholders of the exchangeable bond. This is to ensure that the exchange option is not lost in case of default. We start the evaluation at maturity of the debt where we can assess the debt in closed form. Next we proceed backward and evaluate the bond at every payment date. On the one hand, the firm survives in each step if it can meet its financial commitments to pay coupons and principal amounts to the bondholders. In this case, bondholders of exchangeable bond will compare what they receive to the value of the underlying shares, and exercise the exchange option if it is beneficial. If the option is exercised, the firm again reassess its situation: the total value drops if it no longer owns the underlying shares, and default occurs if senior bondholders cannot be paid. On the other hand, the firm defaults if it cannot honor its commitments to the bondholders. Those of the exchangeable bond will then compare their recovered amount to the value of the underlying shares and exercise, if favorable to them. Upon exercise, the firm can still survive if it can pay the senior bondholders and avoid default.

The paper is organized as follows: Section 3.2 presents our valuation framework, Section 3.3 describes our dynamic program, Section 3.4 shows our numerical investigation, and Section 3.5 concludes our paper.

2 Valuation framework

The issuer credit risk is modeled using a structural model. We consider that the assets' value V_t moves according to geometric-Brownian motion

$$\frac{dV_t}{V_t} = (r - \delta_1)dt + \sigma_V dW_t^1,$$

where r is the constant risk-free rate, δ_1 is the payout rate and σ_V is its volatility. The capital structure of the issuer contains a portfolio of a straight debt and an exchangeable debt, as well as a common stock. The straight debt is a senior debt. The firm is committed to making coupon payments to the bondholders which results in collecting tax benefits. Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n, \ldots, t_N\}$ be a set of payment dates. At each date t_n, the firm is committed to pay $d_n = d_n^e + d_n^{\bar{e}}$ to its creditors, where d_n^e and $d_n^{\bar{e}}$ are the payments due to the bondholders of the exchangeable bond and to the other bondholders respectively. These payments include principal as well as coupon payments. The interest payments are noted C_n^e and $C_n^{\bar{e}}$ respectively. The last payment dates for the debts are indicated by $T^{\bar{e}} \leq T^e = T$. The tax benefits at each payment date t_n are denoted by $tb_n = tb_n^e + tb_n^{\bar{e}}$ where $tb_n^e = r^c C_n^e$, $tb_n^{\bar{e}} = r^c C_n^{\bar{e}}$, and $r^c \in [0, 1]$ is the corporate tax rate. The firm also pays bankruptcy costs in case of default proportional to the remaining assets' value, i.e. wV, where $w \in [0, 1]$ is a constant fraction.

The model assumes that the stockholders determine the time of default by maximizing the firm's total value subject to the limited liability constraint. In addition, the bondholders of the exchangeable debt have the possibility to exchange their bond for a set number of another company's shares at any date until maturity and in case of default. These shares move according to geometric-Brownian motion as follows:

$$\frac{dS_t}{S_t} = (r - \delta_2)dt + \sigma_S dW_t^2,$$

where δ_2 is a continuous dividend rate, σ_S is the shares' volatility, and (W^1, W^2) is a bivariate correlated Brownian motion with

$$\mathrm{Cor}(W_t^1, W_t^2) = \rho, \quad \text{for all } t > 0.$$

We suppose that the shares are pledged to the bondholders of the exchangeable bond, which prevents the exchange option from being lost. We assume the strict priority rule under default. The non-exchangeable bondholders are paid before the exchangeable bondholders unless the later exercise their right. We also suppose that the shares underlying the exchangeable bond are protected against bankruptcy costs.

The balance-sheet equality at time t_n is then

$$v + s + TB_n(v, s) - BC_n(v, s) = D_n^{\bar{e}}(v, s) + D_n^e(v, s) + \mathcal{E}_n(v, s), \tag{1}$$

where $V_n = v$ and $S_n = s$. The functions $TB_n(v, s)$ and $BC_n(v, s)$ are the value of the tax benefits and the value of the bankruptcy costs at date t_n, respectively. $D_n^{\bar{e}}(v, s)$ is the value of the straight bond, $D_n^e(v, s)$ is the value of the exchangeable bond, and $\mathcal{E}_n(v, s)$ is the equity value of the issuer at date t_n. These corporate securities are seen as financial derivatives on the firm's assets value and the exchangeable bond's underlying shares.

At each payment/decision date, several scenarios can happen, depending on the exchange option holder's decision (holding/exercise) and the firm's status (survival/default). We indicate by $F^{e+\bar{e}}$ the firm under holding with its overall exchangeable and non-exchangeable debt and by $F^{\bar{e}}$ the firm just after exercise with its remaining non-exchangeable debt. The scenarios at maturity are as follows:

Case 1: Holding under survival

The holding condition is

$$s \le D_N^e(v,s) = d_N^e.$$

The balance-sheet equality of $F^{e+\bar{e}}$ is

$$v + s + tb_N - 0 = d_N^e + d_N^{\bar{e}} + (v + s + tb_N - d_N),$$

which results in the survival condition

$$v + s + tb_N - d_N > 0.$$

The value functions are

$$\begin{aligned} TB_N(v,s) &= tb_N, \\ BC_N(v,s) &= 0, \\ D_N^{\bar{e}}(v,s) &= d_N^{\bar{e}}, \\ D_N^e(v,s) &= d_N^e, \\ \mathcal{E}_N(v,s) &= v + s + tb_N - d_N. \end{aligned}$$

Case 2: Holding under default

The default condition of $F^{e+\bar{e}}$ is

$$v + s + tb_N - d_N \le 0,$$

as explained in case 1, while its balance-sheet equality is

$$v + s + 0 - wv = \min((1-w)v + s, d_N^{\bar{e}}) + \max((1-w)v + s - d_N^{\bar{e}}, 0) + 0,$$

and the value functions are

$$\begin{aligned} TB_N(v,s) &= 0, \\ BC_N(v,s) &= wv, \\ D_N^{\bar{e}}(v,s) &= \min((1-w)v + s, d_N^{\bar{e}}), \\ D_N^e(v,s) &= \max((1-w)v + s - d_N^{\bar{e}}, 0), \\ \mathcal{E}_N(v,s) &= 0, \end{aligned}$$

which result in the holding condition

$$s \le D_N^e(v,s) = \max((1-w)v + s - d_N^{\bar{e}}, 0)$$

All in all, one has

$$\begin{aligned} D_N^{\bar{e}}(v,s) =& d_N^{\bar{e}}, \\ D_N^e(v,s) =& (1-w)v + s - d_N^{\bar{e}}, \end{aligned}$$

The straight bondholders are fully paid and the exchangeable bondholders are partially paid, while exercising the option is suboptimal.

Case 3: Exercise: $F^{e+\bar{e}}$ and $F^{\bar{e}}$ survive

The survival condition of $F^{e+\bar{e}}$ is

$$v + s + tb_N - d_N > 0,$$

as explained in case 1, while the exercise condition is

$$D_N^e(v,s) = s > d_N^e.$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The balance-sheet equality for the $F^{\bar{e}}$ becomes

$$v + tb_N^{\bar{e}} - 0 = d_N^{\bar{e}} + (v + tb_N^{\bar{e}} - d_N^{\bar{e}})$$

The survival condition of $F^{\bar{e}}$ is then

$$v + tb_N^{\bar{e}} - d_N^{\bar{e}} > 0.$$

The value functions are

$$\begin{aligned} TB_N(v,s) &= tb_N^{\bar{e}}, \\ BC_N(v,s) &= 0, \\ D_N^{\bar{e}}(v,s) &= d_N^{\bar{e}}, \\ D_N^{e}(v,s) &= s, \\ \mathcal{E}_N(v,s) &= v + tb_N^{\bar{e}} - d_N^{\bar{e}}. \end{aligned}$$

Case 4: Exercise: $F^{e+\bar{e}}$ survives and $F^{\bar{e}}$ defaults

The survival condition of $F^{e+\bar{e}}$ is

$$v + s + tb_N - d_N > 0,$$

as explained in case 1, while the exercise condition is

$$D_N^e(v,s) = s > d_N^e.$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The firm $F^{\bar{e}}$ defaults if

$$v + tb_N^{\bar{e}} - d_N^{\bar{e}} \leq 0,$$

as explained in case 3, and its balance-sheet equality becomes

$$v + 0 - wv = (1-w)v + 0.$$

It's worth noticing that $d_N^{\bar{e}} \geq v + tb_N^{\bar{e}} \geq v \geq (1-w)v$. The value functions are

$$\begin{aligned} TB_N(v,s) &= 0, \\ BC_N(v,s) &= wv, \\ D_N^{\bar{e}}(v,s) &= (1-w)v, \\ D_N^{e}(v,s) &= s, \\ \mathcal{E}_N(v,s) &= 0. \end{aligned}$$

Exchanging the bond provokes default.

Case 5: Exercise: $F^{e+\bar{e}}$ defaults and $F^{\bar{e}}$ survives

The firm $F^{e+\bar{e}}$ would have defaulted if the exchange option has not been exercised. The default condition is

$$v + s + tb_N - d_N \leq 0.$$

and the exercise condition is

$$D_N^e(v,s) = s > (1-w)v + s - d_N^{\bar{e}},$$

as explained in case 2. After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The balance-sheet equality of $F^{\bar{e}}$ becomes

$$v + tb_N^{\bar{e}} + 0 = d_N^{\bar{e}} + (v + tb_N^{\bar{e}} - d_N^{\bar{e}}),$$

and the survival condition is

$$v + tb_N^{\bar{e}} - d_N^{\bar{e}} > 0.$$

The value functions are

$$\begin{aligned} TB_N(v,s) &= tb_N^{\bar{e}}, \\ BC_N(v,s) &= 0, \\ D_N^{\bar{e}}(v,s) &= d_N^{\bar{e}}, \\ D_N^{e}(v,s) &= s, \\ \mathcal{E}_N(v,s) &= v + tb_N^{\bar{e}} - d_N^{\bar{e}}. \end{aligned}$$

Exercising the exchange option prevents the firm from default.

Case 6: Exercise: $F^{e+\bar{e}}$ and $F^{\bar{e}}$ default

The default condition of $F^{e+\bar{e}}$ is

$$v + s + tb_N - d_N \leq 0.$$

and the exercise condition is

$$D_N^e = s > (1-w)v + s - d_N^{\bar{e}},$$

as explained in case 2. After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The firm $F^{\bar{e}}$ defaults if

$$v + tb_N^{\bar{e}} - d_N^{\bar{e}} \leq 0,$$

as explained in case 5, and its balance-sheet equality becomes

$$v + 0 - wv = (1-w)v + 0.$$

The value functions are

$$\begin{aligned} TB_N(v,s) &= 0, \\ BC_N(v,s) &= wv, \\ D_N^{\bar{e}}(v,s) &= (1-w)v, \\ D_N^{e}(v,s) &= s, \\ \mathcal{E}_N(v,s) &= 0. \end{aligned}$$

At any decision date t_n, we apply a similar reasoning. The firm defaults if it cannot meet its financial commitments. The exchangeable bondholders will exercise the exchange option whenever the value of the underlying shares is greater than the promised payments in case of survival, or the recovered amount in case of default. The scenarios at any date t_n are as follows:

Case 1: Holding under survival

The holding condition is

$$s \leq D_n^e(v,s) = d_n^e + \mathbb{E}\left[D_{n+1}^e(V_{n+1}, S_{n+1})|F_n\right].$$

The survival condition for $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n - d_n > 0.$$

The value functions are

$$\begin{aligned} TB_n(v,s) &= tb_n + \mathbb{E}\left[TB_{n+1}(V_{n+1}, S_{n+1})|F_n\right], \\ BC_n(v,s) &= \mathbb{E}\left[BC_{n+1}(V_{n+1}, S_{n+1})|F_n\right], \\ D_n^{\bar{e}}(v,s) &= d_N^{\bar{e}} + \mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1}, S_{n+1})|F_n\right], \\ D_n^{e}(v,s) &= d_n^e + \mathbb{E}\left[D_{n+1}^{e}(V_{n+1}, S_{n+1})|F_n\right], \\ \mathcal{E}_n(v,s) &= \mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n - d_n. \end{aligned}$$

Case 2: Holding under default

The default condition of $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n-d_n\leq 0$$

The holding condition is

$$s\leq D_n^e(v,s)=(1-w)v+s-d_n^{\bar{e}}-\mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1},S_{n+1})|F_n\right].$$

The value functions are

$$\begin{aligned}
TB_n(v,s)&=0,\\
BC_n(v,s)&=wv,\\
D_n^{\bar{e}}(v,s)&=d_n^{\bar{e}}+\mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1},S_{n+1})|F_n\right],\\
D_n^{e}(v,s)&=(1-w)v+s-d_n^{\bar{e}}-\mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1},S_{n+1})|F_n\right],\\
\mathcal{E}_n(v,s)&=0.
\end{aligned}$$

Case 3: Exercise: $F^{e+\bar{e}}$ and $F^{\bar{e}}$ survive

The survival condition of $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n-d_n>0.$$

while the exercise condition is

$$D_n^e(v,s)=s>d_n^e+\mathbb{E}\left[D_{n+1}^{e}(V_{n+1},S_{n+1})|F_n\right].$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The survival condition of $F^{\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n^{\bar{e}}-d_n^{\bar{e}}>0.$$

The value functions are

$$\begin{aligned}
TB_n(v,s)&=tb_n^{\bar{e}}+\mathbb{E}\left[TB_{n+1}(V_{n+1},S_{n+1})|F_n\right],\\
BC_n(v,s)&=\mathbb{E}\left[BC_{n+1}(V_{n+1},S_{n+1})|F_n\right],\\
D_n^{\bar{e}}(v,s)&=d_n^{\bar{e}}+\mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1},S_{n+1})|F_n\right],\\
D_n^{e}(v,s)&=s,\\
\mathcal{E}_n(v,s)&=\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n^{\bar{e}}-d_n^{\bar{e}}.
\end{aligned}$$

Case 4: Exercise: $F^{e+\bar{e}}$ survives and $F^{\bar{e}}$ defaults

The survival condition of $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n-d_n>0.$$

while the exercise condition is

$$D_n^e(v,s)=s>d_n^e+\mathbb{E}\left[D_{n+1}^{e}(V_{n+1},S_{n+1})|F_n\right].$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The default condition of $F^{\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1},S_{n+1})|F_n\right]+tb_n^{\bar{e}}-d_n^{\bar{e}}\leq 0.$$

The value functions are

$$\begin{aligned}
TB_n(v,s)&=0,\\
BC_n(v,s)&=wv,\\
D_n^{\bar{e}}(v,s)&=(1-w)v,\\
D_n^{e}(v,s)&s,\\
\mathcal{E}_n(v,s)&=0.
\end{aligned}$$

Exchanging the bond provokes default.

Case 5: Exercise: $F^{e+\bar{e}}$ defaults and $F^{\bar{e}}$ survives

The firm $F^{e+\bar{e}}$ would have defaulted if the exchange option has not been exercised. The default condition of $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n - d_n \leq 0$$

The exercise condition is

$$s = D_n^e(v, s) > (1-w)v + s - d_n^{\bar{e}} - \mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1}, S_{n+1})|F_n\right].$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The survival condition of $F^{\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n^{\bar{e}} - d_n^{\bar{e}} > 0.$$

The value functions are

$$\begin{aligned}
TB_n(v,s) &= tb_n^{\bar{e}} + \mathbb{E}\left[TB_{n+1}(V_{n+1}, S_{n+1})|F_n\right], \\
BC_n(v,s) &= \mathbb{E}\left[BC_{n+1}(V_{n+1}, S_{n+1})|F_n\right], \\
D_n^{\bar{e}}(v,s) &= d_n^{\bar{e}} + \mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1}, S_{n+1})|F_n\right], \\
D_n^e(v,s) &= s, \\
\mathcal{E}_n(v,s) &= \mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n^{\bar{e}} - d_n^{\bar{e}}.
\end{aligned}$$

Exercising the exchange option prevents the firm from default.

Case 6: Exercise: $F^{e+\bar{e}}$ and $F^{\bar{e}}$ default

The default condition of $F^{e+\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n - d_n \leq 0$$

The exercise condition is

$$s = D_n^e(v, s) > (1-w)v + s - d_n^{\bar{e}} - \mathbb{E}\left[D_{n+1}^{\bar{e}}(V_{n+1}, S_{n+1})|F_n\right].$$

After exercise, the exchangeable debt no longer belongs to the firm's debt portfolio. The default condition of $F^{\bar{e}}$ is

$$\mathbb{E}\left[\mathcal{E}_{n+1}(V_{n+1}, S_{n+1})|F_n\right] + tb_n^{\bar{e}} - d_n^{\bar{e}} \leq 0.$$

The value functions are

$$\begin{aligned}
TB_n(v,s) &= 0, \\
BC_n(v,s) &= wv, \\
D_n^{\bar{e}}(v,s) &= (1-w)v, \\
D_n^e(v,s) &= s, \\
\mathcal{E}_n(v,s) &= 0.
\end{aligned}$$

This model can be studied under the assumption that the issuer of the exchangeable bond does not own the equity shares against which the debt can be exchanged. In this case, if the bondholders decide to exercise the option, the firm has to purchase the shares to deliver them. The exchange option can thus be lost if the firm is in distress and cannot deliver the shares. Under this hypothesis, the issuer's default probability increases and the exchangeable bond is less valuable than the previous case as the bondholders are taking more risk. This case is treated in Realdon (2004) but we do not consider it because, as explained in the latter, the issuance of exchangeable bonds when the issuer does not own the underlying shares is discouraged. This supports the affirmation that the issuer usually owns the shares and issues exchangeable bonds as a divesting strategy to dispose of the underlying shares in his possession.

3 Dynamic programming

Let $\mathcal{G}$ be a set of grid points $\{(a_1, b_1), (a_1, b_2), \ldots, (a_p, b_q)\}$ such that $\max(\Delta a_k, \Delta b_l) \to 0$ and $\mathbb{Q}[(V_t, r_t) \in [a_p, \infty) \times \mathbb{R}^*_+ \cup \mathbb{R}^*_+ \times [b_q, \infty)] \to 0$, when p and $q \to \infty$. Let $a_0 = b_0 = 0$ and $a_{p+1} = b_{q+1} = \infty$. The rectangle $[a_i, a_{i+1}) \times [b_j, b_{j+1})$ is designated by R_{ij}. Let $\Delta t = t_{n+1} - t_n$ a constant.

Dynamic programming acts as follows:

1. At date $t_N = T$, the value functions are known in closed form and are computed as described in Section 2.
2. At each date t_n, suppose that an approximation of each value function is available at a future decision date t_{n+1} on $\mathcal{G}$, indicated by $\tilde{f}_{n+1}(a_k, b_l)$, for $k = 1, \ldots, p$ and $l = 1, \ldots, q$, where f_n represents TB_n, BC_n, $D_n^{\bar{e}}$, D_n^e, or $\mathcal{E}_n$. Use a bilinear piecewise polynomial and interpolate each value function $\tilde{f}_{n+1}$ from $\mathcal{G}$ to the overall state space $[0, \infty)^2$ by setting:

$$\hat{f}_{n+1}(x, y) = \sum_{i=0}^{p} \sum_{j=0}^{q} \left(\alpha_{ij}^{n+1} + \beta_{ij}^{n+1} x + \gamma_{ij}^{n+1} y + \delta_{ij}^{n+1} xy\right) \mathbb{I}\left((x, y) \in R_{ij}\right),$$

   where the local coefficients of each value function f_{n+1}, α_{ij}^{n+1}, β_{ij}^{n+1}, γ_{ij}^{n+1}, and δ_{ij}^{n+1}, for $i = 0, \ldots, p$ and $j = 0, \ldots, q$, are the coefficients of the bilinear interpolation.
3. Approximate the expectation of every value function at t_n on $\mathcal{G}$:

$$\begin{aligned} &= \mathbb{E}\left[e^{-r\Delta t} \hat{f}_{n+1}(V_{t_{n+1}}, S_{t_{n+1}}) \mid (V_{t_n}, S_{t_n}) = (a_k, b_l)\right] \\ &= e^{-r\Delta t} \sum_{i,j} \left(\alpha_{ij}^{n+1} T_{klij}^{00} + \beta_{ij}^{n+1} T_{klij}^{10} + \gamma_{ij}^{n+1} T_{klij}^{01} + \delta_{ij}^{n+1} T_{klij}^{11}\right), \end{aligned}$$

   where the transition tables T^{00}, T^{10}, T^{01}, and T^{11} are defined as follows:

$$\begin{aligned} T_{klij}^{\nu\mu} \quad = \quad & \mathbb{E}\big[(V_{t_{n+1}})^{\nu} (S_{t_{n+1}})^{\mu} \mathbb{I}\left((V_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij}\right) \mid \\ & (V_{t_n}, S_{t_n}) = (a_k, b_l)\big], \quad \text{for } \nu \text{ and } \mu \in \{0, 1\}. \end{aligned}$$

   For example, T_{klij}^{00} represents the transition probability that the Markov process (V, S) moves from (a_k, b_l) at t_n and visits the rectangle R_{ij} at t_{n+1}. Closed-form solutions for these transition tables are given in Appendix A.
4. Compute the value functions at t_n on $\mathcal{G}$ as described in Section 2.
5. Go to step 2 and repeat until $n = 0$.

This procedure is time consuming as we have a two-dimensional problem. Therefore, it makes sense to try to use parallel computing to accelerate the procedure. We parallelize our dynamic program by submitting the computation tasks associated to a given number of grid points to each available CPU. The algorithm used to parallelize our dynamic program is described in details in Appendix B. This approach allows us to drastically reduce the computation time to a reasonable level.

We use the supercomputer Briarée managed by Calcul Québec and Compute Canada.[1] The code lines are written in C and compiled with GCC. We use the MPI library to access parallel computing.

4 Numerical investigation

In this section, we examine some characteristics of the exchangeable debt. For comparison purposes, we consider an exchangeable bond with the same parameters as in Realdon (2004). Considering an ordinary debt and an exchangeable bond both with a 5 year maturity and principal amount $P = 1$. The annual

[1]The operation of this supercomputer is funded by the Canada Foundation for Innovation (CFI), Ministère de l'Économie, de la Science et de l'Innovation du Québec (MESI) and the Fonds de recherche du Québec - Nature et technologies (FRQ-NT).

coupon rate is 3% for the ordinary debt and 4.7% for the exchangeable bond. Our numerical investigation presented here are based on a grid size of 300^2. A price calculation takes in average two minutes using parallel computing.

Figure 1 plots the value of the exchangeable bond as a function of the initial shares' level S_0 and the initial assets' value V_0. The exchangeable bond value is an increasing function of both variables. In fact, high values for the firm's assets represent less risky firms making the exchangeable bond more valuable. We also notice that the increase in the shares' value has more significant impact on the exchangeable bond value than the increase in the firm's assets value. In fact, as S_0 increases, the bondholders are more likely to exercise the exchange option and the exchangeable bond value increases.

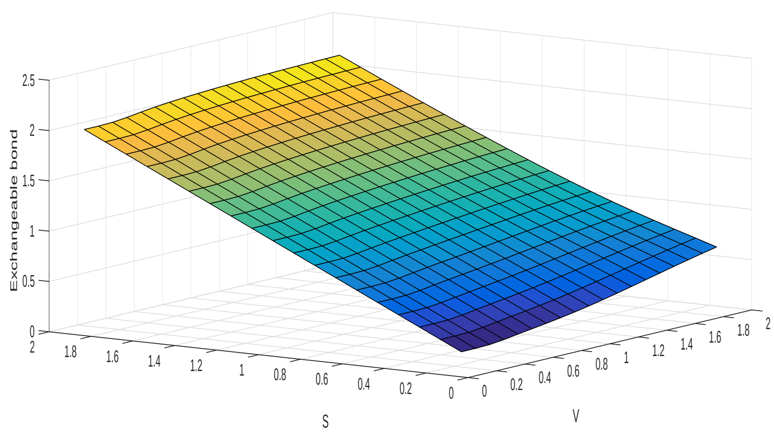

Figure 1: Exchangeable debt value as a function of the shares' value and the firm's assets value. The parameters used are $r = 0.04, \delta_1 = 0.3, \sigma_V = 02, \delta_2 = 0, \sigma_S = 0.3, \rho = 0, w = 0.2,$ **and** $r^c = 0.35$.

Figure 2 presents the exchangeable bond value as a function of the initial shares' value S_0 as the volatility of the firm's assets σ_V is changed. The exchangeable bond value decreases when the latter increases since the firm is more risky. For low values of the shares, the exchangeable bond value is more sensitive to changes in the firm risk-level; when the shares' value is low, the exchange option is less likely to be used and the bondholders are more exposed to the issuer risk.

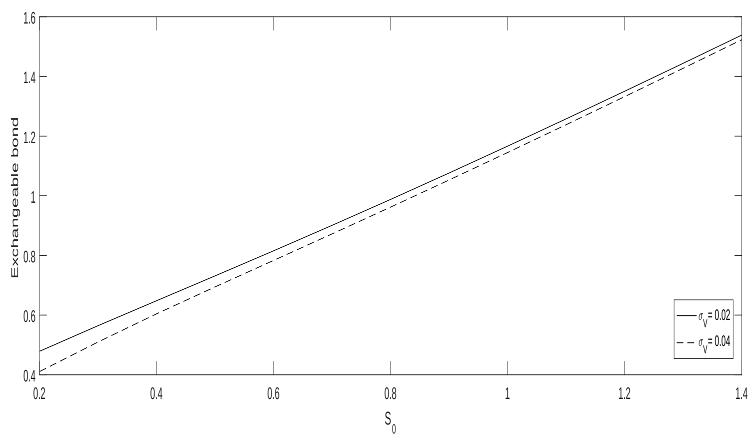

Figure 2: Exchangeable debt value as a function of the shares' value as the assets's volatility is changed. The parameters used are $r = 0.04, V_0 = 1.5, \delta_1 = 0.3, \delta_2 = 0, \sigma_s = 0.3, \rho = 0, w = 0.2,$ **and** $r^c = 0.35$.

Figure 3 illustrates that the exchangeable bond increases in the shares' volatility σ_S as the exchange option becomes more valuable. Figure 4 shows that the exchangeable bond rises when correlation rises. In fact, as explained by Realdon (2004), for negative values of ρ, high values of the issuer's assets are most likely associated to low values of the underlying shares, and vice versa. This situation is most valuable for the exchangeable bond. As correlation becomes positive, high values of the issuer's assets are most likely associated with high values of the underlying shares, and vice versa. The first scenario is beneficial to the bondholders, but the second drives down the exchangeable bond value. Besides, increasing the the proportional bankruptcy costs and the nominal amount of the issuer's other outstanding debt decrease the value of the exchangeable bond as it increases the the loss given default.

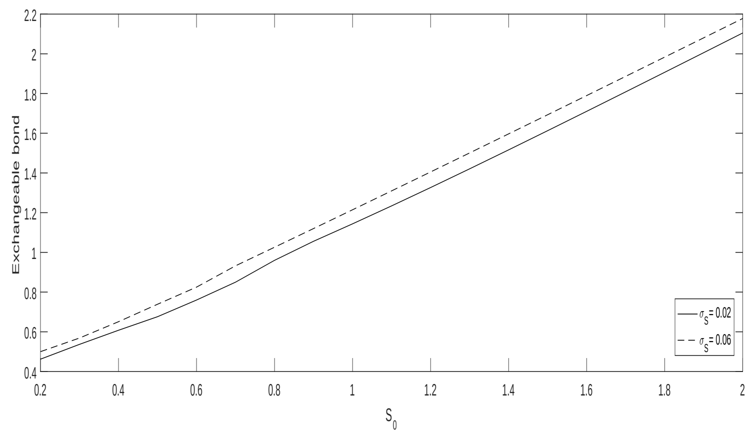

Figure 3: Exchangeable debt value as a function of the shares' value as the shares' volatility is changed. The parameters used are $r = 0.04, V_0 = 1.5, \delta_1 = 0.3, \sigma_V = 0.2, \delta_2 = 0, \rho = 0, w = 0.2,$ **and** $r^c = 0.35$.

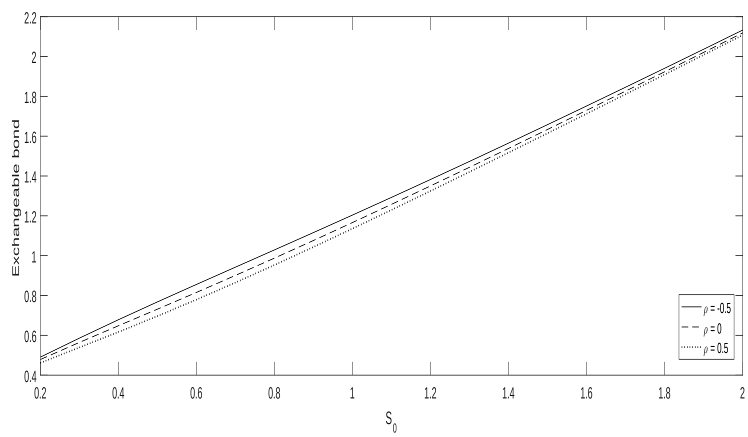

Figure 4: Exchangeable debt value as a function of the shares' value as the correlation is changed. The parameters used are $r = 0.04, \sigma_V = 0.2, \delta 1 = 0.3, V_0 = 1.5, \sigma_S = 0.3, \delta_2 = 0, w = 0.2,$ **and** $r^c = 0.35$.

5 Conclusion

In this paper, we propose a valuation framework for a hybrid-form of convertible debt, namely the exchangeable bond. This structured contract continues to gain popularity in corporate finance but is still less studied in terms of valuation purposes. Hence, we propose a general structural model for valuing exchangeable bonds in a setting that accounts for flexible debt structures, presence of bankruptcy costs and tax benefits. The model is solved using two-dimensional dynamic programming coupled with finite elements and parallel computing. The relevance of our methodology is that it can accommodate other styles of two-dimensional structured financial contracts such as reverse convertible bonds.

Appendix A Transition parameters

The transition parameters $T_{klij}^{\nu\mu}$ for ν and $\mu \in \{0,1\}$, $k \in \{1,\ldots,p\}$, $l \in \{1,\ldots,q\}$, $i \in \{0,\ldots,p\}$, and $j \in \{0,\ldots,q\}$ are calculated as follows:

$$\begin{aligned}
T_{klij}^{00} &= \mathbb{E}^*\left[\mathbb{I}\left((V_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, S_{t_n}) = (a_k, b_l)\right] \\
&= \mathbb{Q}\left[(V_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij} \mid (V_{t_n}, S_{t_n}) = (a_k, b_l)\right] \\
&= \int_{x_{k,i}}^{x_{k,i+1}} \int_{y_{l,j}}^{y_{l,j+1}} \phi(z_1, z_2, \rho) dz_1 dz_2 \\
&= \Phi(x_{k,i+1}, y_{l,j+1}, \rho) - \Phi(x_{k,i}, y_{l,j+1}, \rho) - \Phi(x_{k,i+1}, y_{l,j}, \rho) + \Phi(x_{k,i}, y_{l,j}, \rho),
\end{aligned}$$

where

$$\begin{aligned}
x_{k,i} &= \left(\log\left(a_i/a_k\right) - \left(r - d_1 - \sigma_V^2/2\right)\Delta t\right) / \left(\sigma_V \sqrt{\Delta t}\right) \\
y_{l,j} &= \left(\log\left(b_j/b_l\right) - \left(r - d_2 - \sigma_S^2/2\right)\Delta t\right) / \left(\sigma_S \sqrt{\Delta t}\right).
\end{aligned}$$

The functions $\phi(\cdot,\cdot,\rho)$ and $\Phi(\cdot,\cdot,\rho)$ are respectively the density and the cumulative density functions of the bivariate standard normal distribution with correlation coefficient ρ. The function $\Phi(\cdot,\cdot,\rho)$ is computed according to Genz (2004).

$$\begin{aligned}
T_{klij}^{10} &= \mathbb{E}^*\left[V_{t_{n+1}}\mathbb{I}\left((V_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, S_{t_n}) = (a_k, b_l)\right] \\
&= \int_{x_{k,i}}^{x_{k,i+1}} \int_{y_{l,j}}^{y_{l,j+1}} a_k \exp\left((r - d_1 - \sigma_V^2/2)\Delta t + \sigma_V \sqrt{\Delta t} z_1\right)\phi(z_1, z_2, \rho) dz_1 dz_2 \\
&= w_k^1 \int_{x_{k,i}-\sigma_V\sqrt{\Delta t}}^{x_{k,i+1}-\sigma_V\sqrt{\Delta t}} \int_{y_{l,j}-\rho\sigma_V\sqrt{\Delta t}}^{y_{l,j+1}-\rho\sigma_V\sqrt{\Delta t}} \phi(u_1, u_2, \rho) du_1 du_2 \\
&= w_k^1\big[\Phi(x_{k,i+1} - \sigma_V\sqrt{\Delta t}, y_{l,j+1} - \rho\sigma_V\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i} - \sigma_V\sqrt{\Delta t}, y_{l,j+1} - \rho\sigma_V\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i+1} - \sigma_V\sqrt{\Delta t}, y_{l,j} - \rho\sigma_V\sqrt{\Delta t}, \rho) + \\
&\quad \Phi(x_{k,i} - \sigma_V\sqrt{\Delta t}, y_{l,j} - \rho\sigma_V\sqrt{\Delta t}, \rho)\big],
\end{aligned}$$

where $w_k^1 = a_k \exp\left((r - d_1 - \sigma_V^2/2)\Delta t + \sigma_V^2 \Delta t/2\right)$.

$$\begin{aligned}
T_{klij}^{01} &= \mathbb{E}^*\left[S_{t_{n+1}}\mathbb{I}\left((S_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij}\right) \mid (S_{t_n}, S_{t_n}) = (a_k, b_l)\right] \\
&= \int_{x_{k,i}}^{x_{k,i+1}} \int_{y_{l,j}}^{y_{l,j+1}} b_l \exp\left((r - d_2 - \sigma_S^2/2)\Delta t + \sigma_S \sqrt{\Delta t} z_2\right)\phi(z_1, z_2, \rho) dz_1 dz_2 \\
&= w_l^2 \int_{x_{k,i}-\rho\sigma_S\sqrt{\Delta t}}^{x_{k,i+1}-\rho\sigma_S\sqrt{\Delta t}} \int_{y_{l,j}-\sigma_S\sqrt{\Delta t}}^{y_{l,j+1}-\sigma_S\sqrt{\Delta t}} \phi(u_1, u_2, \rho) du_1 du_2 \\
&= w_{l}^2\big[\Phi(x_{k,i+1} - \rho\sigma_S\sqrt{\Delta t}, y_{l,j+1} - \sigma_S\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i} - \rho\sigma_S\sqrt{\Delta t}, y_{l,j+1} - \sigma_S\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i+1} - \rho\sigma_S\sqrt{\Delta t}, y_{l,j} - \sigma_S\sqrt{\Delta t}, \rho) + \\
&\quad \Phi(x_{k,i} - \rho\sigma_S\sqrt{\Delta t}, y_{l,j} - \sigma_S\sqrt{\Delta t}, \rho)\big],
\end{aligned}$$

where $w_l^2 = b_l \exp\big((r - d_2 - \sigma_S^2/2)\Delta t + \sigma_S^2 \Delta t/2\big)$.

$$
\begin{aligned}
T_{klij}^{11} &= \mathbb{E}^* \Big[V_{t_{n+1}} S_{t_{n+1}} \mathbb{I}\big((V_{t_{n+1}}, S_{t_{n+1}}) \in R_{ij}\big) \mid (V_{t_n}, S_{t_n}) = (a_k, b_l) \Big] \\
&= \int_{x_{k,i}}^{x_{k,i+1}} \int_{y_{l,j}}^{y_{l,j+1}} a_k \exp\big((r - d_1 - \sigma_V^2/2)\Delta t + \sigma_V \sqrt{\Delta t} z_1\big) \times \\
&\quad b_l \exp\big((r - d_2 - \sigma_S^2/2)\Delta t + \sigma_S \sqrt{\Delta t} z_2\big) \phi(z_1, z_2, \rho) dz_1 dz_2 \\
&= w_k^1 w_l^2 \exp(\rho \sigma_V \sigma_S \Delta t) \times \\
&\quad \int_{x_{k,i} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}}^{x_{k,i+1} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}} \int_{y_{l,j} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}}^{y_{l,j+1} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}} \phi(u_1, u_2, \rho) du_1 du_2 \\
&= w_k^1 w_l^2 \exp(\rho \sigma_V \sigma_S \Delta t) \times \\
&\quad \big[\Phi(x_{k,i+1} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}, y_{l,j+1} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}, y_{l,j+1} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}, \rho) - \\
&\quad \Phi(x_{k,i+1} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}, y_{l,j} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}, \rho) + \\
&\quad \Phi(x_{k,i} - (\sigma_V + \rho\sigma_S)\sqrt{\Delta t}, y_{l,j} - (\rho\sigma_V + \sigma_S)\sqrt{\Delta t}, \rho)\big].
\end{aligned}
$$

Appendix B Parallel computing algorithm

Parallel computing uses multiple central processing units (CPUs) simultaneously to speed-up complex computations. The Message Passing Interface (MPI) library allows the computing process to exchange information between the running CPU environments in order to achieve a given job. Each CPU has access to a certain memory space. MPI requires case-sensitive programming changes from the serial code to its parallel version.

The easiest way to parallelize DP is to submit the computation tasks associated to a given grid point (a_k, b_l), for $k = 1, \dots, p$ and $l = 1, \dots, q$, to a single CPU. Our parallel code acts as follows.

1. This single CPU computes once and locally stores the overall grid points (a_i, b_j) and each value function values $f_N(a_i, b_j)$, for $i = 1, \dots, p$ and $j = 1, \dots, q$.
2. It also computes once and locally stores the $4 \times (p+1)(q+1)$ transition parameters T_{klij}^{00}, T_{klij}^{10}, T_{klij}^{01}, and T_{klij}^{11}, for $i = 0, \dots, p$ and $j = 0, \dots, q$.
3. It computes and stores at step $n+1$ the local coefficients α_{ij}^{n+1}, β_{ij}^{n+1}, γ_{ij}^{n+1}, and δ_{ij}^{n+1}, for each value function f_{n+1}, for $i = 0, \dots, p$ and $j = 0, \dots, q$.
4. It computes and stores at step n every value function $\tilde{f}_n(a_k, b_l)$.
5. The same CPU exports $\tilde{f}_n(a_k, b_l)$ to a selected CPU, the so-called master CPU.
6. The master CPU collects $\tilde{f}_n(a_k, b_l)$, for $k = 1, \dots, p$ and $l = 1, \dots, q$, and sends them back to all running CPUs.
7. Go to step 3 and repeat until $n = 0$.

Since the number of CPUs available to the analyst is usually less than the grid size pq, we submit the same number of grid points to each CPU.

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