

Cospectrality of graphs with respect to distance matrices

M. Aouchiche,
P. Hansen

G-2016-127

December 2016

Cette version est mise à votre disposition conformément à la politique de libre accès aux publications des organismes subventionnaires canadiens et québécois.

Avant de citer ce rapport, veuillez visiter notre site Web (<https://www.gerad.ca/fr/papers/G-2016-127>) afin de mettre à jour vos données de référence, s'il a été publié dans une revue scientifique.

This version is available to you under the open access policy of Canadian and Quebec funding agencies.

Before citing this report, please visit our website (<https://www.gerad.ca/en/papers/G-2016-127>) to update your reference data, if it has been published in a scientific journal.

Les textes publiés dans la série des rapports de recherche *Les Cahiers du GERAD* n'engagent que la responsabilité de leurs auteurs.

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2016
– Bibliothèque et Archives Canada, 2016

The authors are exclusively responsible for the content of their research papers published in the series *Les Cahiers du GERAD*.

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Legal deposit – Bibliothèque et Archives nationales du Québec, 2016
– Library and Archives Canada, 2016

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Télec. : 514 340-5665
info@gerad.ca
www.gerad.ca

Cospectrality of graphs with respect to distance matrices

Mustapha Aouchiche^a

Pierre Hansen^a

^a GERAD & HEC Montréal, Montréal (Québec)
Canada, H3T 2A7

mustapha.aouchiche@gerad.ca
pierre.hansen@gerad.ca

December 2016

Les Cahiers du GERAD
G–2016–127

Copyright © 2016 GERAD

Abstract: The distance, distance Laplacian and distance signless Laplacian spectra of a connected graph G are the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of G . Two graphs are said to be cospectral with respect to the distance (resp. distance Laplacian or distance signless Laplacian) matrix if they share the same distance (resp. distance Laplacian or distance signless Laplacian) spectrum. If a graph G does not share its spectrum with any other graph, we say G is defined by its spectrum. In this paper we are interested in the cospectrality with respect to the three distance matrices. First, we report on a numerical study in which we looked into the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of all the connected graphs on up to 10 vertices. Then, we prove some theoretical results about what we can deduce about a graph from these spectra. Among other results we identify some of the graphs defined by their distance Laplacian or distance signless Laplacian spectra.

Keywords: Distance matrices, Laplacian, signless Laplacian, cospectrality, spectra, graph

1 Introduction and definitions

We begin by recalling some definitions. In this paper, we consider only simple, undirected and finite graphs, *i.e.*, undirected graphs on a finite number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$ the complete bipartite graph and by K_n the complete graph, each on n vertices. The graph obtained from a star S_n , $n \geq 3$, by adding an edge is well defined and here denoted by S_n^+ .

The *adjacency matrix* A of G is a 0–1 $n \times n$ -matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \dots, \lambda_n)$ the A -spectrum of G , *i.e.*, the spectrum of the adjacency matrix of G , and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. For more results about the A -spectra of graphs, see the book [14].

The matrix $L = \text{Deg} - A$, where Deg is the diagonal matrix whose diagonal entries are the degrees in G , is called *the Laplacian* of G . Denote by $(\mu_1, \mu_2, \dots, \mu_n)$ the L -spectrum of G , *i.e.*, the spectrum of the Laplacian of G , and assume that the eigenvalues are labeled such that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$. For a survey about the Laplacian matrices of graphs see [34].

The matrix $Q = \text{Deg} + A$ is called *the signless Laplacian* of G . Denote by (q_1, q_2, \dots, q_n) the Q -spectrum of G , *i.e.*, the spectrum of the signless Laplacian of G , and assume that the eigenvalues are labeled such that $q_1 \geq q_2 \geq \dots \geq q_n$. For more details about the signless Laplacian of graphs see [15, 16, 17].

Given two vertices u and v in a connected graph G , $d(u, v) = d_G(u, v)$ denotes the *distance* (the length of a shortest path) between u and v . The *Wiener index* $W(G)$ of a connected graph G is defined to be the sum of all distances in G , *i.e.*,

$$W(G) = \frac{1}{2} \sum_{u, v \in V} d(u, v).$$

The *transmission* $t(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G , *i.e.*,

$$t_v = t(v) = \sum_{u \in V} d(u, v).$$

A connected graph $G = (V, E)$ is said to be *k -transmission regular* if $t(v) = k$ for every vertex $v \in V$. The transmission regular graphs are exactly the *distance-balanced* graphs introduced in [28]. They are also called *self-median* graphs in [9].

The *distance matrix* \mathcal{D} of a connected graph G is the matrix indexed by the vertices of G where $\mathcal{D}_{i,j} = d(v_i, v_j)$, and $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j . Let $(\partial_1, \partial_2, \dots, \partial_n)$ denote the spectrum of \mathcal{D} . It is called the *distance spectrum* of the graph G . Assume that the distance eigenvalues are labeled such that $\partial_1 \geq \partial_2 \geq \dots \geq \partial_n$. For a survey and references about the distance spectra of graphs, see [4].

Similarly to the (adjacency) Laplacian $L = \text{Deg} - A$, we defined in [3] the *distance Laplacian* of a connected graph G as the matrix $\mathcal{D}^L = \text{Tr} - \mathcal{D}$, where Tr denotes the diagonal matrix of the vertex transmissions in G . Let $(\partial_1^L, \partial_2^L, \dots, \partial_n^L)$ denote the spectrum of \mathcal{D}^L and assume that the eigenvalues are labeled such that $\partial_1^L \geq \partial_2^L \geq \dots \geq \partial_n^L = 0$. We call it the *distance Laplacian spectrum* of the graph G . Some properties of the distance Laplacian eigenvalues are discussed in [1]. In [35], Nath and Paul studied the second smallest distance Laplacian eigenvalue ∂_{n-1}^L and characterized some families of graphs for which $\partial_{n-1}^L = n + 1$. They [35] also studied the distance Laplacian spectrum of the path P_n .

Also in [3], and similarly to the (adjacency) signless Laplacian $L = \text{Deg} + A$, we introduced the *distance signless Laplacian* of a connected graph G to be $\mathcal{D}^Q = \text{Tr} + \mathcal{D}$. Let $(\partial_1^Q, \partial_2^Q, \dots, \partial_n^Q)$ denote the spectrum of \mathcal{D}^Q and assume that the eigenvalues are labeled such that $\partial_1^Q \geq \partial_2^Q \geq \dots \geq \partial_n^Q$. We call it the *distance signless Laplacian spectrum* of the graph G .

For a given real number x and a matrix M , we denote $\mu_M(x)$ the multiplicity of x as an eigenvalue of M . Evidently, $\mu_M(x) = 0$ whenever x does not belong to the spectrum of M .

Graphs with the same spectrum with respect to an associated matrix M are called *cospectral* graphs with respect to M , or *M -cospectral graphs*. Two M -cospectral non-isomorphic graphs G and H are called *M -cospectral mates* or *M -mates*. If one considers more than one matrix associated to graphs, say M_1, M_2, \dots, M_k , then two graphs are said to be (M_1, M_2, \dots, M_k) -mates if they are cospectral with respect to all the matrices M_1, M_2, \dots, M_k simultaneously.

In the next section, and after a brief review on the cospectrality with respect to A , L and Q , we report on a numerical study in which we looked into the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of all the connected graphs on up to 10 vertices. To achieve our objective we generated the desired graphs using *Nauty* (a computer program for generating graphs available at <http://cs.anu.edu.au/~bdm/nauty/>) and then calculated the different spectra using the third version of AutoGraphX (AGX III) [10].

In the last section, we prove some theoretical results about what we can deduce about a graph from these spectra. Among others results we identify some of the graphs defined by their distance Laplacian or distance signless Laplacian spectra.

2 Experiments

The question “Which graphs are determined by their A -spectrum?” raised by Günthard and Primas [24] in 1956 in a paper relating spectral theory of graphs and Hückel’s theory from chemistry. Cospectrality plays an important role in isomorphism theory. Actually, it is not yet known if testing isomorphism of two graphs is a hard problem or not, while determining whether two graphs are cospectral can be done in a polynomial time. Thus checking isomorphism is done among cospectral graphs only. It was conjectured [24] that there are no A -cospectral mates. A year later, the conjecture was refuted by Collatz and Sinogowitz [12] giving two A -cospectral mates, which are in fact, the smallest such trees (see Figure 1 (a)). For the class of general graphs, the A -cospectral mates with the smallest order, first given by Cvetković [13], are illustrated in Figure 1 (b). The A -cospectral mates with smallest order among connected graphs, first given by [6], are illustrated in Figure 1 (c).

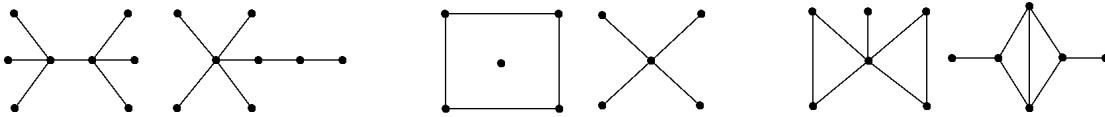


Figure 1: (a) Two A -cospectral trees.

(b) Two A -cospectral graphs.

(c) Two A -cospectral connected graphs.

Several constructions of A -cospectral mates were proposed in the literature. The first infinite family of pairs of A -cospectral mates, among trees, was constructed by Schwenk [37]. For more construction methods see for example [18, 19, 22, 23, 26, 30]. The number of graphs with a mate also attracted much attention. Schwenk [37] proved that asymptotically every tree has a mate, *i.e.*, the proportion of trees (among the class of trees) with mates tends to 1 when the order n tends to $+\infty$. Such a statement is neither proved nor refuted for the class of graphs in general. Till now, computational experiments were done on the set of all graphs on up to 12 vertices [7]. Here, we partially reproduce the table, from [7], containing the number of graphs with a mate. Recall that there are no A -cospectral graphs with less than 5 vertices.

Table 1: statistics from [7].

| Number of vertices | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|------------------------------|----|-----|------|-------|--------|----------|------------|--------------|
| Number of graphs | 34 | 156 | 1044 | 12346 | 274668 | 12005168 | 1018997864 | 165091172592 |
| Number of graphs with a mate | 2 | 10 | 110 | 1722 | 51039 | 2560606 | 215331676 | 31067572481 |

Cospectrality with respect to other graph matrices were also the subject of many publications. The L -cospectrality is studied in [25, 26, 27, 33, 34, 39]. The smallest L -cospectral graphs, with respect to the order, contain 6 vertices, and are given in Figure 2 (a). The Q -cospectrality is studied in [18, 26] (see also [15]). The Q -cospectral graphs with the smallest order are illustrated in Figure 2 (b). Another Laplacian matrix of a graph is defined by $\mathcal{L} = (\text{Diag}(\text{Deg}))^{-\frac{1}{2}} L (\text{Diag}(\text{Deg}))^{-\frac{1}{2}}$ (see [11] for more information). The \mathcal{L} -cospectrality is studied in [8], and the smallest cospectral graphs, with respect to the order, are shown in Figure 2 (c).

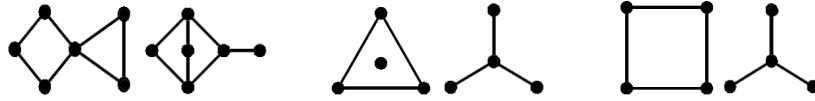


Figure 2: (a) Two L -cospectral graphs. (b) Two Q -cospectral graphs. (c) Two \mathcal{L} -cospectral connected graphs.

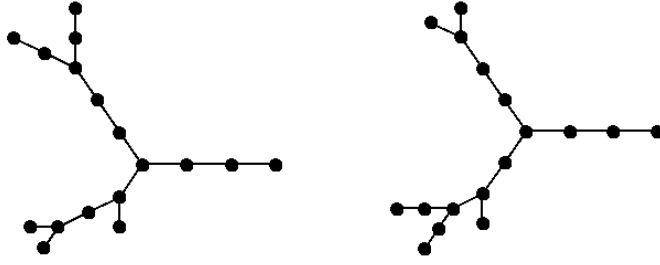
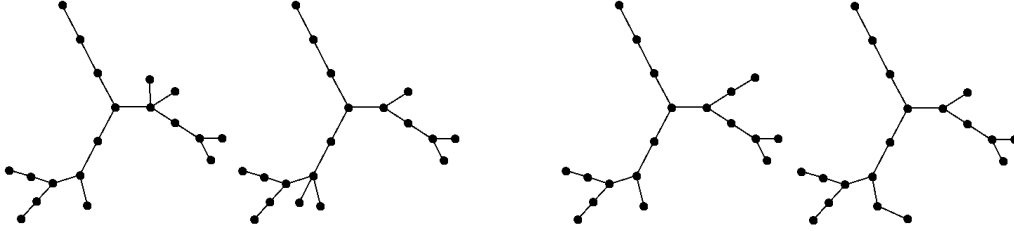
We mentioned above that Schwenk [37] proved that the proportion of trees with A -mate is asymptotically 1. A similar result about \mathcal{D} -mates was established by McKay [32]. The smallest (see Figure 3) \mathcal{D} -cospectral trees contain 17 vertices, and belong to an infinite family of pairs of \mathcal{D} -mates that can be constructed using McKay's method described in [32]. In fact, these two trees are the only \mathcal{D} -cospectral trees on 17 vertices.

Using *Nauty*, we generated all trees with at most 20 vertices and tested cospectrality with respect to \mathcal{D} , \mathcal{D}^L and \mathcal{D}^Q . Among the 123867 trees on 18 vertices, there are two pairs of \mathcal{D} -mates. Note that these pairs can be obtained using McKay's method. Among the 317955 trees on 19 vertices, there are six pairs of \mathcal{D} -mates four of which can be obtained using McKay's method. The remaining two pairs are given in Figure 4. There are 14 pairs of \mathcal{D} -mates, over all the 823065 trees on 20 vertices, nine of which can be obtained using McKay's method (see [3] for more details).

In [3], the authors enumerated all 1346023 trees on at most 20 vertices: no \mathcal{D}^L -mates and no \mathcal{D}^Q -mates were found. In the present paper, we continue the enumeration considering all connected graphs (generated using *Nauty*) on up to 10 vertices. The involved matrices and invariants were calculated using AutoGraphiX III, a conjecture making system in graph theory available at <https://www.gerad.ca/~gillesc/> (see also [10]). Unlike the case of trees, in the case of connected graphs, we found \mathcal{D}^Q -mates with only 5 vertices. They are given in Figure 5 and their \mathcal{D}^Q spectrum is $(11, 4^{(3)}, 3)$. There are 3 pairs of \mathcal{D}^Q -cospectral graphs on 6. There are no \mathcal{D} -mates or \mathcal{D}^L -mates with less than 7 vertices. On 7 vertices, there are 11 pairs of \mathcal{D} -cospectral graphs, 20 pairs of \mathcal{D}^L -cospectral graphs, and one set of 3 graphs with the same \mathcal{D}^L -spectrum. The three graphs are given in Figure 6 and their common \mathcal{D}^L -spectrum is $(11, 10^{(2)}, 8^{(2)}, 7, 0)$.

Detailed statistics about all connected graphs on up to 10 vertices are summarized in Table 2, where (as well as in the other tables) n , N , $N(\cdot)$ and $M(\cdot)$ denote the number of vertices, the number of connected graphs, the number of graphs with a mate and the number of graphs in maximal sets of mates sharing the same spectrum. The distribution of the number of graphs of order 10 according to their spectra is detailed in Table 3. The largest family of graphs on 10 vertices sharing the same \mathcal{D} -spectrum contains 21 graphs. They are illustrated in Figure 7 and their common \mathcal{D} -spectrum is $(13.1138, 0.687669, 0.17603, -0.375826, -0.882076, -1, -2.2877, -2.48748, -3.28521, -3.65925)$. The size of a largest family of graphs on 10 vertices sharing the same \mathcal{D}^L -spectrum is 16. There are two such families and the 16 graphs of one of them are given in Figure 8. Their common \mathcal{D}^L -spectrum is $(17.6591, 16.7142, 16.4689, 15.8805, 15, 14.2691, 13.5169, 12.5202, 11.971, 0)$. The largest family of graphs, shown in Figure 9, on 10 vertices sharing the same \mathcal{D}^Q -spectrum contains 9 graphs. Their common \mathcal{D}^Q -spectrum is $(26.1104, 14.1387, 13.5021, 12.518, 12.2561, 11.5155, 10.7634, 10.1476, 9.8958, 9.15244)$.

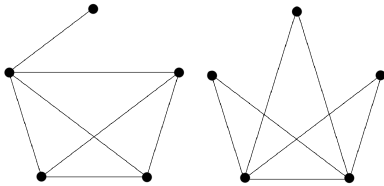
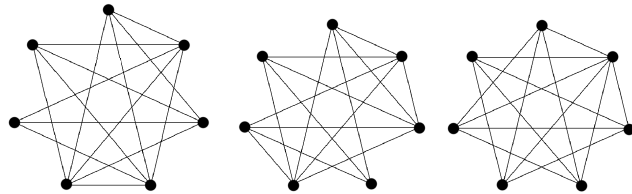
The proportions of graph mates, with respect to \mathcal{D} , \mathcal{D}^L or \mathcal{D}^Q , among the connected graphs with given order $n \in \{3, 4, \dots, 10\}$ are given in Table 4. The proportion of graphs with a \mathcal{D} -mate increases with respect to the number of vertices. The proportion of graphs with a \mathcal{D}^Q -mate decreases with respect to the number of vertices. We do not know if the tendencies remain the same for these matrices or not. For \mathcal{D}^L , the

Figure 3: The smallest \mathcal{D} -cospectral trees.Figure 4: Two pairs of \mathcal{D} -cospectral trees on 19 vertices.

behavior is different. Actually, the proportion of graphs with a \mathcal{D}^L -mate increases till $n = 9$ and decreases from 0.075754558, for $n = 9$, to 0.067242455, for $n = 10$. Besides the proportion of cospectral graphs, we can speak about the number of different spectra (which corresponds to the number of different characteristic polynomials) according to number of vertices and with respect to the three matrices under study: \mathcal{D} , \mathcal{D}^L and \mathcal{D}^Q . These numbers are given in Table 5.

Table 2: Number of cospectral graphs for given order.

| n | N | $N(\mathcal{D})$ | $M(\mathcal{D})$ | $N(\mathcal{D}^L)$ | $M(\mathcal{D}^L)$ | $N(\mathcal{D}^Q)$ | $M(\mathcal{D}^Q)$ |
|-----|----------|------------------|---|--------------------|-------------------------------|--------------------|---------------------|
| 3 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| 4 | 6 | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 | 21 | 0 | 1 | 0 | 1 | 2 | 2 |
| 6 | 112 | 0 | 1 | 0 | 1 | 6 | 2 |
| 7 | 853 | 22 | 2 | 43 | (1×) 3 | 38 | 2 |
| 8 | 11117 | 658 | (8×) 3 | 745 | (23×) 3 (3×) 4 | 453 | (11×) 3 |
| 9 | 261080 | 25058 | (663×) 3 (164×) 4 (24×) 5 (13×) 6 (1×) 7,8,10 | 19778 | (157×) 4 (12×) 6 (1×) 8 | 8168 | (152×) 3 (20×) 4 |
| 10 | 11716571 | 1389984 | See Table 3 | 787851 | See Table 3 | 319324 | See Table 3 |

Figure 5: The two \mathcal{D}^Q -cospectral graphs on 5 vertices.Figure 6: The set of 3 \mathcal{D}^L -cospectral graphs on 7 vertices.

After the study of each of the three distance matrices alone, we considered the problem of cospectrality with respect to 2 matrices simultaneously, and then with respect to the 3 together. The results are summarized in Table 6.

Table 3: Details of the size of cospectral graphs set on 10 vertices.

| Family size | \mathcal{D} | \mathcal{D}^L | \mathcal{D}^Q |
|-------------|---------------|-----------------|-----------------|
| 2 | 583922 | 345065 | 148101 |
| 3 | 46300 | 20010 | 5978 |
| 4 | 14369 | 6947 | 1138 |
| 5 | 1905 | 819 | 87 |
| 6 | 1714 | 580 | 26 |
| 7 | 288 | 138 | 4 |
| 8 | 283 | 82 | 1 |
| 9 | 45 | 30 | 1 |
| 10 | 64 | 17 | 0 |
| 11 | 33 | 6 | 0 |
| 12 | 10 | 5 | 0 |
| 13 | 2 | 5 | 0 |
| 14 | 4 | 2 | 0 |
| 15 | 3 | 1 | 0 |
| 16 | 2 | 2 | 0 |
| 21 | 1 | 0 | 0 |
| Total | 1389984 | 787851 | 319324 |

Table 4: Proportions of the graphs with a mate.

| n | \mathcal{D} | \mathcal{D}^L | \mathcal{D}^Q |
|-----|---------------|-----------------|-----------------|
| 3 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0.095238095 |
| 6 | 0 | 0 | 0.053571429 |
| 7 | 0.025791325 | 0.050410317 | 0.044548652 |
| 8 | 0.05918863 | 0.067014482 | 0.040748403 |
| 9 | 0.095978244 | 0.075754558 | 0.03128543 |
| 10 | 0.118634027 | 0.067242455 | 0.027254049 |

Table 5: Numbers of different spectra with respect to the order.

| n | # of graphs | # of \mathcal{D} -spectra | # of \mathcal{D}^L -spectra | # of \mathcal{D}^Q -spectra |
|-----|-------------|-----------------------------|-------------------------------|-------------------------------|
| 3 | 2 | 2 | 2 | 2 |
| 4 | 6 | 6 | 6 | 6 |
| 5 | 21 | 21 | 21 | 20 |
| 6 | 112 | 112 | 112 | 109 |
| 7 | 853 | 842 | 831 | 834 |
| 8 | 11117 | 10784 | 10730 | 10885 |
| 9 | 261080 | 247984 | 251007 | 256900 |
| 10 | 11716571 | 10975532 | 11302429 | 11552583 |

Table 6: Cospectrality with respect to 2 or 3 matrices.

| n | $N(\mathcal{D}, \mathcal{D}^L)$ | $M(\mathcal{D}, \mathcal{D}^L)$ | $N(\mathcal{D}, \mathcal{D}^Q)$ | $M(\mathcal{D}, \mathcal{D}^Q)$ | $N(\mathcal{D}^L, \mathcal{D}^Q)$ | $M(\mathcal{D}^L, \mathcal{D}^Q)$ | $N(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ | $M(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ |
|-------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|-----------------------------------|-----------------------------------|--|--|
| 3 – 7 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 | 0 | 1 | 0 | 1 | 90 | 2 | 0 | 1 |
| 9 | 32 | 2 | 0 | 1 | 1965 | (7×) 3 | 0 | 1 |
| 10 | 9449 | (15×) 3 | 7712 | (4×) 3 | 61909 | (343×) 3 (19×) 4 | 7622 | (4×) 3 |

Regarding the cospectrality with respect to $(\mathcal{D}, \mathcal{D}^L)$, there exist no mates with fewer than 9 vertices. There are exactly 16 pairs of mates (*i.e.*, 261064 different pairs of $(\mathcal{D}, \mathcal{D}^L)$ spectra) on 9 vertices, and 4702 pairs and 15 triplets of mates (*i.e.*, 11711841 different pairs of $(\mathcal{D}, \mathcal{D}^L)$ spectra) on 10 vertices.

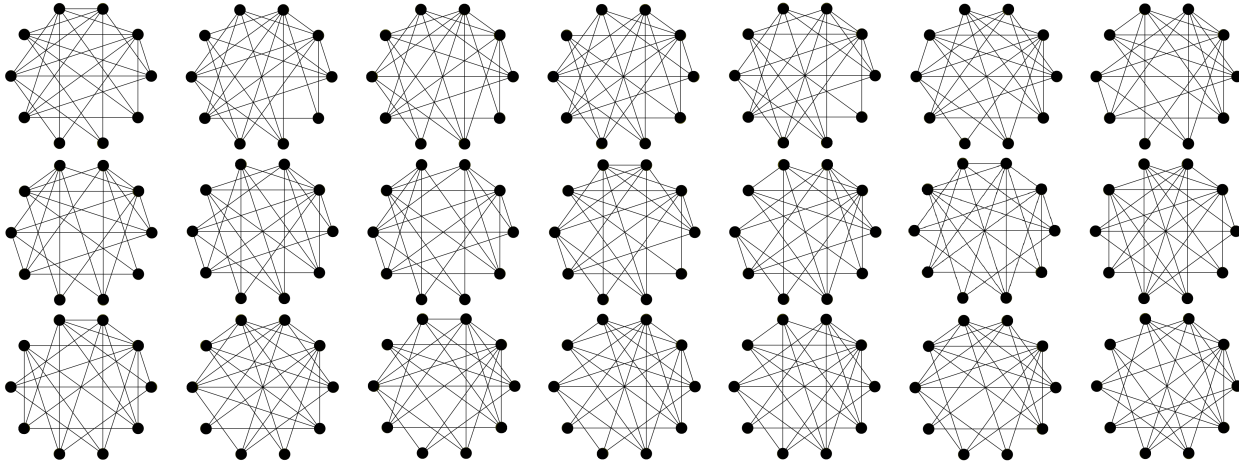


Figure 7: The 21 graphs on 10 vertices with the same \mathcal{D} -spectrum.

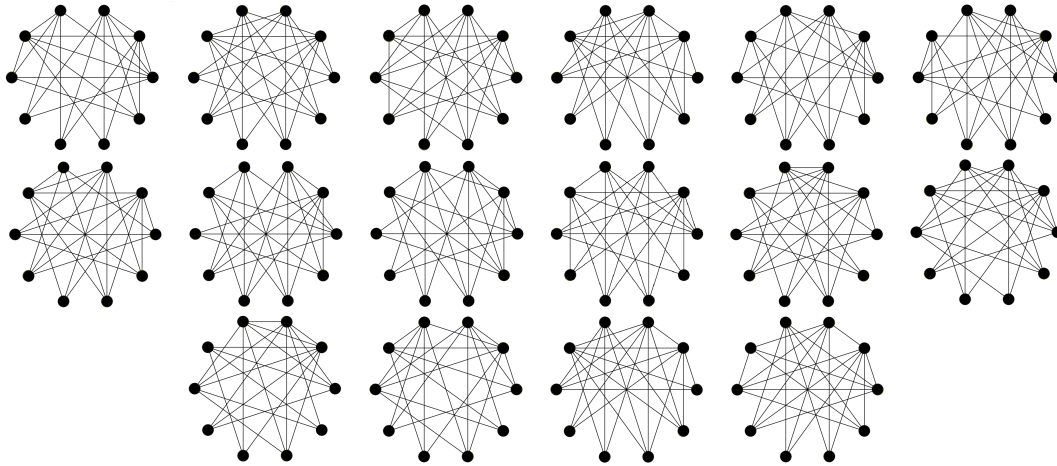


Figure 8: A set of 16 graphs on 10 vertices with the same \mathcal{D}^L -spectrum.

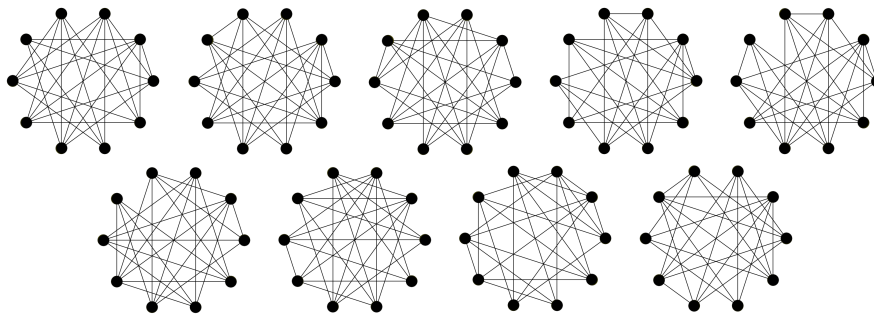


Figure 9: The 9 graphs on 10 vertices with the same \mathcal{D}^Q -spectrum.

Regarding the cospectrality with respect to $(\mathcal{D}, \mathcal{D}^Q)$, there exist no mates with fewer than 10 vertices. There are exactly 3850 pairs and 4 triplets of mates (*i.e.*, 11712713 different pairs of $(\mathcal{D}, \mathcal{D}^Q)$ spectra) on 10 vertices.

Regarding the cospectrality with respect to $(\mathcal{D}^L, \mathcal{D}^Q)$, there exist no mates with fewer than 8 vertices. There are exactly 45 pairs of mates (*i.e.*, 11072 different pairs of $(\mathcal{D}^L, \mathcal{D}^Q)$ spectra) on 8 vertices, 972 pairs

and 7 triplets of mates (*i.e.*, 260094 different pairs of $(\mathcal{D}^L, \mathcal{D}^Q)$ spectra) on 9 vertices, and 30402 pairs, 343 triplets and 19 quadruplets of mates (*i.e.*, 11686169 different pairs of $(\mathcal{D}^L, \mathcal{D}^Q)$ spectra) on 10 vertices.

Now, considering the three matrices together, we observed that there are no mates with respect to $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ with fewer than 10 vertices. There are exactly 3805 pairs and 4 triplets of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates (*i.e.*, 260094 different triplets of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ spectra) on 10 vertices. Note that the number of pairs of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates is close to that of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates, while the number of triplets of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates is equal to that of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates, and evidently, the graphs are exactly the same. The 4 triplets are given in Figure 10, each triplet in a column. The common spectra for each triplet are as follows:

First column,

\mathcal{D} – spectrum : (13.0145, 0.860806, 0.632632, -0.189468, -0.745898, -2, -2, -2.81825, -3.11491, -3.63938),
 \mathcal{D}^L – spectrum : (16.9545, 16.1402, 15.6776, 15, 15, 13.7947, 13.2357, 12.1972, 12, 0),
 \mathcal{D}^Q – spectrum : (26.0578, 13.8909, 13.4771, 13.1016, 12.216, 11, 11, 10.061, 9.77111, 9.42446);

Second column,

\mathcal{D} – spectrum : (13.2112, 0.958622, 0.386055, -0.057418, -0.981982, -1.65446, -2, -2.93591, -3.25412, -3.67196),
 \mathcal{D}^L – spectrum : (17.2429, 16.3028, 16.0878, 15, 15, 14.2351, 13.3409, 12.6972, 12.0934, 0),
 \mathcal{D}^Q – spectrum : (26.4465, 14.0719, 13.582, 13.3028, 12.3508, 11.5818, 11, 10.1736, 9.79338, 9.69722);

Third column,

\mathcal{D} – spectrum : (13.8103, 0.671932, 0.253572, -0.0667908, -1, -1.3536, -2.02291, -2.9435, -3.38921, -3.95979),
 \mathcal{D}^L – spectrum : (17.9066, 17.3028, 16.6591, 15.7649, 15, 15, 13.9122, 13.6972, 12.7571, 0),
 \mathcal{D}^Q – spectrum : (27.6396, 14.3028, 14.2052, 13.8212, 13, 12.4019, 11.6211, 10.6972, 10.3938, 9.91717);

Fourth column,

\mathcal{D} – spectrum : (14.0126, 0.63458, 0.114908, -0.186188, -1, -1, -2.2541, -2.82053, -3.6405, -3.86081),
 \mathcal{D}^L – spectrum : (18, 17.8028, 16.7643, 16.2053, 15, 15, 14.3224, 13.8598, 13.0455, 0),
 \mathcal{D}^Q – spectrum : (28.0505, 14.5514, 14.2257, 13.9277, 13, 13, 11.7822, 10.8678, 10.4903, 10.1045).

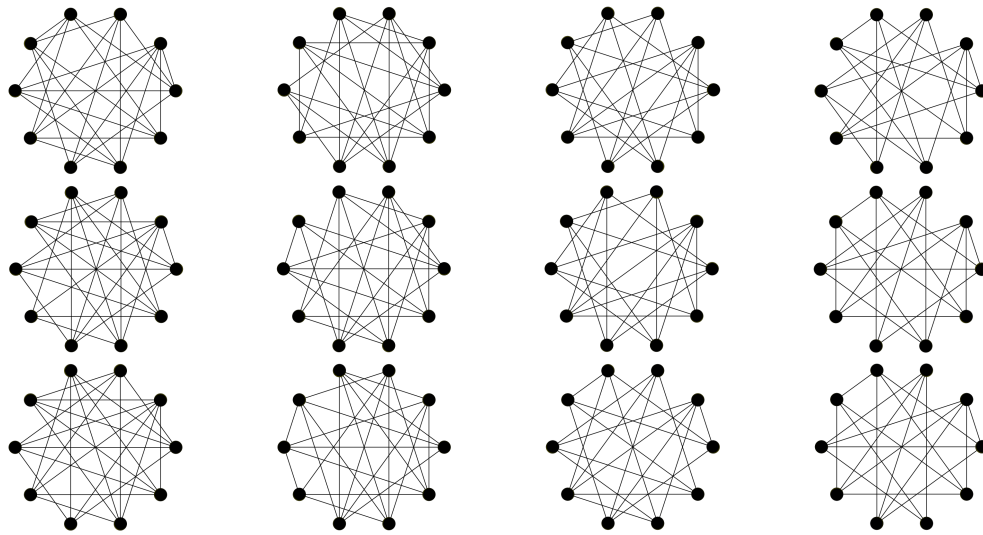


Figure 10: The 4 triplets of $(\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q)$ -mates on 10 vertices.

3 Theoretical results

In this section, we prove some theoretical results related to \mathcal{D} , \mathcal{D}^L and \mathcal{D}^Q spectra. First, we give results about what can be inferred from one of these spectra, regarding the graph, without knowing the graph. Thereafter, we prove, most often as corollaries, that some graphs are determined by their \mathcal{D} , \mathcal{D}^L or \mathcal{D}^Q spectra.

Using only its spectrum of a graph, we can get a lot of information about its structure. For instance recall some such results gathered in the following theorem.

Theorem 1 ([18]) *For the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix of a graph G , the following can be deduced from the spectrum:*

- (i) *The number of vertices.*
- (ii) *The number of edges.*
- (iii) *Whether G is regular.*
- (iv) *Whether G is regular with any fixed girth.*

For the adjacency matrix the following follows from the spectrum:

- (v) *The number of closed walks of any fixed length.*
- (vi) *Whether G is bipartite.*

For the Laplacian matrix the following follows from the spectrum:

- (vii) *The number of components.*
- (viii) *The number of spanning trees.*

We can prove similar results regarding the distance Laplacian and distance signless Laplacian matrices. First, we need to recall the following theorem from [3].

Theorem 2 ([3]) *Let G be a connected graph on n vertices. Then $\partial_{n-1}^L \geq n$ with equality holding if and only if \overline{G} is disconnected. Furthermore, the multiplicity of n as an eigenvalue of \mathcal{D}^L is one less than the number of components of \overline{G} .*

Now, considering the distance Laplacian, we have the following result.

Theorem 3 *From distance Laplacian spectrum of a connected graph G , we can deduce the following:*

- (i) *The number n of vertices of G .*
- (ii) *The Wiener index of G .*
- (iii) *The number of connected components of the complement \overline{G} .*

Proof. (i) The number n of vertices of G is equal to the number of eigenvalues.
(ii) The Wiener index of G is half the sum of the diagonal entries (transmissions) of the distance Laplacian matrix, which is the sum of the eigenvalues.
(iii) The number of connected components of the complement \overline{G} , as stated in Theorem 2, is the multiplicity of n in the spectrum plus 1. □

Using follows from Theorem 3 (i) and (ii), we can deduce following results.

Corollary 1 *The following graphs are determined by their distance Laplacian spectra:*

- a) *the complete graph K_n ;*
- b) *the graph $K_n - e$ obtained from K_n by the deletion of an edge;*
- c) *the path P_n .*

Proof. In all cases the number of vertices n is the number of eigenvalues.

- a) It was proved in [20, 21] that the Wiener index reaches its minimum, over the all connected graphs on n vertices, only for K_n with $W(K_n) = n(n-1)/2$. Thus K_n is the only graph on n vertices with

$$\sum_{i=1}^n \partial_i^L = n(n-1).$$

- b) The graph $K_n - e$ is uniquely from K_n by a removal of an edge, and this operation decreases the value of the Wiener index by 1. Any further deletion of edges strictly decreases the value of the Wiener index. Thus K_n is the only graph on n vertices with

$$\sum_{i=1}^n \partial_i^L = n(n-1) + 2 = n^2 - n + 2.$$

- c) It was proved in [20, 21] that the Wiener index reaches its maximum, over the all connected graphs on n vertices, only for P_n with $W(P_n) = (n-1)n(n+1)/6$. Thus P_n is the only graph on n vertices with

$$\sum_{i=1}^n \partial_i^L = \frac{(n-1)n(n+1)}{3}.$$

□

To state another corollary of Theorem 3, we need to prove a few lemmas. First, recall that a *comet* $Co_{n,\Delta}$ is the tree obtained from a star S_Δ and a path P_n , by joining an endpoint of the path to a pendent vertex of the star. See Figure 11 for $Co_{n,3}$.

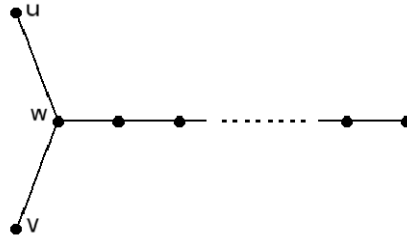


Figure 11: A comet with $\Delta = 3$.

Lemma 1 For a comet $Co_{n,\Delta}$, we have

$$W(Co_{n,\Delta}) = \frac{n^3 - 7n + 18}{6}.$$

Proof. It was proved in [20] that for any tree T on n vertices,

$$W(T) = \frac{(n-1)n(n+1)}{2} - \tau(T), \quad (1)$$

where $\tau(T)$ denotes the number of 3-subsets of non collinear vertices (which do not belong to a same path) in T . For a comet $Co_{n,3}$, any 3-subset of non collinear vertices must contain (see Figure 11) u , v and any other vertex but w . Thus $Co_{n,3}$ contains exactly $n-3$ such subsets. A substitution leads to the result. □

Lemma 2 Let T^* be the graph on n vertices for which the Wiener index attains its second largest value over all graphs of order n . Then T^* is a tree.

Proof. If T^* is not a tree, since the deletion of an edge increases the Wiener index then it must be a unicyclic graph, *i.e.*, a connected graph with $m = n$ edges. In addition, the only spanning tree in T^* must be the path. Indeed, if T is a spanning tree of T^* , then $W(T^*) < W(T) < W(P_n)$ which would be a contradiction with the fact that T^* is the graph with second largest value of W over all graphs of order n .

Now and according to [38], the graph $PK_{n,n}$ consisting of a triangle with an appended path, called path complete on n vertices and n edges, is the only graph with the maximum Wiener index over all the unicyclic graphs on n vertices. For $PK_{n,n}$, we have (see [5])

$$W(PK_{n,n}) = \frac{n^3 - 7n + 12}{6} < W(Co_{n,\Delta}) = \frac{n^3 - 7n + 18}{6}.$$

Since $Co_{n,\Delta}$ is a tree, T^* must also be a tree. □

Before the statement of the next result, we need to recall a few definitions. A vertex of degree at least 3 in a tree is called a *branching vertex*. If v is a branching vertex in a tree T , then a *branch* of T is attached to v is the connected component, that does not contain v , of the graph obtained from T by the deletion of an edge incident to v . The number of branches that can be associated to a branching vertex equals its degree.

Lemma 3 *The second largest value of the Wiener index, over all connected graphs on $n \geq 4$ vertices, is reached uniquely for the comet $Co_{n,\Delta}$.*

Proof. According to Lemma 2, it suffices to prove the result for the class of trees.

It is easy to check that the result is true for $n = 4$ and $n = 5$. So assume that $n \geq 6$, and let T be a tree on n vertices such that $T \not\cong P_n$ and $T \not\cong Co_{n,3}$. Under these conditions, T contains at least a branching vertex v to which at least three branches are attached, two of which contain at least two vertices each. Let V_1 and V_2 two branches attached at v such that $n_1 = |V_1| \geq 2$ and $n_2 = |V_2| \geq 2$. Let $V_3 = V - (V_1 \cup V_2 \cup \{v\})$, where V is the set of all vertices of T . If we take one vertex from each of V_1 , V_2 and V_3 , we get 3 non collinear vertices. So T contains at least $n_1 \cot n_2 \cdot (n - n_1 - n_2 - 1)$ 3-subsets of non collinear vertices. Note that the expression $n_1 \cot n_2 \cdot (n - n_1 - n_2 - 1)$ reaches its minimum value when n_1 and n_2 as small as possible, and then $n_1 \cdot n_2 \cdot (n - n_1 - n_2 - 1) \geq 2 \cdot 2 \cdot (n - 2 - 2 - 2) = 4(n - 5)$. Now, using the formula (1) and Lemma 1, we have

$$\begin{aligned} W(T) &= \frac{(n-1)n(n+1)}{2} - \tau(T) \leq \frac{(n-1)n(n+1)}{2} - n_1 \cdot n_2 \cdot (n - n_1 - n_2 - 1) \\ &\leq \frac{(n-1)n(n+1)}{2} - 4(n-5) < \frac{(n-1)n(n+1)}{2} - (n-3) = W(Co_{n,3}) \quad \text{for all } n \geq 6. \end{aligned}$$

□

The following corollary is an immediate consequence of Theorem 3 and Lemma 3.

Corollary 2 *The comet $Co_{n,3}$ is determined by its distance Laplacian spectrum.*

It was proved in [29] that a complete k -partite graph is determined by its distance spectrum. We prove a similar result with respect to the distance Laplacian spectrum. For that purpose, we need to recall two results from [3] and then prove lemmas.

Theorem 4 *Let G be a graph on n vertices. If $S = \{v_1, v_2, \dots, v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, then $\partial = t(v_i) = t(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\partial + 2$ is an eigenvalue of \mathcal{D}^L with multiplicity at least $p - 1$.*

We use the above theorem to prove the following lemma.

Lemma 4 *The distance Laplacian characteristic polynomial of K_{n_1, n_2, \dots, n_k} , the k -partite graph on $n = n_1 + n_2 + \dots + n_k$ vertices, is given by*

$$P_L^{K_{n_1, n_2, \dots, n_k}}(t) = t(t - n)^{k-1} \prod_{i=1}^k (t - (n + n_i))^{n_i-1}.$$

Proof. It is trivial that 0 is a distance Laplacian eigenvalue of K_{n_1, n_2, \dots, n_k} . Also, since the complement of K_{n_1, n_2, \dots, n_k} contains k connected components, it follows from Theorem 2 that n is a distance Laplacian eigenvalue of K_{n_1, n_2, \dots, n_k} with multiplicity $k - 1$.

Let v be any vertex in the independent set of K_{n_1, n_2, \dots, n_k} that contains n_i vertices. It is easy to see that

$$t(v) = n + n_i - 2.$$

Using Theorem 4 for each $i \in \{1, 2, \dots, k\}$, we conclude that $n + n_i$ is a distance Laplacian eigenvalue of K_{n_1, n_2, \dots, n_k} with multiplicity at least $n_i - 1$.

Summarizing, the distance Laplacian spectrum of K_{n_1, n_2, \dots, n_k} is

$$\left(n + n_1^{(n_1-1)}, n + n_2^{(n_2-1)}, \dots, n + n_k^{(n_k-1)}, n^{(k-1)}, 0 \right).$$

□

The following theorem from [3] is about the behaviour of the the distance Laplacian spectrum of a graph G , when an edge is deleted from G .

Theorem 5 ([3]) *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph \tilde{G} obtained from G by the deletion of an edge. Let $(\partial_1^L, \partial_2^L, \dots, \partial_n^L)$ and $(\tilde{\partial}_1^L, \tilde{\partial}_2^L, \dots, \tilde{\partial}_n^L)$ denote the distance Laplacian spectra of G and \tilde{G} respectively. Then $\tilde{\partial}_i^L \geq \partial_i^L$ for all $i = 1, \dots, n$.*

The above theorem can be improved by adding the following result.

Lemma 5 *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph \tilde{G} obtained from G by the deletion of an edge. Let $(\partial_1^L, \partial_2^L, \dots, \partial_n^L)$ and $(\tilde{\partial}_1^L, \tilde{\partial}_2^L, \dots, \tilde{\partial}_n^L)$ denote the distance Laplacian spectra of G and \tilde{G} respectively. Then, there exists $i \in \{1, 2, \dots, n-1\}$ such that $\tilde{\partial}_i^L > \partial_i^L$.*

Proof. Denote W and \tilde{W} the Wiener indices of G and \tilde{G} respectively. Since the deletion of an edge in a connected graph increases strictly some distances and lets unchanged the others (but never decreases the distance between any pair of vertices), we have

$$\sum_{j=1}^n \tilde{\partial}_j^L = 2\tilde{W} > 2W = \sum_{j=1}^n \partial_j^L.$$

Now, using Theorem 5 and the fact that $\tilde{\partial}_n^L = \partial_n^L = 0$, we get the result. □

Theorem 6 *The k -partite graph on $n = n_1 + n_2 + \dots + n_k$ vertices, K_{n_1, n_2, \dots, n_k} , is determined by its distance Laplacian spectrum.*

Proof. Let G be a graph with same distance Laplacian spectrum as K_{n_1, n_2, \dots, n_k} ,

$$\left(n + n_1^{(n_1-1)}, n + n_2^{(n_2-1)}, \dots, n + n_k^{(n_k-1)}, n^{(k-1)}, 0 \right).$$

Since n a distance Laplacian eigenvalue of G with multiplicity $k - 1$, then, according to Theorem 2, the complement \bar{G} of G contains k connected components. Thus, G is a spanning subgraph of K_{n_1, n_2, \dots, n_k} .

Now, assume that $G \not\cong K_{n_1, n_2, \dots, n_k}$. Then, G can be obtained from K_{n_1, n_2, \dots, n_k} using a sequence of edge deletions. According to Lemma 5, there exists i such that $\partial_i(G) > \partial_i(K_{n_1, n_2, \dots, n_k})$, which contradicts the assumption that G and K_{n_1, n_2, \dots, n_k} share the same distance Laplacian spectrum. \square

Now, let us turn to results related to the distance signless Laplacian.

Theorem 7 *From signless distance Laplacian spectrum of a connected graph G , we can deduce the following:*

- (i) *The number n of vertices of G .*
- (ii) *The Wiener index of G .*
- (iii) *Whether G is transmission regular.*

Proof. (i) and (ii) are proved as in Theorem 3.

(iii) It was proved in [2] that for a connected graph G ,

$$2Tr_{min} \leq 2\overline{Tr} \leq \partial_1^Q(G) \leq 2Tr_{max}$$

with equalities if and only if G is a transmission regular graph, where Tr_{min} , \overline{Tr} and Tr_{max} denote the minimum, average and maximum transmissions, respectively. Now considering the fact that the sum of the transmissions in G is exactly the sum of the distance signless Laplacian eigenvalues of G , we conclude that G is transmission regular if and only if

$$\partial_1^Q = \frac{2tr(\mathcal{D}^Q)}{n}.$$

Thus whenever the above equality holds, the graph is k -transmission regular with $k = \partial_1^Q$. \square

Using Theorem 7 (i) and (ii), we can deduce the following results.

Corollary 3 *The following graphs are determined by their distance signless Laplacian spectra:*

- a) *the complete graph K_n ;*
- b) *the graph $K_n - e$ obtained from K_n by the deletion of an edge;*
- c) *the path P_n ;*
- d) *the comet $Co_{n,3}$.*

The proof of the above corollary is similar to that of Corollary 1.

Next, we prove that the cycle C_n is determined by its signless distance Laplacian. For that purpose, we recall a result from [36] and prove a lemma, but first recall that a connected graph is said k -vertex connected if the deletion of any set of (at most) $k - 1$ vertices does not disconnect the graph.

Lemma 6 ([36]) *Let G be a 2-vertex-connected graph on n vertices with Wiener index W . Then we have*

$$W \leq \frac{n}{2} \left\lfloor \frac{n^2}{4} \right\rfloor$$

with equality if and only if G is the cycle C_n .

Lemma 7 *Let G be a transmission regular graph on $n \geq 3$ vertices. Then G is 2-vertex-connected.*

Proof. Let $G = (V, E)$ be a transmission regular graph on $n \geq 3$ vertices and assume that it contains a vertex v whose removal disconnects G . Denote by V_1 and V_2 the sets of vertices of two connected components of $G - v$ (the graph obtained from G by the deletion of v). Let $V_3 = V - (V_1 \cup V_2 \cup \{v\})$ (note that this set can be empty while V_1 and V_2 are not), and consider the partition $\{V_1, V_2, V_3, \{v\}\}$ of the vertex set of G . Let $v_1 \in V_1$ and $v_2 \in V_2$ with $vv_1, vv_2 \in E$. It is easy to see that

- if $u \in V_1$, then $d(u, v_1) \in \{d(u, v) - 1, d(u, v), d(u, v) + 1\}$ and $d(u, v_2) = d(u, v) + 1$;
- if $u \in V_2$, then $d(u, v_2) \in \{d(u, v) - 1, d(u, v), d(u, v) + 1\}$ and $d(u, v_1) = d(u, v) + 1$;
- if $u \in V_3$, then $d(u, v_1) = d(u, v) + 1$ and $d(u, v_2) = d(u, v) + 1$.

For $i = 1, 2$, let $a_i = |\{u \in V_i : d(u, v_i) = d(u, v) - 1\}|$ and $b_i = |\{u \in V_i : d(u, v_i) = d(u, v) + 1\}|$. Since $v_i \in V_i$ and $d(v_i, v_i) = d(v_i, v) - 1$. Now, let us express the transmissions of v_1 and v_2 using that of v . We have

$$\begin{aligned}
 t(v_1) &= \sum_{u \in V_1} d(u, v_1) + \sum_{u \in V_2} d(u, v_1) + \sum_{u \in V_3} d(u, v_1) + d(v, v_1) \\
 &= \sum_{u \in V_1} d(u, v) + b_1 - a_1 + \sum_{u \in V_2} d(u, v) + |V_2| + \sum_{u \in V_3} d(u, v) + |V_3| + 1 \\
 &= t(v) + b_1 - a_1 + |V_2| + |V_3| + 1;
 \end{aligned}$$

and (similarly)

$$t(v_2) = t(v) + b_2 - a_2 + |V_1| + |V_3| + 1.$$

Since the graph is assumed to be transmission regular, we have $t(v) = t(v_1) = t(v_2)$ and therefore

$$\begin{cases} |V_2| + |V_3| = a_1 - b_1 - 1 \leq a_1 - 1 < |V_1| \\ |V_1| + |V_3| = a_2 - b_2 - 1 \leq a_2 - 1 < |V_2|, \end{cases}$$

which is a contradiction. Thus G does not contain a disconnecting vertex, and therefore it is necessarily a 2-vertex-connected graph. \square

Theorem 8 *The cycle C_n is determined by its signless Laplacian distance spectrum.*

Proof. The result follows from Theorem 7, Lemma 6 and Lemma 7. In fact, a graph G on $n \geq 3$ vertices is the cycle C_n if and only if the double equality

$$\partial_1^Q(G) = \frac{2}{n} \text{tr}(\mathcal{D}^Q(G)) = 2 \left\lfloor \frac{n^2}{4} \right\rfloor$$

holds. \square

References

- [1] M. Aouchiche and P. Hansen, Some properties of the distance Laplacian eigenvalues of a graph. Czech. Math. Journal, in press.
- [2] M. Aouchiche and P. Hansen, On the distance signless Laplacian of a graph. To appear in Linear and Multilinear Algebra.
- [3] M. Aouchiche and P. Hansen, Two Laplacians for the Distance Matrix of a Graph. Linear Algebra Appl. 439 (2013) 21–33.
- [4] M. Aouchiche and P. Hansen, Distance spectra of graphs: A survey. Linear Algebra Appl. 458 (2014) 301–386.
- [5] M. Aouchiche, P. Hansen, Automated results and conjectures on average distance in graphs. Graph Theory in Paris, Trends in Mathematics VI (2007) 21–36.
- [6] G. A Baker, Drum shapes and isospectral graphs. J. Math. Phys. 7 (1966) 2238–2242.
- [7] A. E. Brouwer and E. Spence, Cospectral graphs on 12 vertices. Electronic Journal of Combinatorics 16 (2009) Paper 20.
- [8] S. Butler and J. Grout, A construction of cospectral graphs for the normalized Laplacian. Electron. J. Combin. 18 (2011) Paper 231.
- [9] S. Cabello and P. Lukšić, The complexity of obtaining a distance-balanced graph. Electron. J. Combin. 18, 1 (2011) Paper 49.
- [10] G. Caporossi, Variable neighborhood search for extremal vertices: The system AutoGraphiX-III. Les Cahiers du GERAD, G-2015-09, 2015.
- [11] F. N. K. Chung, Spectral Graph Theory. CBMS Lecture Notes, AMS, Providence, RI, 1997.

- [12] L. Collatz and U. Sinogowitz, *Spektren endlicher grafen*. Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.
- [13] D. M. Cvetković, *Graphs and their spectra* (Thesis) Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 354–356 (1971) 1–50.
- [14] D. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs—Theory and Applications*, 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.
- [15] D. Cvetković and S. K. Simić, *Towards a Spectral Theory of Graphs Based on the Signless Laplacian*. I. Publ. Inst. Math. (Beograd) 85(99) (2009) 19–33.
- [16] D. Cvetković and S. K. Simić, *Towards a Spectral Theory of Graphs Based on the Signless Laplacian*. II. Linear Algebra Appl. 432 (2010) 2257–2272.
- [17] D. Cvetković and S. K. Simić, *Towards a Spectral Theory of Graphs Based on the Signless Laplacian*. III. Appl. Anal. Discrete Math. 4 (2010) 156–166.
- [18] E.R. van Dam and W.H. Haemers, *Which graphs are determined by their spectrum?* Linear Algebra Appl. 373 (2003) 241–272.
- [19] E. R. van Dam, W. H. Haemers and J. H. Koolen, *Cospectral graphs and the generalized adjacency matrix*. Linear Algebra Appl. 423 (2007) 33–41.
- [20] J. K. Doyle and J. E. Graver, *Mean distance in a graph*. Discrete Mathematics, 17 (1977) 147–154.
- [21] R. C. Entringer, D. E. Jackson and D. A. Snyder, *Distance in graphs*. Czechoslovak Math. J. 26 (1976) 283–296.
- [22] C. D. Godsil and B. McKay, *Products of graphs and their spectra*. Lecture Notes in Math., Vol. 560, Springer, Berlin, 1976, 61–72.
- [23] C. D. Godsil and B. McKay, *Constructing cospectral graphs*. Aequationes Mathematicae 25 (1982) 257–268.
- [24] Hs. H. Günthard and H. Primas, *Zusammenhang von graphtheorie und MO-theorie von molekeln mit systemen konjugierter bindungen*. Helv. Chim. Acta 39 (1956) 1645–1653.
- [25] H. Fujii and A. Katsuda, *Isospectral graphs and isoperimetric constants*. Discrete Math. 207 (1999) 33–52.
- [26] W. H. Haemers and E. Spence, *Enumeration of cospectral graphs*. European Journal of Combinatorics 25 (2004) 199–211.
- [27] L. Halbeisen and N. Hungerbühler, *Generation of isospectral graphs*. J. Graph Theory 31 (1999) 255–265.
- [28] K. Handa, *Bipartite graphs with balanced (a, b) -partitions*. Ars Combin. 51 (1999) 113–119.
- [29] Y.-L. Jin and X.-D. Zhang, *Complete multipartite graphs are determined by their distance spectra*. Linear Algebra Appl. 448 (2014) 285–291.
- [30] C.R. Johnson and M. Newman, *A note on cospectral graphs*. J. Combin. Theory B 28 (1980) 96–103.
- [31] H. Lin, Y. Hong, J. Wang and J. Shu, *On the distance spectrum of graphs*. Linear Algebra Appl. 439 (2013) 1662–1669.
- [32] B. D. McKay, *On the spectral characterisation of trees*. Ars Combinatoria 3 (1977) 219–232.
- [33] R. Merris, *Large families of Laplacian isospectral graphs*. Linear and Multilinear Algebra 43 (1997) 201–205.
- [34] R. Merris, *Laplacian matrices of graphs: a survey*. Linear Algebra Appl. 197/198 (1994) 143–176.
- [35] M. Nath and S. Paul, *On the distance Laplacian spectrum of graphs*. Submitted for publication.
- [36] J. Plesník, *On the sum of all distances in a graph or digraph*. Journal of Graph Theory, 8 (1984) 1–21.
- [37] A. J. Schwenk, *Almost all trees are cospectral*. In: F. Harary (Ed.), *New Directions in the Theory of Graphs*, Academic Press, New York, 1973, 275–307.
- [38] L. Soltés, *Transmission in Graphs: a Bound and Vertex Removing*. Math. Slovaca 41 (1991) 11–16.
- [39] J. Tan, *On isospectral graphs*. Interdisciplinary Information Sciences 4 (1998) 117–124.