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Abstract: In this paper, we summarize some properties of the Cartesian product of graphs related to degree and distance-based invariants. Then, we investigate how much a single edge or vertex removal in the Cartesian product of two connected graphs impacts: the distance between any pair of nodes, the average distance, and the diameter in the remaining graph.

Key Words: Cartesian product graphs, distance-based invariants, edge removal impact, vertex removal impact.

1   Introduction

The Cartesian product of graphs was introduced by G. Sabidussi [1, 2] in 1957, and it has been studied since 1972 in the context of communication networks [3, 4]. Cartesian product graphs are well suited for network design and analysis, regarding scalability, performance, and fault-tolerance [5], due to the following properties. The Cartesian product of two connected graphs G and H provides a way of building a graph $G \square H$ much larger than the first ones, while keeping relatively small diameter and maximum degree. That is, whereas the order of $G \square H$ is given by the product of the orders of G and H, its diameter corresponds to the sum of the diameters of G and H, and its maximum degree corresponds to the sum of the maximum degrees of G and H. Besides this, the (edge) connectivity of $G \square H$ is never less than the sum of the (edge) connectivities of G and H [1]. Thus, whatever the (edge) connectivities of G and H, $G \square H$ will remain connected after the removal of any single edge or vertex. Such a removal can however impact distance-based invariants of the graph. How much a single edge or vertex removal in $G \square H$ impacts: the distance between any pair of nodes, the average distance, and the diameter in the remaining graph? This paper answers these questions in Section 3, after providing the needed background on the properties of the Cartesian product of graphs in Section 2. Our results are summarized in Section 4. For the uniformity of notation, Section 2 presents proofs of some known results. For an overview of Cartesian product graphs, we refer the reader to [6].

2   Background

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two connected graphs and denote $G \square H$ the Cartesian product of G and H. We denote $n_G = |V(G)|$ and $n_H = |V(H)|$ as the number of vertices of G and H, respectively, whereas $m_G = |E(G)|$ and $m_H = |E(H)|$ are their number of edges. In this paper, we will assume that $n_G \geq 3$ and $n_H \geq 3$. If $u \in V(G)$ and $v \in V(H)$, then we denote $uv \in V(G \square H)$ the vertex of $G \square H$ associated to $u \in G$ and $v \in H$. By definition of the Cartesian product, $(uv, u'v') \in E(G \square H)$ if and only if $u = u'$ and $(v, v') \in E(H)$ or $(u, u') \in E(G)$ and $v = v'$.

A first property that directly follows the Cartesian product definition is:

Property 1 *The number of vertices of $G \square H$ is $n_{G \square H} = n_G n_H$, and its number of edges is $m_{G \square H} = m_G n_H + m_H n_G$.*

To simplify the notation, we will denote by G^v the induced subgraph of $G \square H$ on the vertices $uv \quad \forall u$ and say that G^v is the copy of G associated to the vertex $v \in V(H)$. Conversely, H^u denotes the induced subgraph on the vertices $uv \quad \forall v$ and is called the copy of H associated to the vertex $u \in V(G)$.

Evidently, G^v is isomorphic to G and H^u is isomorphic to H, and the different copies of G are connected only by edges in copies of H and vice-versa.

A direct consequence of the above definition is that the degree δ_{uv} is equal to:

$$\delta_{uv} = \delta_u^G + \delta_v^H.$$

Property 2 *Let $\Delta(G)$ denote the maximum degree of G, then we have:*

$$\Delta(G \square H) = \Delta(G) + \Delta(H).$$

Proof.

$$
\begin{aligned}
\Delta(G \square H) &= \max_{uv} \delta_{uv} = \max_u \max_v (\delta_u^G + \delta_v^H) \\
&= \max_u \delta_u^G + \max_v \delta_v^H = \Delta(G) + \Delta(H).
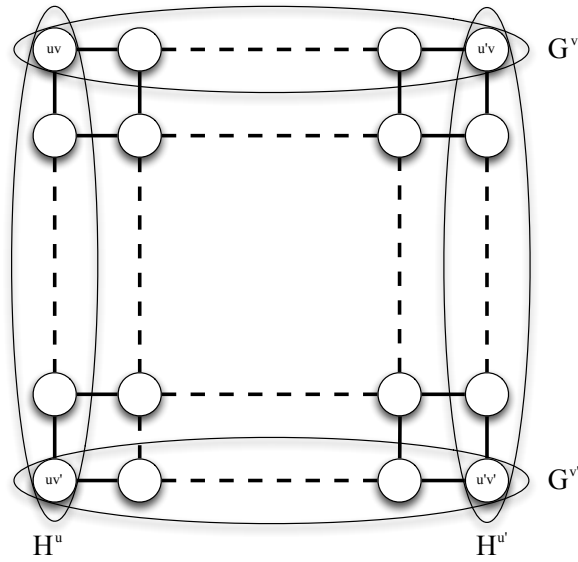\end{aligned}
$$

$\square$

Figure 1: Vertices that may be involved in a shortest path between uv and $u'v'$.

Let $d^G(u, u')$ denote the geodesic distance between vertices u and u' in G, and $d^H(v, v')$ the one between v and v' in H.

Property 3 *The distance between vertices uv and $u'v'$ in $G \square H$ is given by:*

$$d(uv, u'v') = d^G(u, u') + d^H(v, v'),$$

and there exists at least two vertex disjoint shortest paths between uv and $u'v'$ if $u \neq u'$ and $v \neq v'$.

Proof. Each induced subgraph G^v is associated to a vertex $v \in E(H)$, and each induced subgraph H^u is associated to a vertex $u \in V(G)$. Furthermore, two subgraphs G^{v_1} and G^{v_2} are adjacent if and only if $(v_1, v_2) \in E(H)$, which means that the corresponding vertices in the copies of G associated to v_1 and v_2 are adjacent. A path joining a vertex of G^{v_1} to a vertex of G^{v_2} will therefore involve copies of vertices in a path from v_1 to v_2. One shortest path from a vertex of G^v to a vertex of $G^{v'}$ will therefore involve copies of edges in a shortest path from v to v' in H. Similarly, one shortest path from a vertex of H^u to a vertex of $H^{u'}$ will involve copies of edges in a shortest path from u to u' in G. Since $uv \in V(H^u)$, $uv \in V(G^v)$, $u'v' \in V(H^{u'})$, $u'v' \in V(G^{v'})$, we have:

$$d(uv, u'v') \quad = \quad d^G(u, u') + d^H(v, v').$$

Furthermore,

$$\begin{aligned} d(uv, u'v') \quad &= \quad d(uv, uv') + d(uv', u'v') & (1) \\ &= \quad d(uv, u'v) + d(u'v, u'v'). & (2) \end{aligned}$$

Equation (1) involves vertices both from H^u and $G^{v'}$, whereas Equation (2) involves vertices of G^v and vertices of $H^{u'}$. These paths are vertex disjoint if $u \neq u'$ and $v \neq v'$ (they will only share the vertices uv and $u'v'$). $\qquad \square$

Given a shortest path $u, u_1, \ldots, u'$ between u and u', and a shortest path $v, v_1, \ldots, v'$ between v and v', the induced subgraph on the vertices that may belong to a shortest path between uv and $u'v'$ is illustrated in Figure 1.

Property 4 *Let $D(G)$ denote the diameter of G, then we have:*

$$D(G\square H) = D(G) + D(H).$$

Proof.

$$
\begin{aligned}
D(G\square H) &= \max_{uv,u'v'} d(uv, u'v') \\
&= \max_{uv,u'v'} [d(uv, u'v) + d(u, v, u'v')] \\
&= \max_{uv,u'v'} \left[d^G(v, v') + d^H(u, u') \right] \\
&= \max_v d^G(v, v') + \max_u d^H(u, u') \qquad (3) \\
&= D(G) + D(H).
\end{aligned}
$$

$\square$

Property 5 *Let $t_u^G = \sum_{u'} d^G(u, u')$ denote the transmission of the vertex $u \in G$ and t_v^H the transmission of the vertex $v \in H$, then the transmission t_{uv} is:*

$$t_{uv} = n_G t_v^H + n_H t_u^G.$$

Proof.

$$
\begin{aligned}
t_{uv} &= \sum_{u'v' \in G\square H} d(uv, u'v') \\
&= \sum_{u'v' \in G\square H} [d(uv, u'v) + d(u'v, u'v')] \\
&= \sum_{u'v' \in G\square H} \left[d^G(u, u') + d^H(v, v') \right] \\
&= \sum_{u' \in G} \sum_{v' \in H} \left[d^G(u, u') + d^H(v, v') \right] \\
&= \sum_{u' \in G} \left[n_H d^G(u, u') + \sum_{v' \in H} d^H(v, v') \right] \\
&= \sum_{u' \in G} \left[n_H d^G(u, u') + t_v^H \right] \\
&= n_G t_v^H + \sum_{u' \in G} \left[n_H d^G(u, u') \right] \\
&= n_G t_v^H + n_H t_u^G.
\end{aligned}
$$

$\square$

Let $W(G) = \sum_{u,u' \in V(G)} d^G(u, u')$ denote the Wiener index of the graph G, or the sum of the distances between pairs of vertices of G.

Theorem 1 *(Theorem 5.9 in [7]) The Wiener index of $G\square H$ is:*

$$W(G\square H) = n_H^2 W(G) + n_G^2 W(H).$$

Proof.

$$
\begin{aligned}
W(G\Box H) &= \frac{1}{2}\sum_{uv\in G\Box H} t_{uv} \\
&= \frac{1}{2}\sum_{uv\in G\Box H}\left[n_G t_v^H + n_H t_u^G\right] \\
&= \frac{1}{2}\sum_{u\in G}\sum_{v\in H}\left[n_G t_v^H + n_H t_u^G\right] \\
&= \frac{1}{2}\sum_{u\in G}\left[n_H^2 t_u^G + \sum_{v\in H} n_G t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G}\left[n_H^2 t_u^G + n_G \sum_{v\in H} t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G}\left[n_H^2 t_u^G + n_G W(H)\right] \\
&= \frac{1}{2}\sum_{u\in G} n_H^2 t_u^G + \frac{1}{2}\sum_{u\in G} n_G W(H) \\
&= n_H^2 W(G) + n_G^2 W(H).
\end{aligned}
$$

$\square$

Corollary 1 *Let $\bar{d}(G)$ denote the average distance of G, then we have:*

$$
\begin{aligned}
\bar{d}(G\Box H) &= \frac{2(n_H^2 W(G) + n_G^2 W(H))}{(n_G n_H)(n_G n_H - 1)} \\
&= \frac{2n_H W(G)}{n_G(n_G n_H - 1)} + \frac{2n_G W(H)}{n_H(n_G n_H - 1)} \\
&= \frac{n_H(n_G - 1)\bar{d}(G)}{n_G n_H - 1} + \frac{n_G(n_H - 1)\bar{d}(H)}{n_G n_H - 1} \\
&= \frac{n_H(n_G - 1)\bar{d}(G) + n_G(n_H - 1)\bar{d}(H)}{n_G n_H - 1}.
\end{aligned}
$$

$\square$

3   Main results

3.1   Distance sensitivity to an edge removal

Consider an edge $(uv, uv') \in G\Box H$, which means that $(v, v') \in H$. In this section, we will study the impact of the removal of (uv, uv') on the distance metrics presented in Section 2.

Theorem 2 *After the removal of an edge, the distance between any pair of vertices of $G\Box H$ cannot increase by more than 2.*

Proof. Suppose without lose of generality that the edge (uv, uv') is removed from $G\Box H$. First, due to the Property 3, there exists at least two distinct shortest paths between any pair of vertices $u''v''$ and $u'''v'''$ if $u'' \neq u'''$ and $v'' \neq v'''$, therefore the inequality holds in this case. The case $v'' = v''' = v$ is obvious as the shortest path between $u''v$ and $u'''v$ does not contain the edge (uv, uv'). Let us thus consider the case $u'' = u''' = u$. Suppose that (uv, uv') lies in the shortest path between uv'' and uv''' (otherwise the distance $d(uv'', uv''')$ would not be affected by the removal of (uv, uv'), and the inequality holds). Suppose that $(uv, u'v) \in E(G\Box H)$, or equivalently suppose $(u, u') \in E(G)$. We also have $(u'v, u'v') \in$

$E(G\Box H)$ as $(v,v') \in E(H)$ (since $(uv,uv') \in E(G\Box H)$). Then, the edge (uv,uv') may be replaced by the path $(uv,u'v) - (u'v,u'v') - (u'v',uv')$. There exists then a path from uv'' to uv''' whose length is $d(uv'',uv''') + 2$. □

Theorem 3 *After the removal of an edge $(uv,uv') \in E(H^u)$, the average distance of $G\Box H$ does not increase by more than:*

$$\frac{n_H}{n_G(n_G n_H - 1)}.$$

Proof. Suppose without lose of generality that the edge (uv,uv') is removed from $G\Box H$. As shown in the proof of Theorem 2, the distance between pairs of vertices may only be affected if they both belong to H^u, and the distance increase is bounded by 2. Furthermore, the number of shortest paths affected by the removal of (uv,uv') is less than or equal to $(n_H/2)^2$ [8]. Therefore, the total distance cannot increase by more than $2(n_H/2)^2 = n_H^2/2$. Hence,

$$\begin{aligned}
\bar{d}(G\Box H \setminus (uv,uv')) - \bar{d}(G\Box H) &\leq \frac{n_H^2}{n_G n_H(n_G n_H - 1)} \\
&= \frac{n_H}{n_G(n_G n_H - 1)}.
\end{aligned}$$

□

Theorem 4 *The diameter of $G\Box H$ remains unchanged after an edge removal.*

Proof. Let $D(G\Box H)$ denote the diameter of $G\Box H$. Consider uv and $u'v'$ two vertices such that $d(uv,u'v') = D(G\Box H)$. According to Equation (3), we have:

$$D(G\Box H) = \max_{v,v'} d^G(v,v') + \max_{u,u'} d^H(u,u').$$

As $n_G \geq 3$ and $n_H \geq 3$, we deduce that $u \neq u'$ and $v \neq v'$. According to Property 3, there are at least two disjoint shortest paths between uv and $u'v'$. Removing an edge will therefore leave $d(uv,u'v')$ unchanged. □

3.2   Distance sensitivity to a vertex removal

Theorem 5 *After the removal of a vertex, the distance between any pair of vertices of $G\Box H$ cannot increase by more than 2.*

Proof. Suppose the removal of the vertex uv that is on a shortest path between vertices $u'v' \neq uv$ and $u''v'' \neq uv$. Then, two edges adjacent to uv are on that shortest path. Two cases are possible:

1. One edge $(uv^i, uv) \in E(H^u)$, and the other $(uv, u^j v) \in E(G^v)$. In this case, replacing $(uv^i, uv) - (uv, u^j v)$ by $(u^j v, u^j v^i) - (u^j v^i, u^j v)$ will provide a path with exactly the same length as the previous one; and

2. Both edges belong to the same subgraph, either $(uv^i, uv) \in E(H^u)$ and $(uv, uv^j) \in E(H^u)$, or $(u^i v, uv) \in E(G^v)$ and $(uv, u^j v) \in E(G^v)$. Without loss of generality, suppose the path contains $(uv^i, uv) \in E(H^u)$ and $(uv, uv^j) \in E(H^u)$. Then it is possible to replace the edges $(uv^i, uv) - (uv, uv^j)$ by the path $(uv^i, u^k v^i) - (u^k v^i, u^k v) - (u^k v, u^k v^j) - (u^k v^j, uv^j)$, which will result in an increase of the length of the path from $u'v'$ to $u''v''$ by 2.

As a result, the distance between $u'v'$ to $u''v''$ will not increase by more than 2. □

Theorem 6 *After the removal of a vertex uv, the average distance of $G\Box H$ does not increase by more than:*

$$\frac{(n_H - 1)^2 + (n_G - 1)^2 - 2t_{uv}}{(n_G + n_H - 1)(n_G + n_H - 2)}.$$

Proof. The argument is similar to that of Theorem 3, except that paths between vertices of G^v and between vertices of H^u may both be affected, and the removal of the vertex uv must be considered as well. The total distance will not increase by more than:

$$\frac{(n_H - 1)^2 + (n_G - 1)^2}{2} - t_{uv}.$$

Hence,

$$\bar{d}(G\square H \setminus uv) - \bar{d}(G\square H) \quad \leq \quad \frac{(n_H - 1)^2 + (n_G - 1)^2 - 2t_{uv}}{(n_G + n_H - 1)(n_G + n_H - 2)}.$$

□

Theorem 7 *The diameter of $G\square H$ remains unchanged after a vertex removal.*

Proof. This proof is very similar to that of Theorem 4. Let $D(G\square H)$ denote the diameter of $G\square H$. Consider uv and $u'v'$ two vertices such that $d(uv, u'v') = D(G\square H)$. According to Equation (3), we have:

$$D(G\square H) \quad = \quad \max_{v,v'} d^G(v,v') + \max_{u,u'} d^H(u,u').$$

As $n_G \geq 3$ and $n_H \geq 3$, we deduce that $u \neq u'$ and $v \neq v'$. According to Property 3, there are at least two vertex disjoint shortest paths between uv and $u'v'$. Removing a vertex will therefore leave $d(u'v', u''v'')$ unchanged (except, of course, if that vertex is $u'v'$ or $u''v''$). Suppose now that uv is at distance $D(G\square H)$ from another vertex, say $u'v'$, then we could consider that $D(G\square H)$ may decrease by 1, but according to Property 3, we have:

$$
\begin{aligned}
d(u'v, uv') \quad &= \quad d^G(u, u') + d^H(v, v') \\
&= \quad d(uv, u'v') \\
&= \quad D(G\square H),
\end{aligned}
$$

and the distance $d(u'v, uv')$ is not affected by the removal of uv because there are at least two vertex disjoint shortest paths between $u'v$ and uv' (Property 3). □

4 Summary

Bounds were obtained for the impact of one edge or vertex removal on distance-based invariants for the Cartesian product $G\square H$ of two connected graphs G and H of order greater than 2. In particular, it was shown that, after the removal of any single edge or vertex, the distance between any pair of vertices of $G\square H$ cannot increase by more than 2, and the diameter of $G\square H$ remains unchanged. These results apply to fault tolerance analysis of communication networks modeled by Cartesian product graphs.

References

[1] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Canad. J. Math. 9 (1957) 515–525.

[2] G. Sabidussi, Graph multiplication, Mathematische Zeitschrift 72(1) (1960) 446–457.

[3] B.R. Myers, Class of optimal damage-resistant communication nets, Electronics Letters 8(11) (1972) 285–286.

[4] K.W. Cattermole, Class of communication networks with optimal connectivity, Electronics Letters 8(12) (1972) 316–318.

[5] E. Parsonage, H.X.N.R. Bowden, S. Knight, N. Falkner, M. Roughan, Generalized graph products for network design and analysis, in: Network Protocols (ICNP), 2011 19th IEEE International Conference on, 2011, pp. 79–88.

[6] W. Imrich, S. Klavzar, D.F. Rall, Topics in Graph Theory: Graphs and Their Cartesian Product, AK Peters Ltd, 2008.

[7] A. Graovac, T. Pisanski, On the Wiener index of a graph, Journal of Mathematical Chemistry 8(1) (1991) 53–62.

[8] G. Caporossi, M.H.M. Paiva, D. Vukicevic, M.E.V. Segatto, Centrality and betweenness: Vertex and edge decomposition of the Wiener index, MATCH, Communications in Mathematical and in Computer Chemistry 68(1) (2012) 293–302.