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The dominant of a matrix
Application to the classification of tropical modules

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Abstract: Tropical algebra is the algebra constructed over the tropical semifield $\mathbb{R}_{max}$. After revisiting the classification of 2-dimensional $\mathbb{R}_{max}$ semimodules, we define here the concept of dominant of a matrix and use it to show that every m-dimensional tropical module M over $\mathbb{R}_{max}$ with strongly independent basis can be embedded into $\mathbb{R}^m_{max}$. We also show that – up to matrix equivalence – the right residuate of a matrix by itself characterises the isomorphy class of the semimodule generated by its columns. The strong independence condition also yields a significant improvement to the Whitney embedding for tropical torsion modules published earlier. We also show that the strong independence of the basis of M is equivalent to the unique representation of elements of M. The results are illustrated with numerous examples.

Key Words: Idempotent semiring module, tropical module, embedding, classification.

Résumé : L'algèbre tropicale est l'algèbre construite sur le demi-corps idempotent $\mathbb{R}_{max}$. Après avoir revisité la classification des modules tropicaux de dimension 2, on introduit la notion de dominant d'une matrice, qui nous permet ensuite de montrer que, si les colonnes d'une matrice rectangulaire A sont fortement indépendantes, la classe d'équivalence de la matrice carrée $A\backslash A$ (la résiduée à droite de A par rapport à elle-même) caractérise la classe d'isomorphie du module tropical engendré par les colonnes de A. La condition d'indépendence au sens fort fournit aussi une amélioration importante du Théorème de Whitney pour les modules tropicaux publiée précédemment. On montre également que notre condition d'indépendance au sens fort est équivalente à la condition de représentation unique des éléments de M. De nombreux exemples illustrent nos résultats.

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1 Introduction

Idempotent and tropical mathematics arose from applications. Basically, we could say from the modelling and analysis of man-made systems, and from mathematical physics. Man-made systems include in particular computers, and production systems. After the celebrated paper by Kleene [15], many authors used idempotent mathematics: semigroups in language theory [23], semirings in network routing problems [10]. From the mathematical point of view, these idempotent structures have been widely investigated by Cuninghame-Green [12], and applications to control and optimization of production systems have been developed [1, 11], to mention only a few.

Some applications to automatic control can be found in [2, 3], and [14]. In [24], O. Viro arose the interest of the mathematical community to the topic by constructing a piecewise linear geometry of a special kind of polyhedra in finite dimensional Euclidean space [5, 6, 8, 9, 13].

Subsequently, the tropical approach raised increased interest in the algebraic geometry community [7, 18, 21, 22]. The reader will find a more detailed introduction to the topic in [16] and [17] (see also [4]).

The classification of modules over a principal ideal domain is given by their decomposition into a direct sum of free and torsion modules. No such result exists for tropical modules. One reason is that the direct sum decomposition of tropical modules is trivial, on the one hand, and that this classification problem received scant attention in the other. In a previous approach, we showed that although the direct sum decomposition misses the target, we can introduce the weaker concept of *semi-direct sum* [27], a concept more closely related to the algebraic structure of tropical modules, which are to idempotent abelian monoïds (i.e. semilattices) what modules are to abelian groups. Also in [27], we show that every general tropical module may be decomposed into a semi-direct sum of four sub-semimodules: free, Boolean, semi-Boolean, and torsion tropical module, respectively.

The aim of this paper is to prove a classification result for tropical modules with a strongly independent basis. Our main result (Theorem 1) shows that this problem can be completely solved when the basis satisfies a strong independence condition. To make it short, the aim of the paper is to fill the gap in the table below.

Algebraic invariants

category	specify	char.
vector space	field F, n	F^n
module	PID	free $\oplus$ torsion.
tropical module	?	?

The paper is organised as follows. In Section 2 below, we recall the basic properties of tropical modules. In Section 3, we revisit the classification of two-dimensional tropical modules, define the concept of dominant of a matrix, and use it to prove the classification theorem for finite dimensional tropical modules with strongly independent basis. Some examples are then provided in Section 4. The completed table of algebraic invariants then concludes the paper.

2 Idempotent semirings and semiring modules

The tropical semifield $S = \mathbb{R}_{max} = (\underline{\mathbb{R}}, \vee, \cdot, \underline{\mathbb{0}}, \mathbb{1})$ is defined as follows:

- $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, with $(\underline{\mathbb{R}}, \vee, \underline{\mathbb{0}})$ a commutative monoid, where $\vee$ stands for the max operator, with neutral $\underline{\mathbb{0}} = \{-\infty\}$.
- $\cdot$ stands for usual addition, with $\mathbb{1}$ as neutral (the real number 0)
- $\cdot$ distributes over $\vee$, and $\underline{\mathbb{0}}$ is also absorbing for $\cdot$, i.e.
- $\forall \sigma \in \underline{\mathbb{R}}$, $\underline{\mathbb{0}} \cdot \sigma = \sigma \cdot \underline{\mathbb{0}} = \underline{\mathbb{0}}$ ($-\infty$ is absorbing for addition)
- Since $(\mathbb{R}, \cdot, \mathbb{1}) = (\mathbb{R}, +, 0)$ is a group, this makes S a semifield.

Note that S is endowed with an order relation given by $\sigma \leq \mu \iff \sigma \vee \mu = \mu$. Since $\underline{\mathbf{0}}$ is the neutral element of $\vee$, it follows that $\underline{\mathbf{0}}$ is the bottom element of S, i.e $\forall \sigma \in S\,,\ \underline{\mathbf{0}} \leq \sigma$.

Dually, replacing $\mathbb{R}_{max}$ by $\mathbb{R}_{min}$, we define the semiring $(\overline{\mathbb{R}}, \wedge, \cdot, \overline{\mathbf{0}}, \mathbb{1})$, with top element $\overline{\mathbf{0}}$ as neutral for the min (written $\wedge$). We will also consider the extended (idempotent) semiring with bottom $\underline{\mathbf{0}}$ ($\underline{\mathbf{0}} \leq \sigma$), and top $\overline{\mathbf{0}}$ ($\sigma \leq \overline{\mathbf{0}}$) for all $\sigma \in S$.

By abuse of language, the structure $(S, \vee, \wedge, \underline{\mathbf{0}}\,,\ \overline{\mathbf{0}}\,,\ \cdot\mathbb{1})$ will also be called a semiring (or semifield, or dioïd).

Note that both $\vee$ and $\wedge$ are idempotent.

2.1 Notation

Since idempotent semirings are at the intersection of linear algebra and ordered structures, and – as will be seen in the sequel – we will often need the use of both the min and max operators using the $+$ or $\oplus$ notation (as some auhors do) would soon become awkward, whence the use of the lattice and ordered structures notation (i.e. $\vee$ for max and $\wedge$ for min) will be more convenient. Unless necessary, the notation $\cdot$ will usually be omitted.

Matrix multiplication: Let A, B be two matrices of appropriate sizes with entries $(A)_{ik}$ – written a_{ik} – (resp $(B)_{kj}$ –written b_{kj}–) in S.

Define $(A \cdot B)_{ij} = \bigvee_k a_{ik} b_{kj}$, and $(A \star B)_{ij} = \bigwedge_k a_{ik} b_{kj}$.

Also, we write A^t for the transpose of $A\,,\ A^-$ for the matrix with entries a_{ij}^{-1}, and A^{-t} for $(A^t)^- = (A^-)^t$, where a^{-1} is the multiplicative inverse of $a \in S \setminus \{\underline{\mathbf{0}}, \overline{\mathbf{0}}\}$.

2.2 Semimodules over an idempotent semiring

Left (right) $\vee$-semimodule over a semiring is defined similarly as module over a ring:

1. $(M, \vee)$ is a monoid with neutral also written $\underline{\mathbf{0}}$
2. There is a map $S \times M \ \to M$ called exterior multiplication, satisfying: $(\sigma\ ,\ x) \mapsto \sigma x$.
   i) $(\sigma \vee \mu,\, x) = (\sigma\, x \vee \mu\, x)$,
   ii) $(\sigma\ ,\ x \vee y) = (\sigma\, x \vee \sigma\, y)$
   iii) $(\underline{\mathbf{0}},\ x) = (\sigma\ ,\ \underline{\mathbf{0}}) = \underline{\mathbf{0}}$.

Note also that $x \leq y \iff x \vee y = y$ defines an order relation on $\vee$-semimodules.

Since the semiring (semifield) is idempotent, then so is the semimodule: $x \vee x = \mathbb{1}x \vee \mathbb{1}x = (\mathbb{1} \vee \mathbb{1})\, x = \mathbb{1}\, x = x$ (and similarly for $\wedge$).

The first composition laws $\vee$ and $\wedge$ in S extend to vector and matrices in a natural way. Also exterior multiplication by a scalar $\lambda \in S$ is defined componentwise (resp. entrywise) for vectors (matrices). This makes S^n and the set of matrices with entries in S, left (or right) $\vee$-semimodules over S.

Notwithstanding the fact that we consider here $\vee$-semimodules, ($\wedge$-semimodules can be defined similarly), we will however use the $\wedge$ composition whenever required by the developments of the theory..

2.3 Independence

Let M be a S semimodule, and $X = (x_i)_{i \in I} \subset M$. We say that:

$$M_X = \left\{ \bigvee_{i \in I} \lambda_i x_i | x_i \in X\ ,\ \lambda_i \in S\ ,\ \lambda_i = \underline{\mathbf{0}} \text{ except for a finite number of them} \right\}$$

is the semimodule generated by X, and X is the set of generators of M.

In [25], we considered the following concepts of independence for $X \subset S^n$.

1. $\forall Y, Z \subset X \; M_Y \bigcap M_Z = M_{Y \cap Z}$
2. $\forall Y, Z \subset X\,,\; Y \bigcap Z = \emptyset \Rightarrow M_Y \bigcap M_Z = \{\underline{\mathbf{0}}\}$
3. $\forall x \in X\,,\; x \notin M_{X \setminus \{x\}}$.

Note that $1 \Rightarrow 2 \Rightarrow 3$, while the converse does not hold, although they are equivalent in vector spaces.

In [25] (see also [19]), the proof that every finitely generated semimodule has generating set satisfying 3, and that this set is unique up to a homothetic transformation $x_i \mapsto \lambda_i x_i\,,\; x_i \in X\,,\; \lambda_i \in S$ is given.

Let $A \in \mathrm{Hom}(\mathrm{S^m}, \mathrm{S^n})$, i.e. A is a rectangular matrix of size $n \times m$ with entries in S. Clearly, the columns of A generate a finite dimensional semimodule over S. We write M_A for this subsemimodule of S^n. Also, if the columns of A are independent in the sense of 3 above, then $\dim M_A = m$. From the existence and uniqueness theorem mentioned above, follows that for any diagonal and permutation matrices of appropriate sizes D_1, D_2, P_1, P_2 the matrices A and $B = D_1 P_1 A P_2 D_2$ generate isomorphic semimodules. We write in this case $M_A \simeq M_B$ and $A \sim B$.

The problem we address in this paper is twofold. First, is there an algebraic invariant which characterises the isomorphy class of M_A? Second, what is the minimal p such that M_A is isomorphic to a subsemimodule of S^p? In [28, 29], we addressed this problem for semimodules over $S = \mathbb{R}_{\max}$ with finite entries (i.e. $\neq \underline{\mathbf{0}}$) only.

3 The classification theorem

3.1 The 2-dimensional case revisited

In [27], using the order relation in M, we showed that 2-dimensional semimodules can be classified by a 1-parameter family. More precisely, representing each generator as a column vector in S^2, the set of generators of M are necessarily $\left\{\begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}\right\}$, $\left\{\begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}\right\}$, or $\left\{\begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \tau \end{bmatrix}\right\}$, for some $\tau > \mathbb{1}$.

We revisit this classification by assuming each generator to lie in S^n, with $n \geq 2$.

Let $X = \{x, y\}$, with $x\,, y \in S^n$. We consider the following cases:

1. $\exists\, i \neq j$ s.t. $x_i = y_j = \underline{\mathbf{0}}$ (the case $i = j$ is omitted, since then $x, y \in S^{n-1}$).
2. $\exists i$ s.t. $x_i = \underline{\mathbf{0}}$, while $\forall j\,,\; y_j \neq \underline{\mathbf{0}}$.
3. $\forall i, j\,,\; x_i\,, y_j \neq \underline{\mathbf{0}}$.

The generators will be represented as the columns of a $n \times 2$ matrix A.

Case 1

$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \cdots \\ x_n & y_n \end{bmatrix}$. Up to a permutation of the rows of A we may assume that $x_2 = \underline{\mathbf{0}}\,,\; y_1 = \underline{\mathbf{0}}$. Let $D_1 = diag\left[x_1^{-1} \bigvee_{i=1}^{n} x_i \; y_2^{-1} \bigvee_{i=1}^{n} y_i \; \mathbb{1} \ldots \mathbb{1}\right], D_2 = diag\left[(\bigvee_{i=1}^{n} x_i)^{-1} \;\; (\bigvee_{i=1}^{n} y_i)^{-1}\right]$.

We have $D_1 A D_2 = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} \\ x_3(\bigvee_{i=1}^{n} x_i)^{-1} & y_3(\bigvee_{i=1}^{n} y_i)^{-1} \\ \cdots & \cdots \\ x_n(\bigvee_{i=1}^{n} x_i)^{-1} & y_n(\bigvee_{i=1}^{n} y_i)^{-1} \end{bmatrix} \sim A$, which we may rewrite as $B = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} \\ a_3 & b_3 \\ \cdots & \cdots \\ a_n & b_n \end{bmatrix}$, with $a_i, b_i \leq \mathbb{1}\,,\; i = 3, \ldots, n$.

Now $\begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \end{bmatrix} B = I_2$, and the map $\varphi\colon M_B \to \ M_{I_2}$, is an isomorphism.

Clearly φ is surjective. We show it is injective.

Let $u = \xi_1 \begin{bmatrix} \mathbb{1} \\ \underline{\mathbf{0}} \\ a_3 \\ \dots \\ a_n \end{bmatrix} \vee \xi_2 \begin{bmatrix} \underline{\mathbf{0}} \\ \mathbb{1} \\ b_3 \\ \dots \\ b_n \end{bmatrix}$, $v = \lambda_1 \begin{bmatrix} \mathbb{1} \\ \underline{\mathbf{0}} \\ a_3 \\ \dots \\ a_n \end{bmatrix} \vee \lambda_2 \begin{bmatrix} \underline{\mathbf{0}} \\ \mathbb{1} \\ b_3 \\ \dots \\ b_n \end{bmatrix}$ and assume $\varphi(u) = \varphi(v)$.

Clearly $\varphi(u) = \begin{bmatrix} \xi_1 & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \xi_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \lambda_2 \end{bmatrix} = \varphi(v) \ \Rightarrow \ \xi_i = \lambda_i \, , \ i = 1, 2.$

Case 2

$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_n & y_n \end{bmatrix}$, with $y_i \neq \underline{\mathbf{0}}\, , \ 1 \leq i \leq n$. W.l.o.g, we may assume that $x_2 = \underline{\mathbf{0}}$.

Let $\lambda = \bigvee_{i=1}^{n} x_i y_i^{-1}$. We have $x \leq \lambda y$. It is not difficult to see that $\lambda = \bigwedge\{\xi \in S \,|\, x \leq \xi y\}$ (since $x \leq \xi y \iff x_i \leq \xi y_i \, , \ 1 \leq i \leq n$). Now let $z = \lambda y$, We have $x \leq z$ with $\bigwedge\{\xi \in S \,|\, x \leq \xi z\} = \mathbb{1}$.

Moreover $\exists\, i\, [\neq 2]$ s.t. $z_i = x_i$ (for if not, then $x < z$, i.e. $\bigwedge\{\xi \in S \,|\, x \leq \xi z\} > \mathbb{1}$). Up to a permutation of the rows, we may assume that $z_1 = x_1$.

Define the diagonal matrices $D_3 = diag \left[\mathbb{1} \ x_1 z_2^{-1} \ \mathbb{1} \ \dots \ \mathbb{1} \right]$, $D_4 = diag \, [\mathbb{1} \ \lambda]$.

$A \sim B = D_3 A D_4 = \begin{bmatrix} x_1 & x_1 \\ \underline{\mathbf{0}} & x_1 \\ x_3 & z_3 \\ \dots & \dots \\ x_n & zn \end{bmatrix}$ and for $C = x_1^{-1} \begin{bmatrix} \mathbb{1} & \mathbb{1} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \end{bmatrix}$, we get $CB = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix} = E$.

We show that the map $\psi\colon M_B \to M_E$ is an isomorphism. Clearly ψ is surjective.

Let $u = \lambda_1 \begin{bmatrix} x_1 \\ \underline{\mathbf{0}} \\ x_3 \\ \dots \\ x_n \end{bmatrix} \vee \lambda_2 \begin{bmatrix} x_1 \\ x_1 \\ z_3 \\ \dots \\ z_n \end{bmatrix}$ and $v = \xi_1 \begin{bmatrix} x_1 \\ \underline{\mathbf{0}} \\ x_3 \\ \dots \\ x_n \end{bmatrix} \vee \xi_2 \begin{bmatrix} x_1 \\ x_1 \\ z_3 \\ \dots \\ z_n \end{bmatrix}$.

Assuming $\psi(u) = \psi(v)$ yields $\lambda_1 \vee \lambda_2 = \xi_1 \vee \xi_2$, and $\lambda_2 = \xi_2$.

$\lambda_1 \leq \lambda_2 \ \Rightarrow \ \psi(u) = \lambda_2 \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix}$, and $u = \begin{bmatrix} \lambda_2 x_1 \\ \lambda_2 x_1 \\ \lambda_1 x_3 \vee \lambda_2 z_3 \\ \dots \\ \lambda_1 x_n \vee \lambda_2 z_n \end{bmatrix} = \lambda_2 \begin{bmatrix} x_1 \\ x_1 \\ z_3 \\ \dots \\ z_n \end{bmatrix} = v$.

$\lambda_1 > \lambda_2 \ \Rightarrow \ \lambda_1 = \xi_1 \vee \lambda_2 \ \Rightarrow \ \xi_1 = \lambda_1$, and $u = v$.

Case 3

We first consider the case $n = 2$. Let $A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$, $D = diag(x_1^{-1} \ x_2^{-1})$. Then $DA = \begin{bmatrix} \mathbb{1} & x_1^{-1} y_1 \\ \mathbb{1} & x_2^{-1} y_2 \end{bmatrix}$.
Muliplying column 2 by $x_1 y_1^{-1}$, we get the equivalent matrix $B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & x_1 y_2 (x_2 y_1)^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \tau \end{bmatrix}$, with $\tau = x_1 y_2 (x_2 y_1)^{-1}$.

Note that if $\tau < \mathbb{1}$, then multiplying row 2 of B by τ^{-1}, followed by the permutation of the two columns of B yields an equivalent matrix with $\tau^{-1} > \mathbb{1}$.

Another point of view is that of torsion (cf. [26, 28]), which can be defined as follows. Let $\lambda_{12} = \bigwedge\{\xi \in S | x_i \leq \xi_i y_i\ , i = 1, 2\}$, and $\lambda_{21} = \bigwedge\{\xi \in S | y_i \leq \xi_i x_i\ , i = 1, 2\}$. Note that the matrix $\Lambda_A = A^t \cdot A^- = \begin{bmatrix} \mathbb{1} & \lambda_{12} \\ \lambda_{21} & \mathbb{1} \end{bmatrix}$, has the property $\lambda_{12}\lambda_{21} = \tau$, which we call the torsion of M_A. This is an intrincic invariant of M_A.

Note also that $\tau = x_1 y_2 (y_1 x_2)^{-1}$ shows some similarities with the determinant of A, hence, we may call it the semi-determinant of A. In addition, for (say) $\tau > \mathbb{1}$, we have $x_1 y_2 > y_1 x_2$, hence $x_1 y_2 \vee y_1 x_2 = x_1 y_2$.

For $n > 2$, let $A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_n & y_n \end{bmatrix}$, with $\forall i,\ x_i, y_i \neq \underline{\mathbf{0}}$.

We get $\Lambda_A = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} x_1^{-1} & y_1^{-1} \\ x_2^{-1} & y_2^{-1} \\ \dots & \dots \\ x_n^{-1} & y_n^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \bigvee\limits_{i=1}^{n} x_i y_i^{-1} \\ \bigvee\limits_{i=1}^{n} x_i^{-1} y_i & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \lambda_{12} \\ \lambda_{21} & \mathbb{1} \end{bmatrix}$.

Note that $\tau = \lambda_{12}\lambda_{21} = \bigvee\limits_{1 \leq i,j \leq n} x_i y_j (x_j y_i)^{-1}$ corresponds to the maximum of the semi-determinants of the $n(n-1)$ 2 by 2 square submatrices of A (defined up to a permutation of the two rows).

Right multiplication of A by the diagonal matrix $(x_1^{-1} \ \dots \ x_n^{-1})$, yields the equivalent matrix $\begin{bmatrix} \mathbb{1} & x_1^{-1} y_1 \\ \mathbb{1} & x_2^{-1} y_2 \\ \dots & \dots \\ \mathbb{1} & x_n^{-1} y_n \end{bmatrix}$.

Then multiplication of the 2nd column by $\bigvee\limits_{i=1}^{n} x_i y_i^{-1}$, and ordering the new column in nondecreasing order yields $B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & z_2 \\ \dots & \dots \\ \mathbb{1} & \tau \end{bmatrix} \sim A$, (for some z $[\mathbb{1} \leq z_i \leq \tau]$).

The projection map $F = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$, restricted to B yields $FB = C = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \tau \end{bmatrix}$.

In [28] we proved (through a slighly different method) that M_C is isomorphic to M_G. The interested reader may use the approach followed in Cases 1 and 2 above to show that the matrix F yields such an isomorphism.

3.2 Generalisation of the Λ matrix of a matrix

Note that, in Case 3 above, the coefficiends $\lambda_{12} = \bigvee\limits_{i=1}^{n} x_i y_i^{-1}$ and $\lambda_{21} = \bigvee\limits_{i=1}^{n} y_i x_i^{-1}$ are given by the matrix $\Lambda_A = A^t \cdot A^- = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix} \cdot \begin{bmatrix} x_1^{-1} & y_1^{-1} \\ x_2^{-1} & y_2^{-1} \\ \dots & \dots \\ x_n^{-1} & y_n^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \lambda_{12} \\ \lambda_{21} & \mathbb{1} \end{bmatrix}$. Moreover, since $x \leq \lambda_{12} y \leq \lambda_{12}\lambda_{21} x$, we have $\tau = \lambda_{12}\lambda_{21}$.

This matrix has been introduced in [28] for the case of tropical torsion modules, i.e tropical modules over $\mathbb{R}$ (no $\underline{\mathbf{0}}$).

Consider first Case 2 above.

Since $M_A \simeq M_E$, we determine what Λ_E should look like.

A coherent definition of $\underline{\mathbf{0}}^{-1}$ should be $\overline{\mathbf{0}}$. Then $E^{-} = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$ and $\Lambda_E = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} \vee \underline{\mathbf{0}}\overline{\mathbf{0}} & \mathbb{1} \\ \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$. Thus for a coherent value of $\underline{\mathbf{0}} \cdot \overline{\mathbf{0}}$ we should have $\underline{\mathbf{0}}\overline{\mathbf{0}} \leq \mathbb{1}$. Similarly, for $\overline{\mathbf{0}} \cdot \underline{\mathbf{0}}$, we should have $\overline{\mathbf{0}} \cdot \underline{\mathbf{0}} \geq \mathbb{1}$.

The following convention will be used: $\quad \mathbf{C} \quad \underline{\mathbf{0}} \cdot \overline{\mathbf{0}} = \overline{\mathbf{0}} \cdot \underline{\mathbf{0}} = \mathbb{1}$

Note that this convention extends the property $aa^{-} = \mathbb{1}$ to the cases $a = \underline{\mathbf{0}}$, and $a = \overline{\mathbf{0}}$ and amounts to: $(-\infty) + (+\infty) = (+\infty) + (-\infty) = 0$.

Convention $\mathbf{C}$ allows for the extension of the definition of $\Lambda_A = A^t \cdot A^{-}$ of [28] to the case considered here.

The values in Λ_E can be interpreted as follows:

$\lambda_{12} = \mathbb{1}$ simply states that $e_1 \leq e_2$,
$\lambda_{21} = \overline{\mathbf{0}}$ states that there is no finiite $\lambda \in S$ s.t. $e_2 \leq \lambda e_1$.

Consider now Case 1.

As above, it suffices to consider the case of I_2. We have $\Lambda_{I_2} = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \overline{\mathbf{0}} \\ \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \overline{\mathbf{0}} \\ \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$. The interpretation is similar.

3.3 The Γ matrix of a tropical matrix

Let $A \in \mathrm{Hom}(\mathrm{S^m}, \mathrm{S^n})$, and M_A the S-semimodule generated by the columns of A. It is well-known in residuation theory, that the inequation $A \cdot X \leq B$ has a maximal solution $A \backslash B$, called the right residuate of A by B, and we have $A \backslash B = A^{-t} \star B$ (cf. [3], eq. 4.82, [20], or [30]). In particular, for $B = A$, the matrix $A \backslash A$ has been defined in [29] as the Γ-matrix of A, written Γ^A.

In short, $\Gamma^A = \bigvee \{X \in \mathrm{Hom}(\mathrm{S^m}.\mathrm{S^n}) \,|\, AX = A\}$.

Writing $a_{\cdot j}$ (resp. $\gamma_{\cdot j}$) for column j of A (resp. Γ^A), we have: $\gamma_{\cdot j} = \bigvee \{x \in S^m | Ax = a_{\cdot j}\}$.

For any tropical map $\varphi \colon M \rightarrow N$, we have the congruence relation $x \simeq y \iff \varphi(x) = \varphi(y)$, and the diagram below commutes.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & N \\ M\,|\simeq \arrow[ur] & \end{tikzcd}$$

Clearly, the map $M|_{\simeq} \rightarrow Im\varphi$ is an isomorphism, and the semimodule $\mathrm{DOMINJ}_A = \{\overline{x} | x \in \mathbb{R}^m\}$ is isomorphic to $M|_{\simeq}$, where $\overline{x} = \bigvee \{y \in S^m | Ay = Ax\}$.

Let $e_1, e_2, \ldots, e_m$ stand for the canonical basis of S^m. It is easy to see that $\gamma_{\cdot j} = \overline{e}_j$, $j = 1, \ldots, m$.

Example 3.1 *We revisit Case 3 considered above.*

We have shown that $A = \begin{bmatrix} \mathbb{1} & a_1 \\ \mathbb{1} & a_2 \\ \cdots & \cdots \\ \mathbb{1} & a_n \end{bmatrix}$, *where* $\mathbb{1} = a_1 \leq a_2 \leq \cdots \leq a_n$ *is isomorphic to* $B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \tau \end{bmatrix}$, *with* $\tau = a_n$ *the torsion coefficient of A.*

Then $M_A \simeq \{\xi \in S^2 | \xi_2 \leq \xi_1 \leq \tau\xi_2\} = M_B$, *and* $\Lambda_B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \tau & \mathbb{1} \end{bmatrix}, \Gamma^B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \mathbb{1} \end{bmatrix}$.

Now $M_{\Gamma^B} = \{\xi \in S^2 | \tau^{-1}\xi_1 \leq \xi_2 \leq \xi_1\}$.

Clearly $M_B \simeq M_{\Gamma^B}$*, since* $diag(\mathbb{1}\ \tau^{-1})B = \Gamma^B$.

Note also that $B\xi = \begin{matrix} \xi_1 \vee \xi_2 \\ \xi_1 \vee \lambda\xi_2 \end{matrix} = \begin{bmatrix} \xi_1 \\ \lambda\xi_2 \end{bmatrix} \iff \xi_1 \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \vee \xi_2 \begin{bmatrix} \mathbb{1} \\ \lambda\xi_2 \end{bmatrix}$ *is non-redundant. Indeed* $\xi_1 = \xi_2$ *or* $\xi_1 = \lambda\xi_2 \Rightarrow \xi_1 \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \vee \xi_2 \begin{bmatrix} \mathbb{1} \\ \lambda\xi_2 \end{bmatrix}$ *is redundant and conversely.*

We have the following statement, which generalises Proposition 5.1 of [29] (stated for torsion tropical modules) to the case considered in this paper.

Proposition 3.1 *For an arbitrary tropical matrix* A*, we have* $\Gamma^A = \Lambda_A^-$.

Proof. We extend the proof given in [29] to the case considered in this paper. Since $\left[(\Lambda_A)^-\right]_{jk} = \left[(A^t \cdot A^-)^-\right]_{jk}$, we consider the cases:

1. $\exists\, i\, (1 \leq i \leq n)$ s.t $a_{ij} = \underline{\mathbf{0}}\, , (i \in I$, say).
2. $\exists\, i\, (1 \leq i \leq n)$ s.t $a_{ik} = \underline{\mathbf{0}}$, while $a_{ij} \neq \underline{\mathbf{0}}$.

Note that we can always assume $j \neq k$ since if $j = k$, then $\gamma_{kk} = \mathbb{1}$.

Case 1

$$\left(\left[(A^t \cdot A^-)\right]^-\right)_{jk} = (\bigvee_{i=1}^{n} a_{ij}a_{ik}^{-1})^{-1} = (\bigvee_{i=1, i \notin I}^{n} a_{ij}a_{ik}^{-1})^{-1} = \bigwedge_{i=1, i \notin I}^{n} a_{ij}^{-1}a_{ik}.$$

On the other hand, by definition, writing $(\Gamma^A)_{jk} = \gamma_{jk}$, we must have $\bigvee_j a_{ij}\gamma_{jk} = a_{ik}$, hence $\gamma_{jk} \leq a_{ik}a_{ij}^{-1}$, $i = 1, \ldots, n$, i.e. $\gamma_{jk} = \bigwedge_i a_{ik}a_{ij}^{-1}$. But $\bigwedge_i a_{ik}a_{ij}^{-1} = \bigwedge_{i, i \notin I} a_{ik}a_{ij}^{-1}$, since $\forall\, i \in I\ a_{ij}^{-1} = \overline{\mathbf{0}}$.

Case 2

Clearly, $a_{ik} = \underline{\mathbf{0}} \Rightarrow (\bigvee_{j=1}^{n} a_{ij}a_{jk}^{-1})^{-1} = \underline{\mathbf{0}}$. Similarly $\bigvee_\ell a_{i\ell}\gamma_{\ell k} = \underline{\mathbf{0}} \Rightarrow a_{i\ell}\gamma_{\ell k} = \underline{\mathbf{0}}\, , \ell = 1, \ldots, n$. Hence $\gamma_{jk} = \underline{\mathbf{0}}$, since for $\ell = j$, we have $a_{ij} \neq \underline{\mathbf{0}}$. $\square$

Example 3.2 $A = \begin{bmatrix} \underline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ 2 & 2 & 5 \\ 6 & \underline{\mathbf{0}} & 3 \\ \mathbb{1} & 1^{-1} & 2 \end{bmatrix}$

$$\Lambda_A = \begin{bmatrix} \underline{\mathbf{0}} & 2 & 6 & \mathbb{1} \\ \mathbb{1} & 2 & \underline{\mathbf{0}} & 1^{-1} \\ \mathbb{1} & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} \overline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ 2^{-1} & 2^{-1} & 5^{-1} \\ 6^{-1} & \overline{\mathbf{0}} & 3^{-1} \\ \mathbb{1} & 1 & 2^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \overline{\mathbf{0}} & 3 \\ \overline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$$

Interpretation of the λ_{ij}*'s (here* $a_{\cdot j}$ *stands for column* j *of* A*):*

$\lambda_{12} = \overline{\mathbf{0}}$*: There is no* ξ *s.t.* $a_{\cdot 1} \leq \xi a_{\cdot 2}$ *(since* $a_{23} = \underline{\mathbf{0}}$*)*
$\lambda_{13} = 3$: $a_{\cdot 1} \leq 3a_{\cdot 3}$.
$\lambda_{21} = \lambda_{23} = \lambda_{32} = \overline{\mathbf{0}}$ *same as for* $\lambda_{12} = \overline{\mathbf{0}}$.

Then: $A\Gamma^A = \begin{bmatrix} \underline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ 2 & 2 & 5 \\ 6 & \underline{\mathbf{0}} & 3 \\ \mathbb{1} & 1^{-1} & 2 \end{bmatrix} \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} & 3^{-1} \\ \underline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} & \mathbb{1} & \mathbb{1} \\ 2 & 2 & 5 \\ 6 & \underline{\mathbf{0}} & 3 \\ \mathbb{1} & 1^{-1} & 2 \end{bmatrix} = A.$

Note that $a_{\cdot 1} \vee 7a_{\cdot 2} = \begin{bmatrix} 7 \\ 9 \\ 6 \\ 6 \end{bmatrix} = 7a_{\cdot 2} \vee 3a_{\cdot 3} \in M_{12} \bigcap M_{23} \neq \underline{\mathbf{0}}$, *where* M_{ij} *stands for the semimodule generated by columns* i *and* j *of* A.

But $\{a_{\cdot 1}, a_{\cdot 2}\} \cap \{a_{\cdot 2}, a_{\cdot 3}\} = \{a_{\cdot 2}\} \in M_2$, *while* $\begin{bmatrix} 7 \\ 9 \\ 6 \\ 6 \end{bmatrix} \notin M_2$. *Thus the columns of* A *are not strongly independent.*

The following statements (Propositions 5.1–5.4, as well as Theorem 1 in [29]), stated for tropical torsion matrices extend to the case considered in this paper. Matrices are always assumed to have independent columns (in the sense of 3 above).

Proposition 3.2 *For an arbitrary matrix* A, *we have* $\Lambda_{\Gamma^A} = \Lambda_A$.

Proof. By Proposition 3.1 above, $(\Gamma^A)^- = \Lambda_A$, thus $\Lambda_{\Gamma^A} = (\Gamma^A)^t (\Gamma^A)^- = (\Gamma^A)^t \Lambda_A = (\Gamma^A)^t A^t A^- = (A\Gamma^A)^t A^- = A^t A^- = \Lambda_A$. □

Recall (cf. [3], eq 1.22 for instance) that the Kleene star of a matrix A is defined by $A^* = I \vee A \vee A^2 \vee \ldots$.

We now address the following problem.

Given two matrices A and B, it is clear that equality of their torsion coefficients is a necessary condition for the existence of an isomorphism $\varphi\colon M_A \to M_B$.

Question: Is this condition also sufficient?

Example 3.3 *Let* $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 6 \\ \mathbb{1} & 2 & 2 \\ \mathbb{1} & 7 & 14 \end{bmatrix}$, *then* $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 2 \\ 7^{-1} & \mathbb{1} & \mathbb{1} \\ 14^{-1} & 7^{-1} & \mathbb{1} \end{bmatrix}$, *with* $\tau_{12} = \tau_{13} = 7, \tau_{23} = 12$ *(for both* A *and* Γ^A*).*

Writing $a_{\cdot j}$ *for column* j *of* A, *we have* $[6\ 5\ 14]^t \in M_{a_{\cdot 1}, a_{\cdot 3}} \bigcap M_{a_{\cdot 2}, a_{\cdot 3}}$, *while* $[6\ 5\ 14]^t \notin M_{a_{\cdot 3}} = M_{[a_{\cdot 1} \cap a_{\cdot 3}] \cap [a_{\cdot 2} \cap a_{\cdot 3}]}$. *Hence the columns of* A *are not strongly independent.*

Now, writing $\gamma_{\cdot j}$ *for column* j *of* Γ^A, *it is not difficult to see that there is no nontrivial solution to any of the equations*

$$\xi_1 \gamma_{\cdot 1} \vee \xi_2 \gamma_{\cdot 2} = \lambda_1 \gamma_{\cdot 1} \vee \lambda_2 \gamma_{\cdot 3} \tag{1}$$
$$\xi_1 \gamma_{\cdot 1} \vee \xi_2 \gamma_{\cdot 2} = \lambda_1 \gamma_{\cdot 2} \vee \lambda_2 \gamma_{\cdot 3} \tag{2}$$
$$\xi_1 \gamma_{\cdot 2} \vee \xi_2 \gamma_{\cdot 3} = \lambda_1 \gamma_{\cdot 1} \vee \lambda_2 \gamma_{\cdot 3} \tag{3}$$

which shows that the columns of Γ^A *are strongly independent.*

Note also that $A \begin{bmatrix} \underline{\mathbf{0}} \\ 3 \\ \mathbb{1} \end{bmatrix} = A \begin{bmatrix} 1 \\ 3 \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 14 \end{bmatrix}$, *while* $\Gamma^A \begin{bmatrix} \underline{\mathbf{0}} \\ 3 \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ \mathbb{1} \end{bmatrix} \neq \begin{bmatrix} 5 \\ 3 \\ \mathbb{1} \end{bmatrix} = \Gamma^A \begin{bmatrix} 1 \\ 3 \\ \mathbb{1} \end{bmatrix}$.

We have the following statement:

Proposition 3.3 *Equality of the torsion coeffcients of the independent generators of two tropical modules* M *and* N *is a necessary but not sufficient condition for* N *to be isomorphic to* M.

Example 3.4 $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 12 \\ \mathbb{1} & 5 & 5 \\ \mathbb{1} & 7 & 11 \end{bmatrix}$

We have $9a_{\cdot 1} \vee a_{\cdot 3} = (12\ 9\ 11)^t = 4a_{\cdot 2} \vee a_{\cdot 3} \in M_{1,3} \bigcap M_{2,3}$. *But* $(12\ 9\ 11)^t \notin M_3$. *Hence* $M_{1,3} \bigcap M_{2,3} \neq M_3$, *and the columns of* A *are not strongly independent.*

We have $\Lambda_A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 5 & 7 \\ 12 & 5 & 11 \end{bmatrix} \cdot \begin{bmatrix} \mathbb{1} & \mathbb{1} & 12^{-1} \\ \mathbb{1} & 5^{-1} & 5^{-1} \\ \mathbb{1} & 7^{-1} & 11^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5^{-1} \\ 7 & \mathbb{1} & \mathbb{1} \\ 12 & 12 & \mathbb{1} \end{bmatrix}$. *The torsion coefficients of* M_A : $\tau_{12} = \lambda_{12}\lambda_{21}, \tau_{13} = \lambda_{13}\lambda_{31}$, *and* $\tau_{23} = \lambda_{23}\lambda_{32}$ *are equal to* $7, 7, 12$ *respectivly.*

Now $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ 7^{-1} & \mathbb{1} & \mathbb{1} \\ 12^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}$ *by Proposition 3.1, and the torsion coefficients of* M_{Γ^A} *are the same as those of* M_A, *by Proposition 3.2.*

Now let $y_1 = [9\ \underline{\mathbf{0}}\ \mathbb{1}]^t$, $y_2 = [\underline{\mathbf{0}}\ 4\ \mathbb{1}]^t$.

We have $\Gamma^A y_1 = [9\ 2\ \mathbb{1}]^t = z_1$, $\Gamma^A y_2 = [5\ 4\ \mathbb{1}]^t = z_2$, *while* $Az_1 = Ay_1 = Az_2 = Ay_2 = [12\ 9\ \mathbb{1}]^t$. *Hence* A *is not injective on* M_{Γ^A}.

Proposition 3.4 *For any square matrix* A , $I \vee A^2 = A \iff A^* = A$.

Proof. The proof is straightforward, and is independent of the fact that A may have some $\underline{\mathbf{0}}$ entries. It is given in [29]. □

Proposition 3.5 *For an arbitrary matrix* A, *we have* $\Gamma^{\Gamma^A} = \Gamma^A$.

Proof. The proof given in [29] for torsion tropical modules extends to the case considered here, since propositions 3.1 and 3.2 provide the desired extension. Then we have: $\Gamma^{\Gamma^A} = \Lambda^-_{\Gamma^A} = \Lambda^-_A = \Gamma^A$. □

Consider a rectangular matrix A of size $n \times m\ (n \geq m)$, $x = (\xi_1, \ldots, \xi_m)^t$, and a nonredundant combination of the columns of A, written as $Ax = \bigvee_{k=1}^{m} \xi_k a_{\cdot k}$. Then for every column j of A, there is at least one i $(1 \leq i \leq n)$ for which $\xi_j a_{ij}$ dominates, i.e. $\bigvee_{k=1, k\neq j}^{m} \xi_k a_{\cdot k} < \xi_j a_{ij}$. For if not, then there would be one column $a_{\cdot j}$ of A and one ξ_j, such that $\xi_j a_{\cdot j}$ never dominates, i.e. $x = \bigvee_{k=1, k\neq j}^{m} \xi_k a_{\cdot k}$, which contradicts our non-redundancy assumption for x. We can state this property as follows.

Proposition 3.6 *Let* A *be a rectangular matrix of size* $n \times m\ (n \geq m)$, *and* $\bigvee_{k=1}^{m} \xi_k a_{\cdot k}$ *be a nonredundant combination of the columns of* A, *then*

$$\exists\, \upsilon \colon \{1, \ldots, n\} \ \to\ \{1, \ldots m\} \text{ surjective s.t}$$

$$\bigvee_{k=1, k\neq \upsilon(i)}^{m} a_{ik}\xi_k < a_{i\upsilon(i)}\xi_{\upsilon(i)}.$$

□

Note that, for $n = m$, we have that $\upsilon \in \mathcal{S}_m$.

3.4 The dominant of a matrix

We define

Definition 3.1 *Let* $A = (a_{ij})$ *be a square matrix of size* m. *We say that*

$$\delta_A = \bigvee_{\sigma \in \mathcal{S}_m} \prod_{i=1}^{m} a_{i\sigma(i)} \quad [= \max_{\sigma \in \mathcal{S}_m} \sum_{i=1}^{m} a_{i\sigma(i)}]$$

is the **dominant** *of* A*.*

In [28], we proved that $\arg(\delta_A)$ is unique, i.e. the permutation realizing the max is unique. It is easy to see that, although stated for matrices with finite entries, this result extends to the case considered here.

For a rectangular $n \times m$ matrix A, up to permutations in $\mathcal{S}_m$ there are $\begin{pmatrix} n \\ m \end{pmatrix}$ square submarices of size m of A. We write S_A for this set, and let $\delta_A = \bigvee_{B \in S_A} \delta_B$.

Example 3.5 $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 3 \\ \mathbb{1} & 1 & 5 \\ \mathbb{1} & 2 & 2 \\ \mathbb{1} & 3 & 4 \end{bmatrix}$, $\delta_{B_{123}} = 7\,,\ \delta_{B_{124}} = 8\,,\ \delta_{B_{134}} = 6\,,\ \delta_{B_{234}} = 8.$

Hence $\delta_A = \delta_{B_{124}} = \delta_{B_{234}} = 8.$

This shows in particular that, for a rectangular matrix, the submatrix realizing the dominant may not be unique.

We have the following statement.

Theorem 1 *If the columns of* A *are strongly independent, then* $M_A \simeq M_{\Gamma^A}$*.*

Proof. Assume the columns of A are strongly independent. Clearly A is surjective on M_{Γ^A} since $\forall x \in M_A\,,\ x = \bigvee_{j=1}^{m} \xi_j a_{\cdot,j} = A\xi$. Then let $y = \Gamma^A \xi$. Since $A\Gamma^A \xi = A\xi$, we have $Ay = A\Gamma^A \xi = A\xi = x$.

We show that A is injective on M_{Γ^A}. Let $y = \bigvee_{\gamma_{\cdot j} \in Y} \xi_j \gamma_{\cdot j}\,,\ z = \bigvee_{\gamma_{\cdot k} \in Z} \lambda_k \gamma_{\cdot k}$ be such that $Az = Ay$.

Then $Ay = \bigvee_{a_{\cdot j} \in Y} \xi_j a_{\cdot j}\,,\ Az = \bigvee_{a_{\cdot k} \in Z} \lambda_k a_{\cdot k}$. Since the columns of A are strongly independent, we must have $Y = Z$. W.l.o.g. we may assume that $Y = \{a_{\cdot 1}, \ldots, a_{\cdot k}\}$, and write A_Y for the submatrix $\begin{bmatrix} a_{11} & \ldots & a_{1k} \\ a_{21} & \ldots & a_{2k} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nk} \end{bmatrix}$ of A.

Let $\mathcal{A}$ stand for the set of the $\begin{pmatrix} n \\ k \end{pmatrix}$ square submarices of size k of A_Y defined up to permutations in $\mathcal{S}_k$, and let A_k stand for any of the matrices such that $\delta_{A_k} = \delta_{\mathcal{A}}$.

By Proposition 3.6, we have $\bigvee_{j=1, j \neq \upsilon(i)}^{k} \xi_j a_{i,j} < a_{i\upsilon(i)} \xi_{\upsilon(i)}$.

But $Az = Ay \ \Rightarrow\ A_k z = A_k y \ \Rightarrow\ a_{i\upsilon(i)} \xi_{\upsilon(i)} = a_{i\upsilon(i)} \lambda_{\upsilon(i)}$ $\forall i$ $(1 \leq i \leq k)$. Hence $z = y$. □

Corollary 3.1 $\in \mathrm{Hom}(S^m, S^n)$*. The following are equivalent:*

i) The columns of A *are strongly independent.*

ii) The representation of any $x \in M_A$ *is unique.*

Proof. The equivalence $i) \iff ii)$ is straightforward. As a matter of fact, the uniqueness of the representation of any $x \in M_A$ is just another way of stating the strong independence of the columns of A. □

Example 3.6 *We have* $A = \begin{bmatrix} 2 & \mathbb{1} & 4 & \mathbb{1} \\ 1 & 1 & 2 & \mathbb{1} \\ 3 & 2 & 3 & \mathbb{1} \\ 4 & 3 & 5 & \mathbb{1} \end{bmatrix} \sim \begin{bmatrix} \mathbb{1} & \mathbb{1} & 3 & 4 \\ \mathbb{1} & 1 & 1 & 3 \\ \mathbb{1} & 1 & 1 & 2 \\ \mathbb{1} & 2 & 2 & 5 \end{bmatrix}$, *with* $\Lambda_A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 1^{-1} & 2^{-1} \\ 2 & \mathbb{1} & \mathbb{1} & 1^{-1} \\ 3 & 3 & \mathbb{1} & \mathbb{1} \\ 5 & 4 & 3 & \mathbb{1} \end{bmatrix}$, *which yields* $\left\{ \begin{array}{l} \tau_{12} = \tau_{13} = 2\,,\ \tau_{14} = \\ \tau_{24} = \tau_{23} = \tau_{34} = 3 \end{array} \right.$.

Permutation of columns 2 *and* 3 *of* A *yields the equivalent matrix:* $B = \begin{bmatrix} 2 & 4 & \mathbb{1} & \mathbb{1} \\ 1 & 2 & 1 & \mathbb{1} \\ 3 & 3 & 2 & \mathbb{1} \\ 4 & 5 & 3 & \mathbb{1} \end{bmatrix} \sim \begin{bmatrix} \mathbb{1} & \mathbb{1} & 3 & 4 \\ \mathbb{1} & 1 & 4 & 6 \\ \mathbb{1} & 1 & 3 & 3 \\ \mathbb{1} & 2 & 2 & 5 \end{bmatrix}$,

with $\Lambda_B = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 2^{-1} & 3^{-1} \\ 2 & \mathbb{1} & \mathbb{1} & 2^{-1} \\ 4 & 3 & \mathbb{1} & \mathbb{1} \\ 6 & 5 & 3 & \mathbb{1} \end{bmatrix}$ *and the same* τ_{ij}*'s, although the* $\lambda_{ij}(A)$ *may differ from the* $\lambda_{ij}(B)$.

The reader may find it interesting to show that the columns of A *are not strongly independent.*

Corollary 3.2 *Strong independence of the columns of a matrix allows for a new equivalence between matrices, namely* $A \sim \Gamma^A$, *which relates a (possibly) rectangular matrix to a square matrix.* □

Corollary 3.3 *If the columns of* $A \in \mathrm{Hom}(S^m, S^n)$ *are strongly independent, then* M_A *can be embedded in* S^m.

Proof. The proof is straightforward, since $M_{\Gamma^A} \in S^m$. □

4 Examples

Example 4.1 *(4.3 of [29]) Let* $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix}$, *with* $5 < a < 8$. *It is not difficult to see that the columns of* A *are strongly independent, and* $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4 \\ 9^{-1} & \mathbb{1} & \mathbb{1} \\ 15^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}$. *The* τ_{ij} *of* A *and* Γ^A *are equal to* $(9, 11, 12)$ *respectively. As stated in Theorem 1,* $M_A \simeq M_{\Gamma^A}$, *and thus can be embedded in* S^3, *independently of the value of* $a \in]5, 8[$. *This is a significant improvement to the statement in [28].*

Note also that, writing A_6 *(resp.* A_7*) for the matrix with* $a = 6$ *(resp. 7), and* M_6 *(resp* M_7*) for the semimodule generated by the columns of* A_6 (A_7), *we have* $M_6 \simeq M_7$. *However, there is no isomorphism* $S^6 \to S^6$ *whose restriction to* M_6 *yields* M_7.

Up to row permutations, there are $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ 3×3 *submatrices of* A.

We have $\delta_A = 24 = \delta_{A_{156}} = \delta_{A_{456}}$, *where* A_{ijk} *stands for the submatrix made of rows* ijk *of* A.

Example 4.2 *Let* $A = \begin{bmatrix} \mathbb{1} & 9 & 5 \\ \mathbb{1} & \mathbb{1} & 2 \\ \mathbb{1} & \underline{\mathbf{0}} & \mathbb{1} \\ \underline{\mathbf{0}} & 4 & \underline{\mathbf{0}} \end{bmatrix}$.

We have $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \overline{\mathbf{0}} \\ 9^{-1} & \mathbb{1} & \overline{\mathbf{0}} & 4^{-1} \\ 5^{-1} & 2^{-1} & \mathbb{1} & \overline{\mathbf{0}} \end{bmatrix} \star \begin{bmatrix} \mathbb{1} & 9 & 5 \\ \mathbb{1} & \mathbb{1} & 2 \\ \mathbb{1} & \underline{\mathbf{0}} & \mathbb{1} \\ \underline{\mathbf{0}} & 4 & \underline{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \underline{\mathbf{0}} & \mathbb{1} \\ \underline{\mathbf{0}} & \mathbb{1} & \underline{\mathbf{0}} \\ 5^{-1} & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$.

The torsion $\tau_{13} = 5$ *in both* M_A*, and* M_{Γ^A}*.*

However, here we have $a_{.,1} \vee a_{.,2} = \begin{bmatrix} 9 \\ \mathbb{1} \\ \mathbb{1} \\ 4 \end{bmatrix} \leq a_{.,2} \vee a_{.,3} = \begin{bmatrix} 9 \\ 2 \\ \mathbb{1} \\ 4 \end{bmatrix} \leq 2 \begin{bmatrix} 9 \\ \mathbb{1} \\ \mathbb{1} \\ 4 \end{bmatrix}$*. More precisely the torsion coefficient* $\tau_{c_1 \vee c_2, c_2 \vee c_3} = 2$ *in* M_A*, while the corresponding coefficient in* M_{Γ^A} *is equal to* 5*. It follows that* M_A*, and* M_{Γ^A} *cannot be isomorphic. Therefore, by Theorem 1 the columns of* A *cannot be strongly independent.*

Indeed, by inspection,we get $M_{c_1,c_2} \bigcap M_{c_2,c_3} \ni \begin{bmatrix} 13 \\ 4 \\ 1 \\ 8 \end{bmatrix} \notin M_{c_2}$*.*

In our next example, we revisit Example 4.1 in which we set $a = \underline{\mathbf{0}}$ and reorder rows and columns for convenience.

Example 4.3 *Let* $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4 \\ \mathbb{1} & 3 & 13 \\ \mathbb{1} & 1 & 12 \\ \mathbb{1} & 5 & 5 \\ \mathbb{1} & 10 & 11 \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}$,

$$\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \underline{\mathbf{0}} \\ \mathbb{1} & 3^{-1} & 1^{-1} & 5^{-1} & 10^{-1} & \underline{\mathbf{0}} \\ 4^{-1} & 13^{-1} & 12^{-1} & 5^{-1} & 11^{-1} & \mathbb{1} \end{bmatrix} \star \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4 \\ \mathbb{1} & 3 & 13 \\ \mathbb{1} & 1 & 12 \\ \mathbb{1} & 5 & 5 \\ \mathbb{1} & 10 & 11 \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4 \\ 10^{-1} & \mathbb{1} & \mathbb{1} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \mathbb{1} \end{bmatrix}.$$

Here M_A *is isomorphic to the semi-direct sum* $M_B \tilde{\oplus} M_C$ *introduced in [27], where* M_B *is generated by the columns of* $B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 3 \\ \mathbb{1} & 1 \\ \mathbb{1} & 5 \\ \mathbb{1} & 10 \end{bmatrix}$ *and* M_C *by the column vector* $C = (4\ 13\ 12\ 5\ 11\ \mathbb{1})^t$*.*

Since $\lambda_{12} = 10\ (= \tau_{12})$*, and* $\lambda_{13} = 4$*,* $\lambda_{23} = \mathbb{1}$*, the sum* $M_B \tilde{\oplus} M_C$ *cannot be a direct sum.*

By Theorem 1 (cf. also Case 3 of Subsection 3.1) $M_B \simeq M_{\Gamma^B}$*, with* $\Gamma^B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ 10^{-1} & \mathbb{1} \end{bmatrix}$*. By Theorem 1* $M_A \simeq M_{\Gamma^A}$*, which is isomorphic to the semi-direct sum* $M_B \tilde{\oplus} M_D$*, with* M_D *generated by the column vector* $D = (4\ \mathbb{1}\ \mathbb{1})^t$*.*

5 Conclusion

We conclude below by exhibiting the short table of (some) algebraic invariants mentioned in the introduction.

Algebraic invariants

category	specify	char.
vector space	field F , n	F^n
module	PID	free $\oplus$ torsion.
idempotent semimodule	strongly indep. basis	Λ-matrix

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