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Abstract: In this article, we characterize incentive equilibrium strategies and their credibility conditions for the classes of linear-state and linear-quadratic dynamic games played over event trees. In such games, the transition from one node to another is nature's decision and cannot be influenced by players' actions. Assuming that two players wish to optimize their joint payoff over a given planning horizon, we show that this outcome can be achieved as an incentive equilibrium, and hence ensures that cooperation will continue from one node onward. Two simple examples illustrate these strategies and the credibility conditions.

Key Words: Dynamic games, incentive equilibria, event tree, cooperation, linear-state dynamic games, linear-quadratic dynamic games.

Résumé : Dans cet article, on caractérise les équilibres en stratégies incitatives ainsi que leur crédibilité dans deux classes de jeux dynamiques et stochastiques joués sur un arbre d'évènement, à savoir, des jeux linéaires en l'état et des jeux linéaires-quadratiques. Dans ces jeux, la transition d'un nœud à un autre est décidée par la nature et est indépendante des actions des joueurs. Supposant que les deux joueurs désirent maximiser leur gain conjoint, on montre que ce résultat peut être supporté par un équilibre incitatif qui assure que la coopération restera en place à partir de tout nœud de l'arbre pour le reste du jeu. On illustre les résultats par deux exemples simples.

Mots clés : Jeux dynamiques, équilibres incitatifs, arbre d'évènements, coopération, jeux dynamiques linéaires, jeux dynamiques linéaires-quadratiques.

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1 Introduction

A main issue in cooperative dynamic games is how to sustain cooperation over time, that is, how to ensure that each player will indeed implement her part of the agreement as time goes by. Breakdowns of long-term agreements before their maturity are empirically observed. Schematically, they will occur if, either all the parties agree, at an intermediate instant of time, to replace the initial agreement by a new one for the remaining periods, or if one of the players finds it (individually) rational to deviate, that is, to switch to her noncooperative strategy from that time onward [8]. The literature in dynamic games suggested mainly three approaches to sustain cooperation over time.

Time consistency: A cooperative agreement is time consistent at initial date and state of a dynamic game, if at any intermediate instant of time the cooperative payoff-to-go of each player dominates, at least weakly, her noncooperative payoff-to-go; see, e.g., [14, 15, 26, 27, 33]. In a nutshell, the idea here is to decompose over time the total individual cooperative payoff in a way such that the requirement in the definition of time consistency is satisfied. Note that a time-consistent payment schedule always exists, and that the cooperative and noncooperative payoffs-to-go are compared along the cooperative state trajectory, which implicitly assumes that the players have so far played cooperatively. An alternative stronger concept is agreeability, which requires the cooperative payoff-to-go to dominate the noncooperative payoff-to-go along *any* state trajectory; see, e.g., [17–20].[1]

Cooperative equilibrium: If the cooperative solution is an equilibrium, then it is self supported and the issue of durability of the agreement is solved. To endow the cooperative solution with an equilibrium property, one approach is to use trigger strategies that punish credibly and effectively any player who deviates from the agreement. See, e.g., [9, 32] and Dockner et al. (2000) for applications in differential games.[2]

Incentive equilibrium: In a two-player dynamic game, incentive strategies can be used to support the cooperative agreement (see, [2–4]). Informally, the incentive strategy of each player is defined as a function of possible deviation of the other player with respect to the coordinated or cooperative solution.

The objective of this paper is to characterize incentive equilibrium strategies and outcomes for the class of dynamic games played over event trees (DGET). In these games, the transition from one node to another is nature's decision and cannot be influenced by the players' actions. Zaccour [34] and Haurie et al. [12] introduced the S-adapted Nash equilibrium as solution concept for DGET. Haurie and Zaccour [11], see also [13], provided a stochastic-control formulation of this class of games, and characterized the S-adapted equilibria through maximum principles and established a link with the theory of open-loop multistage games. Haurie and Roche [10] compared S-adapted and piecewise open-loop information structures, and [6, 7, 28] and [29] used the formalism of DGET to predict equilibrium investment strategies in some energy markets. Recently, [30] and [25] considered cooperative DGET and defined a node-consistent Shapley value and node-consistent core allocations for this class of games. This paper belongs to this research effort aiming at designing mechanisms to sustain cooperation over time in DGET.

We start by writing the model and the incentive strategies in a general setting, and next focus on linear-state and linear-quadratic dynamic games, which are tractable games, that is, they admit closed-form solutions, and are very popular in applications (see, e.g., the books by Engwerda [5] and Haurie et al. [13], and a survey of some applications in [16]). Martín-Herrán and Zaccour [23, 24] characterized incentive strategies and their credibility for the same classes of games (i.e., linear-state and linear-quadratic dynamic games), but in a deterministic and continuous time setting.

The rest of the paper is organized as follows. In Section 2, we briefly recall the ingredients of dynamic games played over event trees (DGET) and derive the coordinated solution. In Section 3, we define the incentive equilibrium strategies for a general model and implement the results in games with special structures in Section 4. We briefly conclude in Section 5.

[1] For a survey of time consistency in cooperative differential games, see [35].

[2] In some models, the cooperative solution happens to be also an equilibrium, and therefore trigger strategies are not needed to enforce it; see, e.g., [1, 22, 31].

2 Elements of the game

We recall the main elements of dynamic games played over event trees.[3] Let $\mathcal{T} = \{0,\ 1,\ \ldots,\ T\}$ be the set periods, and denote by $(\xi(t) : t \in \mathcal{T})$ the exogenous stochastic process represented by an event tree, with a root node n_0 in period 0 and a set of nodes $\mathcal{N}^t$ in period $t = 0,\ 1,\ \ldots,\ T$. Each node $n^t \in \mathcal{N}^t$ represents a possible sample value of the history h^t of the $\xi(.)$ process up to time t. Let $a(n^t) \in \mathcal{N}^{t-1}$ be the unique predecessor of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T$, and denote by $S(n^t) \in \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T-1$. We call a *scenario* a path from node n_0 to a terminal node n^T. Each scenario has a probability and the probabilities of all scenarios sum up to 1. We denote by π^{n^t} the probability of passing through node n^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi^{n_0} = 1$ and π^{n^T} is equal to the probability of the single scenario that terminates in (leaf) node $n^T \in \mathcal{N}^T$.

Denote by $u_i(n^t) \in U_i^{n^t} \subseteq \mathbb{R}^{m_i^{n^t}}$ the decision variable of player i at node n^t, where $U_i^{n^t}$ is the control set, $m_i^{n^t}$ is the dimension of the decision variable for player i, $i = 1, 2$. Let $\mathbf{u}(n^t)$ denote the vector of decision variables of both players at node n^t, i.e., $\mathbf{u}(n^t) = (u_1(n^t), u_2(n^t))$. Let $X \subseteq \mathbb{R}^q$ with q a given positive integer, be a state set. A transition function $f^{n^t}(.,.) : X \times U^{n^t} \to X \subseteq \mathbb{R}^q$, where $U^{n^t} = U_1^{n^t} \times U_2^{n^t}$, is associated with each node n^t, and the state equations are given by

$$\begin{aligned} x\left(n^t\right) &= f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}(a(n^t))\right), \\ \mathbf{u}(a\left(n^t\right)) &\in U^{a\left(n^t\right)}, \quad n^t \in \mathcal{N}^t, t = 1, \ldots, T. \end{aligned}$$

At each node $n^t, t = 0, \ldots, T-1$, the reward to player i is a function of the state and of the controls of all players, given by $\mathrm{L}_i^{n^t}(x\left(n^t\right), \mathbf{u}(\ n^t))$ and is assumed to be twice continuously differentiable. At a terminal node n^T, the reward to Player i is given by the function $\Phi_i^{n^T}(x\left(n^T\right))$ and is also supposed to be twice continuously differentiable. Now, we can define the following multistage game where $x = \{x\left(n^t\right) : n^t \in \mathcal{N}^t, t = 0, \ldots, T\}$ and $\mathbf{u} = \left\{\mathbf{u}^{n^t} : n^t \in \mathcal{N}^t, t = 0, \ldots, T-1\right\}$ and $V_i\left(\mathbf{u}, x^0\right)$ is the total payoff to player i given by:

$$V_i\left(\mathbf{u}, x^0\right) = \max_{u_i(n^t)} \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} L_i^{n^t}\left(x\left(n^t\right), \mathbf{u}(n^t)\right) + \sum_{n^t \in \mathcal{N}^t} \pi^{n^T} \Phi_i^{n^T}\left(x\left(n^T\right)\right), \quad i \in M, \tag{1}$$

subject to:

$$\begin{aligned} x\left(n^t\right) = f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}(a\left(n^t\right))\right), &\qquad x\left(n_0\right) = x^0 \\ \mathbf{u}(a\left(n^t\right)) \in U^{a\left(n^t\right)}, &\qquad n^t \in \mathcal{N}^t, t = 1, \ldots, T. \end{aligned} \tag{2}$$

An admissible S-adapted strategy for player i is a vector $\mathbf{u}_i = \{u_i(n^t) : n^t \in \mathcal{N}^t, t = 0, ..., T-1\}$. In other words, an admissible S-adapted strategy is a plan of actions adapted to the history of the random process represented by the event tree.

2.1 Cooperative solution

Suppose that the two players agree to cooperate and maximize their joint payoff, that is,

$$\max \sum_{i=1}^{2} V_i\left(\mathbf{u}(n^t)\right) = \sum_{i=1}^{2}\left(\sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} L_i^{n^t}\left(x\left(n^t\right), \mathbf{u}\left(n^t\right)\right) + \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} \Phi_i^{n^T}\left(x^{n^T}\right)\right), \tag{3}$$

subject to (2). The solution of the above problem is the one that needs to be supported by incentive strategies. To solve this optimization problem, we introduce the Lagrangian

[3] For a detailed description of DGET, see Haurie et al. (2012).

$$\begin{aligned}\mathcal{L}^C(\lambda,x,\mathbf{u}) = &\sum_{i=1}^{2} L_i^{n_0}\left(x^{n_0},\mathbf{u}(n_0)\right) + \sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi^{n^t}\Big\{\sum_{i=1}^{2} L_i^{n^t}\left(x\left(n^t\right),\mathbf{u}\left(n^t\right)\right)\\ &+\lambda\left(n^t\right)\left(f^{a(n^t)}\left(x\left(a\left(n^t\right)\right),\mathbf{u}\left(a\left(n^t\right)\right)\right)-x\left(n^t\right)\right)\Big\}\\ &+\sum_{n^t\in\mathcal{N}^T}\pi^{n^T}\Big\{\sum_{i=1}^{2}\Phi_i^{n^T}\left(x(n^T)\right)\\ &+\lambda\left(n^T\right)\left(f^{a(n^T)}\Big(x\left(a\left(n^T\right)\right),\mathbf{u}\left(a\left(n^T\right)\right)\Big)-x\left(n^T\right)\right)\Big\},\end{aligned}$$

where $\lambda(\cdot)$ is the vector of costate variables. The necessary optimality conditions are

$$\frac{\partial\mathcal{L}^C}{\partial u_k(n^t)} = 0,\quad k=1,2,\ n^t\in\mathcal{N}^t,\ t=0,1,...,T-1, \tag{4}$$

$$-\lambda^C(n^t) = \frac{\partial\mathcal{L}^C}{\partial x(n^t)},\ t=0,1,...,T. \tag{5}$$

To save on space, we shall not write the full expressions of the above optimality conditions, but we will do it whe we deal with games with special structures. The following proposition recalls the conditions under which there is at least one solution to the above dynamic optimization problem.

Proposition 1 *Assume that $L_i^{n^t}$ and f^{n^t} are concave in $\mathbf{u}(n^t)$ for all $n^t\in\mathcal{N}^t$ and the set of admissible controls is compact. Assume that $\mathbf{u}^C$ is the S-adapted cooperative strategy at x^0, generating the state trajectory $x^C(n^t), n^t\in\mathcal{N}^t, t=0,...,T$ over the event tree for the joint maximization problem defined by (3) subject to (2). Then, there exists a cooperative costate trajectory $\lambda^C(n^t)$ such that the conditions (4) and (5) hold true for $i=1,2$.*

Proof. See, e.g., [21]. □

The optimality conditions in (4)–(5) yield the cooperative solution, that is, $\mathbf{u}^C(n^t)$, $x^C(n^t)$, and $\lambda^C(n^t)$ for all $n^t\in\mathcal{N}^t, t=0,...,T$. Inserting these values in the payoff function in (1) we get the cooperative outcome of each player.

3 S-Adapted incentive equilibria

As mentioned in the introduction, our aim is to design incentive equilibrium strategies to support the cooperative (or coordinated) solution $\mathbf{u}^C(n^t) = \left(u_1^C\left(n^t\right), u_2^C\left(n^t\right)\right)\in U_1^{n^t}\times U_2^{n^t}$. Denote by

$$\Psi_i = \{\psi_i|\psi_i : U_j\to U_i\},\quad i,j=1,2;\ \ i\neq j,$$

the set of admissible strategies over the event tree.

Definition 1 *A strategy $\psi_i\in\Psi_i, i=1,2$ is an incentive equilibrium at $\mathbf{u}^C$, if*

$$V_1(u_1^C,u_2^C)\geq V_1(u_1^C,\psi_2(u_1)),\quad \forall u_1\in U_1,$$
$$V_2(u_1^C,u_2^C)\geq V_2(\psi_1(u_2),u_2^C),\quad \forall u_2\in U_2,$$
$$\psi_1(u_2^C(n^t)) = u_1^C(n^t),\qquad \psi_2(u_1^C(n^t)) = u_2^C(n^t),\quad n^t\in\mathcal{N}^t, t=0,\dots,T-1.$$

The above definition states that if a player implements her part of the agreement, then the best response of the other player is to do the same. To determine the incentive strategies we need to solve two optimal control problems, where in each problem one player assumes that the other player is using her incentive strategy. The optimization problem of player i is as follows:

$$\max V_i\left(\mathbf{u},x^0\right) = \sum_{t=0}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi^{n^t}L_i^{n^t}\left(x\left(n^t\right),\mathbf{u}\left(n^t\right)\right) + \sum_{n^T\in\mathcal{N}^T}\pi^{n^T}\Phi_i^{n^T}\left(x^{n^T}\right),$$

subject to:

$$x\left(n^t\right) = f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}\left(a\left(n^t\right)\right)\right), \qquad x\left(n_0\right) = x^0$$
$$\mathbf{u}\left(a\left(n^t\right)\right) \in U^{a\left(n^t\right)}, \qquad n^t \in \mathcal{N}^t, t = 1, \ldots, T,$$
$$u_j(n^t) = \psi_j(\mathbf{u}_{-j}(n^t)), \quad i, j = 1, 2\ , i \neq j.$$

The Lagrangian of the above problem is

$$\begin{aligned}
\mathcal{L}_i\left(\lambda_i, x, \mathbf{u}\right) = {} & L_i^{n^0}(x(n^0), u_i(n^0), \psi_j(\mathbf{u}_{-i}(n^0))) \\
& + \sum_{t=1}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \Big\{ L_i^{n^t}\left(x\left(n^t\right), u_i(n^t), \psi_j(\mathbf{u}_{-i}(n^t))\right) \\
& + \lambda_i\left(n^t\right)\left(f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u_i(a(n^t)), \psi_j(\mathbf{u}_{-i}(a(n^t)))\right) - x\left(n^t\right)\right)\Big\} \\
& + \sum_{n^t \in \mathcal{N}^T} \pi^{n^T} \Big\{ \Phi_i^{n^T}\left(x\left(n^T\right)\right) \\
& + \lambda_i\left(n^T\right)\left(f^{a\left(n^T\right)}\left(x(a\left(n^T\right)), u_i(a(n^T)), \psi_j(\mathbf{u}_{-i}(a(n^T)))\right) - x\left(n^T\right)\right)\Big\},
\end{aligned}$$

where $\lambda\left(\cdot\right)$ is the vector of costate variables. Assuming an interior solution, the necessary optimality conditions include

$$\frac{\partial \mathcal{L}_i}{\partial u_i}(x, \mathbf{u}, \lambda_i) = 0, \qquad t = 0, 1, ..., T-1, \tag{6}$$

$$-\lambda_i(n^t) = \frac{\partial \mathcal{L}_i}{\partial x(n^t)}, \qquad t = 0, 1, ..., T. \tag{7}$$

Solving the above conditions yields the values of the incentive control and costate variables, that is, $u_i^I(n^t)$ and ${\lambda_i}^I(n^t)$, on which we impose the equality $u_i^I(n^t) = u_i^C(n^t)$. This implies that $u_i^C(n^t)$ must satisfy its associated condition given in (6), and moreover we have

$$\psi_i(\mathbf{u}_{-i}^C(n^t)) = u_i^C(n^t), \quad \text{for } i = 1, 2.$$

The condition (6) can then be written in long as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}_i}{\partial u_i}(x, \mathbf{u}^C, \lambda_i) = {} & \partial_{u_i} L_i^{n^0}(x(n^0), u_i(n^0), \psi_j(\mathbf{u}_{-i}(n^0))) \\
& + \sum_{t=1}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \Bigg\{ \partial_{u_i} L_i^{n^t}\left(x\left(n^t\right), \mathbf{u}^C(n^t)\right) \\
& + \partial_{u_j} L_i^{n^t}\left(x\left(n^t\right), \mathbf{u}^C(n^t)\right) \times \frac{\partial \psi_j}{\partial u_i}(u_i^C(n^t)) \\
& + \lambda_i\left(n^t\right)\Big[\partial_{u_i} f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}^C(a(n^t)\right) \\
& + \partial_{u_j} f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}^C(a(n^t)\right) \times \frac{\partial \psi_j}{\partial u_i}(u_i^C(n^t))\Big]\Bigg\} \\
& + \sum_{n^t \in \mathcal{N}^T} \pi^{n^T} \lambda_i\left(n^T\right) \Bigg\{ \partial_{u_i} f^{a\left(n^T\right)}\left(x(a\left(n^T\right)), \mathbf{u}^C(a(n^T))\right) \\
& + \partial_{u_j} f^{a\left(n^T\right)}\left(x(a\left(n^T\right)), \mathbf{u}^C(a(n^T))\right) \times \frac{\partial \psi_j}{\partial u_i}(u_i^C(n^T))\Bigg\} = 0, \quad i = 1, 2.
\end{aligned} \tag{8}$$

Additionally, $\mathbf{u}^C$ satisfies the condition in (5) that characterizes the cooperative solution. Using equations (5) and (8), one may establish the necessary conditions that must be satisfied by the incentive equilibrium strategies. In the following two sections these necessary conditions will be derived for two tractable classes of games, namely, linear-state and linear-quadratic dynamic games for which we can obtain closed-form equilibrium strategies and outcomes.

One important concern with incentive strategies is their credibility. These strategies are said to be credible if it is in the best interest of each player to implement her incentive strategy if she detects a deviation from the agreed upon solution by the other player. If it is not the case, then a player can freely cheat on the agreement, that is, no punishment is taken place. A formal definition of credibility follows.

Definition 2 *The incentive equilibrium strategy* $(\psi_i^{n^t} \in \Psi_i, \quad \forall i)$ *is credible at* $\mathbf{u}^C \in U_1 \times U_2$ *if the following inequalities are satisfied:*

$$\begin{aligned} V_1(\psi_1(u_2(n^t)), u_2(n^t)) &\geq V_1(u_1^C(n^t), u_2(n^t)), \quad \forall u_2 \in U_2, \\ V_2(u_1(n^t), \psi_2(u_1(n^t))) &\geq V_2(u_1(n^t), u_2^C(n^t)), \quad \forall u_1 \in U_1. \end{aligned} \tag{9}$$

Note that the above definition characterizes the credibility of the equilibrium strategies for any possible deviation in the set $U_1 \times U_2$. As we need the functional forms to proceed, we discuss the credibility conditions for the incentive strategies in more details in the following sections.

4 Implementation in tractable game structures

In this section, we derive incentive equilibrium strategies for the classes of linear-state and linear-quadratic games played over event trees, and discuss the associated credibility conditions.

4.1 Linear-state game

In a linear-state dynamic game the payoff functions and the state dynamics are polynomial of degree one in the state variables and there are no cross terms between the state and the control variables. Consider the game defined by (1) and (2) and let $L_i(x(n^t), \mathbf{u}^{n^t})$, $\Phi_i(x^{n^T})$ and $f^{n^t}(x(n^t), \mathbf{u}^{n^t})$ satisfy the property of linear-state games. In particular, this implies that

$$\frac{\partial^2 \mathcal{L}^i}{\partial u_j^{n^t} \partial x^{n^t}} = 0, \quad i = 1, 2, \ j = 1, 2, \ t = 0, ..., T-1, \ n^t \in \mathcal{N}^t, \tag{10}$$

$$\frac{\partial^2 \mathcal{L}^i}{\partial {x^{n^t}}^2} = 0, \quad i = 1, 2, \ t = 0, ..., T, \ n^t \in \mathcal{N}^t, \tag{11}$$

which shows that the optimality conditions

$$\frac{\partial \mathcal{L}_i}{\partial u_i}(x, \mathbf{u}, \lambda_i) = 0, \qquad t = 0, 1, ..., T-1,$$

are independent of the state, and that the costate equations

$$-\lambda_i(n^t) = \frac{\partial \mathcal{L}_i}{\partial x(n^t)}, \qquad t = 0, 1, ..., T,$$

do not include the state variables. In turn, this implies that the costate and control trajectories may be computed independently from the initial state, which yields the well-known result that an open-loop equilibrium is Markov perfect for this class of games. In this context, it is easy to show that $\lambda^C(n^t) = \sum_{i=1}^{2} \lambda_i(n^t)$.[45] Thus, equation (5) implies that at $\mathbf{u}^C(n^t)$ we have

[4]The proof is straightforward simply because of (10), (11) and the condition $u_i^I = u_i^C, \forall i$.
[5]This condition directs us to,

$$\sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu) = \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \sum_{i=1}^{2} \lambda_i(\nu).$$

$$\pi^{n^t}\partial_{u_1}L_1^{n^t}+\partial_{u_1}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_1(\nu) = -\pi^{n^t}\partial_{u_1}L_2^{n^t}-\partial_{u_1}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_2(\nu), \quad (12)$$

$$\pi^{n^t}\partial_{u_2}L_2^{n^t}+\partial_{u_2}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_2(\nu) = -\pi^{n^t}\partial_{u_2}L_1^{n^t}-\partial_{u_2}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_1(\nu). \quad (13)$$

The proposition below states the conditions to be satisfied by incentive strategies.

Proposition 2 *The pair of strategies $(\psi_1,\psi_2)\in\Psi_1\times\Psi_1$ is an incentive equilibrium at $\mathbf{u}^C$ if it satisfies the following conditions:*[6]

$$\frac{\partial\psi_1}{\partial u_2}(u_2^C(n^t))\times\frac{\partial\psi_2}{\partial u_1}(u_1^C(n^t))=1, \quad (14)$$

and

$$\psi_1'(u_2^C)=-\frac{\pi^{n^t}\partial_{u_2}L_2^{n^t}+\partial_{u_2}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_2(\nu)}{\pi^{n^t}\partial_{u_1}L_2^{n^t}+\partial_{u_1}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_2(\nu)}, \quad (15)$$

$$\psi_2'(u_1^C)=-\frac{\pi^{n^t}\partial_{u_1}L_1^{n^t}+\partial_{u_1}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_1(\nu)}{\pi^{n^t}\partial_{u_2}L_1^{n^t}+\partial_{u_2}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_1(\nu)}, \quad (16)$$

where

$$\pi^{n^t}\partial_{u_j}L_i^{n^t}+\partial_{u_j}f^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^{\nu}\lambda_i(\nu),\quad i,j=1,2,\ i\neq j,$$

are assumed to be nonzero.

Proof. In a linear-state game, equation (8) is simplified to[7]

$$[\pi^{n^t}\partial u_i^{n^t}L_i^{n^t}+\partial u_i^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_i(\nu)]$$
$$+\frac{\partial\psi_j}{\partial u_i}[\pi^{n^t}\partial u_j^{n^t}L_i^{n^t}+\partial u_j^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_i(\nu)]=0,\quad i,j=1,2,\ i\neq j. \quad (17)$$

Replacing the equations (12) in (17) for $i=2,\ j=1$, we get,

$$[\pi^{n^t}\partial u_2^{n^t}L_1^{n^t}+\partial u_2^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)]+\frac{\partial\psi_1}{\partial u_2}[\pi^{n^t}\partial u_1^{n^t}L_1^{n^t}+\partial u_1^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)]=0. \quad (18)$$

Besides, from (17) for $i=1,\ j=2$, we have,

$$[\pi^{n^t}\partial u_1^{n^t}L_1^{n^t}+\partial u_1^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)]=-\frac{\partial\psi_2}{\partial u_1}[\pi^{n^t}\partial u_2^{n^t}L_1^{n^t}+\partial u_2^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)]. \quad (19)$$

Substituting the right-hand side of the last equation in (18) and arranging the terms, we obtain

$$[\pi^{n^t}\partial u_2^{n^t}L_1^{n^t}+\partial u_2^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)][1-\frac{\partial\psi_1}{\partial u_2}\times\frac{\partial\psi_2}{\partial u_1}]=0$$

Since $[\pi^{n^t}\partial u_2^{n^t}L_1^{n^t}+\partial u_2^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1(\nu)]$ is assumed to be nonzero, we have condition (14). Moreover, (19) can be rewritten as (15) and assuming $\pi^{n^t}\partial u_1^{n^t}L_2^{n^t}+\partial u_1^{n^t}f^{n^t}\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_2(\nu)$ nonzero, equation (16) can be derived. □

[6] All functions evaluated at (u_1^C,u_2^C).

[7] To keep the notation simple, the arguments of all functions are omitted.

To get more insight into the results, let us assume the following specific functional forms for the 2-player linear-state game under consideration:

$$\begin{aligned} L_i^{n^t}(x(n^t),\mathbf{u}(n^t)) &= \frac{1}{2}r_i^{n^t}u_i(n^t)^2 - d_i^{n^t}x(n^t), \\ \Phi_i^{n^T} &= -d_i^{n^T}x(n^T), \\ x^{n^t} &= \sum_{i=1}^{2} g_i^{a(n^t)}u_i(a(n^t)) + k^{a(n^t)}x(a(n^t));\quad x(n^0)=x_0. \end{aligned} \tag{20}$$

Note that in the above formulation, the parameters vary in different nodes. Denoting by $u_i^C(n^t)$ the cooperative strategy of player i at node n^t, the conditions (5) and (8) can be rewritten as follows:

$$\begin{aligned} \frac{\partial \mathcal{L}^C}{\partial u_i^{n^t}} &= \pi^{n^t}r_i^{n^t}u_i(n^t) + g_i^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda^C(\nu) = 0 \Rightarrow u_i^C(n^t) = -\frac{g_i^{n^t}}{\pi^{n^t}r_i^{n^t}}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda^C(\nu), \\ \frac{\partial \mathcal{L}_i}{\partial u_i^{n^t}} &= \pi^{n^t}r_i^{n^t}u_i(n^t) + \left(g_i^{n^t} + g_j^{n^t}\frac{\partial\psi_j(\mathbf{u}_i(n^t))}{\partial u_i}\right)\left(\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda_i(\nu)\right) = 0, \\ \Rightarrow u_i(n^t) &= -\frac{\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda_i(\nu)}{\pi^{n^t}r_i^{n^t}}\left(g_i^{n^t} + g_j^{n^t}\frac{\partial\psi_j(\mathbf{u}_i(n^t))}{\partial u_i}\right);\quad i,j=1,2,\ i\neq j. \end{aligned}$$

Similarly, using equations (4) and (7) the cooperative costate variables are given by

$$\begin{aligned} \lambda^C(n^t) &= -\frac{1}{1+\pi^{n^t}}\left(\pi^{n^t}\sum_{i=1}^{2}d_i^{n^t} - k^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda^C(\nu)\right),\ t=0,...,T-1, \\ \lambda^C(n^T) &= -\frac{\pi^{n^T}\sum_{i=1}^{2}d_i^{n^T}}{1+\pi^{n^T}}, \end{aligned}$$

and their noncooperative counterparts by

$$\begin{aligned} \lambda_i(n^t) &= -\frac{1}{1+\pi^{n^t}}\left(\pi^{n^t}d_i^{n^t} - k^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda_i(\nu)\right),\ t=0,...,T-1, \\ \lambda_i(n^T) &= -\frac{\pi^{n^T}d_i^{n^T}}{1+\pi^{n^T}}. \end{aligned}$$

We collect the results for the cooperative case in the following proposition.

Proposition 3 *If the players optimize their joint payoffs, then the optimal control is constant and given by*

$$u_i^C(n^t) = -\frac{g_i^{n^t}}{\pi^{n^t}r_i^{n^t}}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda^C(\nu),$$

where the costate variables are obtained by solving recursively the following equations:

$$\begin{aligned} \lambda^C(n^t) &= -\frac{1}{1+\pi^{n^t}}\left(\pi^{n^t}\sum_{i=1}^{2}d_i^{n^t} - k^{n^t}\sum_{\nu\in\mathcal{S}(n^t)}\pi^\nu\lambda^C(\nu)\right),\ t=0,...,T-1, \\ \lambda^C(n^T) &= -\frac{\pi^{n^T}\sum_{i=1}^{2}d_i^{n^T}}{1+\pi^{n^T}}. \end{aligned}$$

The cooperative state trajectory $x^C(n^t)$ is given by

$$x^C(n^t) = \sum_{i=1}^{2} g_i^{a(n^t)}u_i^C(a(n^t)) + k^{a(n^t)}x^C(a(n^t)), \tag{21}$$

and Player i's optimal payoff by

$$V_i(\mathbf{u}^C) = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \left(\frac{1}{2} r_i u_i^C(n^t)^2 - d_i^{n^t} x^C(n^t) \right) - \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} d_i^{n^T} x^C(n^T). \tag{22}$$

To fully characterize incentive strategies and the credibility conditions, we need to assume a certain functional form for these strategies. Consider the linear strategies given by

$$\psi_1(u_2(n^t)) = u_1^C(n^t) + b_1^{n^t}(u_2(n^t) - u_2^C(n^t)), \tag{23}$$

$$\psi_2(u_1(n^t)) = u_2^C(n^t) + b_2^{n^t}(u_1(n^t) - u_1^C(n^t)). \tag{24}$$

The node-varying parameter $b_i^{n^t}$ represents the penalty that player i imposes on the other player deviation from cooperation at node n^t. Of course, the idea is to have no deviation so that the penalty becomes immaterial. Note that under the linearity assumption of the incentive strategies, it is easy to verify that the conditions in (15)–(16) and (14) become

$$\psi_1'(u_2) = b_1^{n^t}, \quad \psi_2'(u_1) = b_2^{n^t}, \quad b_1^{n^t} \times b_2^{n^t} = 1.$$

The following proposition characterizes the conditions under which these incentive strategies are credible.

Proposition 4 *Consider the game defined by (20) and denote by $(\mathbf{u}^C)$ its cooperative solution. The incentive equilibrium strategy $(\psi_i \in \Psi_i)$ at $u_i^C(n^t)$ for $i = 1, 2$, is credible in $U_1 \times U_2$ if the following conditions hold:*

$$\begin{aligned}
&\frac{1}{2} r_1^{n^0} (u_1^C(n^0)^2 - \psi_1(u_2(n^0))^2) + \sum_{t=1}^{T-1} \sum_{n^t} \pi^{n^t} \Big(\frac{1}{2} r_1^{n^t} (u_1^C(n^t)^2 - \psi_1(u_2(n^t))^2) \\
&- d_1^{n^t} [g_1^{a(n^t)} (u_1^C(a(n^t)) - \psi_1(u_2(a(n^t)))) + k^{a(n^t)} \big(x_1(a(n^t)) - x_2(a(n^t))\big)] \Big) \\
&- \sum_{n^T} \pi^{n^T} d_1^{n^T} [g_1^{a(n^T)} (u_1^C(a(n^T) - \psi_1(u_2(a(n^T)))) \\
&+ k^{a(n^T)} \big(x_1(a(n^T)) - x_2(a(n^T))\big)] \le 0, \ \forall u_2 \in U_2,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2} r_2^{n^0} (u_2^C(n^0)^2 - \psi_2(u_1(n^0))^2) + \sum_{t=1}^{T-1} \sum_{n^t} \pi^{n^t} \Big(\frac{1}{2} r_2^{n^t} (u_2^C(n^t)^2 - \psi_2(u_1(n^t))^2) \\
&- d_2^{n^t} [g_2^{a(n^t)} (u_2^C(a(n^t)) - \psi_2(u_1(a(n^t)))) + k^{a(n^t)} \big(x_3(a(n^t)) - x_4(a(n^t))\big)] \Big) \\
&- \sum_{n^T} \pi^{n^T} d_2^{n^T} [g_2^{a(n^T)} (u_2^C(a(n^T)) - \psi_2(u_1(a(n^T)))) \\
&+ k^{a(n^T)} \big(x_3(a(n^T)) - x_4(a(n^T))\big)] \le 0, \ \forall u_1 \in U_1.
\end{aligned}$$

where $x_1(n^t)$, $x_2(n^t)$, $x_3(n^t)$, and $x_4(n^t)$ are state variables defined by (21) at $(u_1^C(n^t), u_2(n^t))$, $(\psi_1^{n^t}(u_2), u_2(n^t))$, $(u_1(n^t), u_2^C(n^t))$, and $(u_1(n^t), \psi_2^{n^t}(u_1))$ respectively.

Proof. It suffices to compute the expressions of the different payoffs in the inequalities (9) taking into account the expression of player i's payoff in (22). □

4.1.1 Numerical illustration

The credibility conditions involve too long expressions to be amenable to a qualitative analysis. To visualize the set of credible incentive strategies, we shall resort to a simple numerical example. The event tree is depicted in Figure 1, and the parameter values are given in Table 1. The last line in the table specifies the

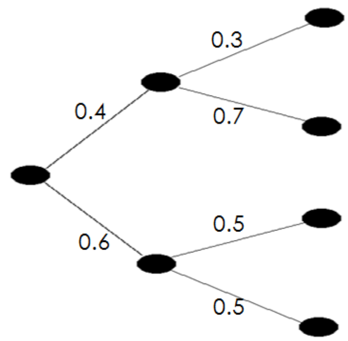

Figure 1: Event tree

Table 1: Parameter values

Node	r_1	r_2	d_1	d_2	g_1	g_2	k
n^0	5	6.5	0.7	1	1	1.5	−0.3
n^1	5.5	7.475	0.721	1.08	1.04	1.635	−0.33
n^2	4.5	5.525	0.697	0.92	0.96	1.365	−0.27
n^3	6.05	8.59625	0.74263	1.1664	1.0816	1.78215	−0.363
n^4	4.95	6.35375	0.69937	0.9936	0.0.9984	1.48785	−0.297
n^5	4.95	6.35375	0.69937	0.9936	0.0.9984	1.48785	−0.297
n^6	4.05	4.69625	0.65863	0.8464	0.9216	1.24215	−0.243
Variation	±0.1	±0.15	±0.03	±0.08	±0.04	±0.09	±0.1

variation of each parameter value with respect to its value at the antecedent node. Note that in a k-level binary tree, each of the two conditions defined in the above proposition contains $2^{k-1}-1$ variables; therefore, three in this example.

The values of the control variables and the penalties at the different nodes are given in Table 2. Observe that $b_1\left(n^t\right) \times b_2\left(n^t\right) = 1$ for all n^t.

Table 2: Control and penalty values

	Controls		Penalty parameters	
Node	u_1	u_2	b_1	b_2
n^0	0.0535	0.0617	0.9202	1.0843
n^1	0.0303	0.0351	0.9205	1.0886
n^2	0.0394	0.0456	0.9546	1.0504

The credibility conditions defined in Proposition 3 correspond to the polyhedron regions, which are not necessarily with flat faces and straight edges, shown in Figures 2 and 3. This representation holds true for any set of parameters in the linear-state game with linear-incentive strategies, that is, the credibility conditions correspond to the area inside a polyhedron. Further, for any set of parameters, we may find the lower and upper bounds for the decision variables for which the polyhedrons are drawn.

4.2 Linear-quadratic game

Linear-quadratic dynamic games are characterized by a linear system of state equations and quadratic objective functions. Suppose that in the game defined by (1) and (2) the functions $L_i(x(n^t), \mathbf{u}(n^t))$ and $\Phi_i(x(n^T))$ are quadratic and $f^{n^t}(x(n^t), \mathbf{u}(n^t))$ is linear in the state variable $x(n^t)$. The optimization problem of player i

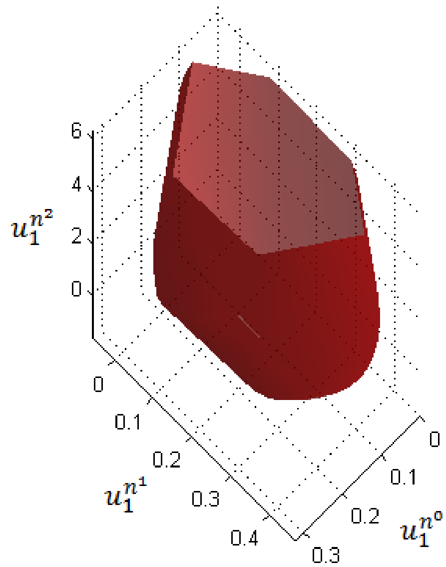

Figure 2: Credibility conditions for player 1

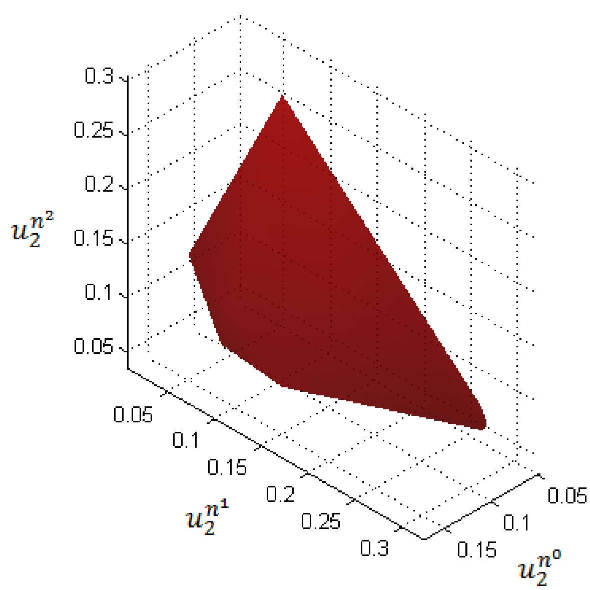

Figure 3: Credibility conditions for player 2

can then be written as follows:

$$\max V_i(x, \mathbf{u}) = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \Big(\frac{1}{2} x'(n^t) Q_i(n^t) x(n^t) + p_i'(n^t) x(n^t) + \frac{1}{2} \sum_{j=1}^{2} u_j'(n^t) R_{ij}(n^t) u_j(n^t) \Big)$$
$$+ \sum_{n^t \in \mathcal{N}^T} \pi^{n^T} \Big(\frac{1}{2} x'(n^T) Q_i(n^T) x(n^T) + p_i'(n^T) x(n^T) \Big),$$

subject to

$$x(n^t) = A(a(n^t))x(a(n^t)) + \sum_{j=1}^{2} B_j(a(n^t))u_j(a(n^t)), \quad x(n^0) = x_0,$$

where $Q_i(n^t) \in \mathbb{R}^{q\times q}$ is a symmetric matrix and $R_{ij}(n^t) \in \mathbb{R}^{m_j^{n^t}\times m_j^{n^t}}$ is a positive definite matrix, $p_i(n^t) \in \mathbb{R}^q$, $A(n^t) \in \mathbb{R}^{q\times q}$ and $B_j(n^t) \in \mathbb{R}^{q\times m_j^{n^t}}$ for all $n^t \in \mathcal{N}^t, t \in T$.

In this setting conditions (4) and (5) can be rewritten as follows:

$$\frac{\partial \mathcal{L}^C}{\partial u_i^{n^t}} = \pi^{n^t} u_i'(n^t) \sum_{j=1}^{2} R_{ji}(n^t) + \lambda^C(S(n^t))B_i^{n^t} = 0, \tag{25}$$

$$\Rightarrow u_i^C(n^t) = -\frac{1}{\pi^{n^t}} \left(\sum_{j=1}^{2} R_{ji}(n^t) \right)^{-1} B_i^{n^t} \lambda^C(S(n^t)).$$

$$-\lambda^C(n^t) = \frac{\partial \mathcal{L}^C}{\partial x^{n^t}} = \pi^{n^t} \sum_{j=1}^{2} (Q_j^{n^t} x^{n^t} + p_j^{n^t}) + A^{n^t} \lambda^C(S(n^t)) - \pi^{n^t} \lambda^C(n^t),$$

$$-\lambda^C(n^T) = \frac{\partial \mathcal{L}^C}{\partial x^{n^T}} = \pi^{n^T} \sum_{j=1}^{2} (Q_j^{n^T} x^{n^T} + p_j^{n^T}),$$

where

$$\lambda^C(S(n^t)) = \sum_{\nu \in S(n^t)} \pi^\nu \lambda^C(\nu).$$

Proposition 5 *The above linear-quadratic game has a cooperative solution, defined by*

$$u_i^C(n^t) = -\frac{1}{\pi^{n^t}} \left(\sum_{j=1}^{2} R_{ji}(n^t) \right)^{-1} B_i^{n^t} \Big(k^\nu (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{j=1}^{2} S_j^{n^t} \alpha^\nu) + \alpha^\nu \Big), \tag{26}$$

where x^C is the associated state trajectory determined by

$$x^C(\nu) = (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{j=1}^{2} S_j^{n^t} \alpha^\nu), \tag{27}$$

with k^{n^t} and α^{n^t} recursively defined by

$$k^{n^t} = \frac{1}{1+\pi^{n^t}} \Big(\pi^{n^t} \sum_{j=1}^{2} Q_j + A^{n^t} k^\nu (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} A^{n^t} \Big), \tag{28}$$

$$\alpha^{n^t} = \frac{1}{1+\pi^{n^t}} \Big(\pi^{n^t} \sum_{j=1}^{2} p_j + A^{n^t} (\alpha^\nu - k^\nu (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} \sum_{j=1}^{2} S_j^{n^t} \alpha^\nu) \Big).$$

where

$$S_i^{n^t} = \frac{1}{\pi^{n^t}} B_i(n^t) (\sum_{j=1}^{2} R_{ij}^{n^t})^{-1} B_i'(n^t), \tag{29}$$

and the matrix $(I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)$ is assumed to be invertible.

Proof. See Appendix. □

As stated before, we need to solve two optimal control problems in order to find the incentive strategies. The problem for player 1 is as follows:[8]

$$V_1(x,\mathbf{u}) = \sum_{t=0}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi^{n^t}\Big(\frac{1}{2}x'(n^t)Q_1(n^t)x(n^t)+p_1'(n^t)x(n^t)+\frac{1}{2}\sum_{j=1}^{2}u_j'(n^t)R_{1j}(n^t)u_j(n^t)\Big)$$
$$+\sum_{n^t\in\mathcal{N}^T}\pi^{n^T}\Big(\frac{1}{2}x'(n^T)Q_1(n^T)x(n^T)+p_1'(n^T)x(n^T)\Big),$$

subject to:

$$x(n^t) = A(a^{n^t})x(a^{n^t})+\sum_{j=1}^{2}B_j(a^{n^t})u_j(a(n^t)),\quad x(n^0)=x_0,$$
$$u_2(n^t)=\psi_2(u_1(n^t)).$$

The corresponding Lagrangian is given by

$$\mathcal{L}_1(\lambda_1,x,u_1) = \frac{1}{2}\Big\{x'(n^0)Q_1^{n^0}x(n^0)+2p_1^{'n^0}x(n^0)+u_1'(n^0)R_{11}^{n^0}u_1(n^0)+\psi'(u_1(n^0))_2R_{12}^{n^0}\psi_2(u_1(n^0))\Big\}$$
$$+\sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\frac{\pi^{n^t}}{2}\Big\{x'(n^t)Q_1^{n^t}x(n^t+2p_1^{'n^t}x(n^t)+u_1'(n^t)R_{11}^{n^t}u_1(n^t)+\psi_2'(u_1(n^t))R_{12}^{n^t}\psi_2(u_1(n^t))\Big\}$$
$$+\sum_{n^T\in\mathcal{N}^T}\frac{\pi^{n^T}}{2}\Big\{x'(n^T)Q_1^{n^T}x(n^T)+2p_1^{'T}x(n^T)\Big\}+\lambda_1(n^0)(x_0-x(n^0))$$
$$+\sum_{t=1}^{T}\sum_{n^t\in\mathcal{N}^t}\pi^{n^t}\lambda_1(n^t)\Big\{A^{a(n^t)}x(a(n^t))+B_1^{a(n^t)}u_1(a(n^t))+B_2^{a(n^t)}\psi_2(u_1(a(n^t)))-x(n^t)\Big\},$$

where $\lambda_1(\cdot)$ represents the vector of Lagrange multipliers. The first-order optimality conditions are

$$\begin{aligned}
\frac{\partial\mathcal{L}_1}{\partial u_1^{n^t}} &= \pi^{n^t}(u_1'(n^t)R_{11}^{n^t}+\psi_2'(u_1(n^t))\frac{\partial\psi_2}{\partial u_1}R_{12}^{n^t})+\sum_{\nu\in S(n^t)}\pi^{\nu}\lambda_1^{\nu}(B_1^{n^t}+B_2^{n^t}\frac{\partial\psi_2}{\partial u_1})=0,\\
\Rightarrow u_1(n^t) &= -\frac{1}{\pi^{n^t}}(R_{11}^{n^t})^{-1}(B_1^{n^t}+B_2^{n^t}\frac{\partial\psi_2}{\partial u_1})\lambda_1(S)-(R_{11}^{n^t})^{-1}\psi_2'(u_1(n^t))\frac{\partial\psi_2}{\partial u_1}R_{12}^{n^t},\\
\lambda_1(n^t) &= \frac{\partial\mathcal{L}_1}{\partial x^{n^t}} = \pi^{n^t}(Q_1^{n^t}x(n^t)+p_1^{n^t})+A^{n^t}\lambda_1(S)-\pi^{n^t}\lambda_1(n^t),\\
\lambda_1(n^t) &= \pi^{n^T}(Q_1^{n^T}x(n^T)+p_1^{n^T}).
\end{aligned}\tag{30}$$

The proposition below states the conditions to be satisfied by incentive strategies.

Proposition 6 *The pair of strategies $(\psi_1,\psi_2)\in\Psi_1\times\Psi_2$ is an incentive equilibrium at $\mathbf{u}^C$ if it satisfies the following conditions:*[9]

$$(R_{1i}(n^t)+R_{2i}(n^t))^{-1}B_i^{n^t}(k^{\nu}x^C(n^t)+\alpha^{\nu}) = R_{ii}^{-1}(n^t)(B_i^{n^t}+B_j^{n^t}\frac{\partial\psi_j}{\partial u_i})(k_i^{\nu}x^C(\nu)+\alpha_i^{\nu})$$
$$+\pi^{n^t}R_{ii}^{-1}R_{ij}^{n^t}\frac{\partial\psi_j}{\partial u_i}\psi_j'(u_i(n^t)),\quad i,j=1,2,\ i\neq j,$$

with $k_i^{n^t}$ and $\alpha_i^{n^t}$ recursively defined by

$$k_i^{n^t} = \frac{1}{1+\pi^{n^t}}\Big(\pi^{n^t}Q_i^{n^t}+A^{n^t}k_i^{\nu}(I+\sum_{j=1}^{2}(S_j^{n^t}+l_j^{n^t})k_i^{\nu})^{-1}A^{n^t}\Big),\tag{31}$$

[8]To keep it as simple as possible and without loss of generality, we write down the optimality conditions for player 1 and then we switch to the general case.

[9]All functions evaluated at (u_1^C,u_2^C).

$$\alpha_i^{n^t} = \frac{1}{1+\pi^{n^t}}\left(\pi^{n^t} p_i^{n^t} + A^{n^t}\left\{\alpha_i^{\nu} - k_i^{\nu}\big(I + \sum_{j=1}^{2}(S_j^{n^t} + l_j^{n^t})k_i^{\nu}\big)^{-1}\sum_{j=1}^{2}\big((S_j^{n^t} + l_j^{n^t})\alpha_i^{\nu} + m_i^{n^t}\big)\right\}\right),$$

where

$$\begin{aligned} S_i^{n^t} &= \frac{1}{\pi^{n^t}} B_i^{n^t}(R_{ii}^{n^t})^{-1} B_i^{n^t}, \\ l_i^{n^t} &= \frac{1}{\pi^{n^t}} B_i^{n^t}(R_{ii}^{n^t})^{-1} B'^{n^t}_j \frac{\partial \psi_j}{\partial u_i}, \\ m_i^{n^t} &= B_i^{n^t}(R_{ii}^{n^t})^{-1}\psi'_j(u_i(n^t))\frac{\partial \psi_j}{\partial u_i} R_{ij}^{n^t}, \quad i,j = 1,2,\ i \neq j. \end{aligned} \tag{32}$$

Proof. See Appendix. □

To obtain more insight, let us assume the following functional forms, all the parameters of the game may vary in different nodes:

$$\begin{aligned} L_i^{n^t}(x(n^t), \mathbf{u}(n^t)) &= \frac{1}{2}\left\{r_i^{n^t} u_i(n^t)^2 - d_j^{n^t} x(n^t)^2 - c_i^{n^t} x(n^t)\right\}, \\ \Phi_i^{n^T} &= -\frac{1}{2}\left\{d_i^{n^T} x(n^T)^2 + c_i^{n^T} x(n^T)\right\}, \\ x(n^t) &= \sum_{j=1}^{2} g_j^{a(x^{n^t})} u_j(a(x^{n^t})) + k^{a(x^{n^t})} x(a(x^{n^t})); \quad x(n^0) = x_0. \end{aligned} \tag{33}$$

It is easy to verify that in this linear-quadratic game, player j's optimal payoff under cooperation is given by

$$\begin{aligned} V_i(\mathbf{u}^C) = &\sum_{t=0}^{T-1}\sum_{n^t\in\mathcal{N}^t}\frac{\pi^{n^t}}{2}\left\{r_i^{n^t} u_i^C(n^t)^2 - d_i^{n^t} x^C(n^t)^2 - c_i^{n^t} x^C(n^t)\right\} \\ &- \sum_{n^T\in\mathcal{N}^T}\frac{\pi^{n^T}}{2}\left\{d_i^{n^T} x^C(n^T)^2 + c_i^{n^T} x^C(n^T)\right\}, \end{aligned} \tag{34}$$

where $u_j^C(n^t)$ and $x^C(n^t)$ are given by (26) and (27) respectively.

Proposition 7 *Consider the game defined by (33). Denote by ($\mathbf{u}^C$) its cooperative solution. The incentive equilibrium strategy ($\psi_i \in \Psi_i$) at $u_i^C(n^t)$ for $i = 1,2$, is credible in $U_1 \times U_2$ if the following conditions hold:*

$$\begin{aligned} &\frac{1}{2} r_1^{n^0}\big(u_1^C(n^0)^2 - \psi_1(u_2(n^0))^2\big) + \sum_{t=1}^{T-1}\sum_{n^t}\frac{\pi^{n^t}}{2}\Big(r_1^{n^t}\big(u_1^C(n^t)^2 - \psi_1(u_2(n^t))^2\big) \\ &- d_1^{n^t}[x\big(u_1^C(n^t), u_2(n^t)\big)^2 - x\big(\psi_1(u_2(n^t)), u_2(n^t)\big)^2] \\ &- c_1^{n^t}[x\big(u_1^C(n^t), u_2(n^t)\big) - x\big(\psi_1(u_2(n^t)), u_2(n^t)\big)]\Big) \\ &- \sum_{n^T}\frac{\pi^{n^T}}{2}\Big(d_1^{n^T}[x\big(u_1^C(n^T), u_2(n^T)\big)^2 - x\big(\psi_1(u_2(n^T)), u_2(n^T)\big)^2] \\ &+ c_1^{n^T}[x\big(u_1^C(n^T), u_2(n^T)\big) - x\big(\psi_1(u_2(n^T)), u_2(n^T)\big)]\Big) \leq 0, \quad \forall u_2 \in U_2, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2} r_2^{n^0}\big(u_2^C(n^0)^2 - \psi_2(u_1(n^0))^2\big) + \sum_{t=1}^{T-1}\sum_{n^t}\frac{\pi^{n^t}}{2}\Big(r_2^{n^t}\big(u_2^C(n^t)^2 - \psi_2(u_1(n^t))^2\big) \\ &- d_2^{n^t}[x\big(u_1(n^t), u_2^C(n^t)\big)^2 - x\big(u_1(u^t), \psi_2(u_1(n^t))\big)^2] \end{aligned}$$

$$- c_2^{n^t}[x\big(u_1(n^t), u_2^C(n^t)\big) - x\big(u_1(n^t), \psi_2(u_1(n^t))\big)]\Big)$$
$$- \sum_{n^T} \frac{\pi^{n^T}}{2}\Big(d_2^{n^T}[x\big(u_1(n^T), u_2^C(n^T)\big)^2 - x\big(u_1(n^T), \psi_2(u_1(n^T))\big)^2]$$
$$+ c_2^{n^T}[x\big(u_1(n^T), u_2^C(n^T)\big) - x\big(u_1(n^T), \psi_2(u_1(n^T))\big)]\Big) \leq 0, \quad \forall u_1 \in U_1.$$

where $x^C(n^t)$ *is the cooperative state variable defined in (27).*

Proof. Similar to the linear-state game, it suffices to compute the expressions of the different payoffs in the inequalities (9) taking into account the expression of player i's payoff along a given decision established in (34). □

4.2.1 Numerical illustration

We retain the event tree in Figure 1 and the parameter values in Table 3. Again, we assume that the incentive strategies are linear, with their expressions being given in (23) and (24). Table 4 provides the optimal control values as well as the penalty terms at the different nodes. Again, as in the linear-state again, we note that the product of the penalty terms at each node is equal to one, i.e., $b_1(n^t) \times b_2(n^t) = 1$ for all n^t. Also, each of the credibility conditions defined in Proposition 5 corresponds to the area inside a polyhedron as shown in Figures 4 and 5. Lower and upper bounds for the decision variables may be found based on the drawn polyhedrons.

Table 3: Parameter values

Node	r_1	r_2	c_1	c_2	d_1	d_2	g_1	g_2	k
n^0	5	6.5	0.7	1	0.3	0.5	1	1.5	−0.3
n^1	5.5	7.475	0.721	1.08	0.303	0.525	1.04	1.635	−0.33
n^2	4.5	5.525	0.697	0.92	0.297	0.475	0.96	1.365	−0.27
n^3	6.05	8.59625	0.74263	1.1664	0.306	0.551	1.0816	1.78215	−0.363
n^4	4.95	6.35375	0.69937	0.9936	0.3	0.499	0.998	1.48785	−0.297
n^5	4.95	6.35375	0.69937	0.9936	0.3	0.499	0.998	1.48785	−0.297
n^6	4.05	4.69625	0.65863	0.8464	0.294	0.451	0.992	1.24215	−0.243
Variation	±0.1	±0.15	±0.03	±0.08	±0.01	±0.05	±0.04	±0.09	±0.1

Table 4: Control and penalty values

	Controls		Penalty parameters	
Node	u_1	u_2	b_1	b_2
n^0	0.0512	0.0591	0.9317	1.0731
n^1	0.0293	0.0338	0.9189	1.0874
n^2	0.0381	0.0441	0.9486	1.0553

5 Conclusion

In this paper, we determined incentive equilibrium strategies in linear-state and linear-quadratic dynamic games played over event trees, and characterized the conditions under which these strategies are credible. We illustrated the implementation of such equilibria on two very simple examples, where we obtained non-empty regions for credibility.

Two extensions of this work are worth considering. First, the results have been obtained under the assumption of linear incentive strategies. Using other forms is clearly possible and it would be of interest to see the impact of having non-linear strategies on the credibility issue. Second, extending the formalism of incentive strategies to more than two players is a challenging and relevant research question.

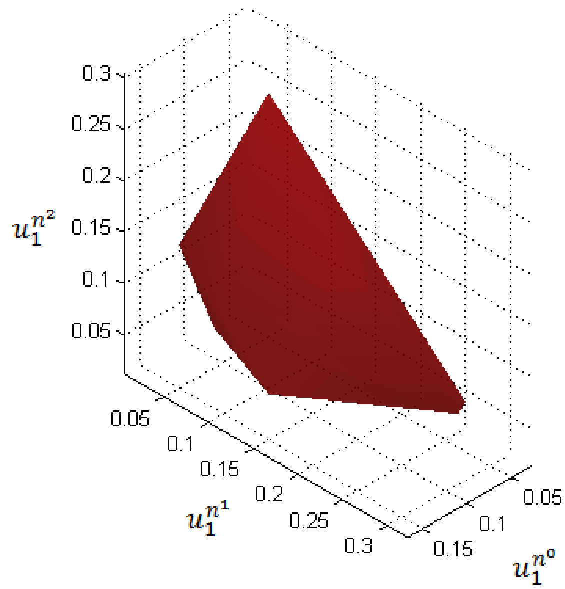

Figure 4: Credibility conditions for player 1

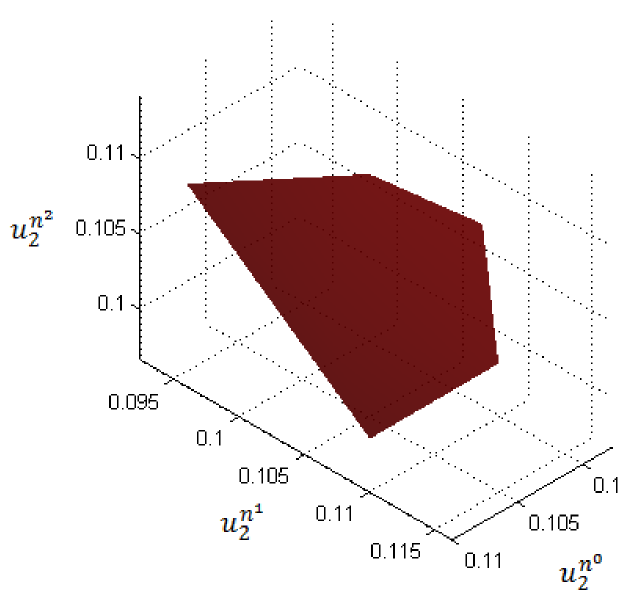

Figure 5: Credibility conditions for player 2

Appendix

Proof of Proposition 4. The first-order optimality conditions are given in (25). If $S_i^{n^t}$ is defined by (29),

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \lambda^C(S(n^t)); \quad x^C(n^0) = x_0;\ \nu \in S(n^t);\ n^t \in \mathcal{N}^t;\ t = 1, ..., T$$

Let us suppose that the costate variables are linear in the state (see [5]), that is,

$$\lambda^C(n^t) = k^{n^t} x^C(n^t) + \alpha^{n^t}; \quad n^t \in \mathcal{N}^t; \forall t$$

which leads to

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{\nu \in S(n^t)} \pi^\nu \sum_{i=1}^{2} S_i^{n^t} k^\nu x^C(\nu) - \sum_{\nu \in S(n^t)} \pi^\nu \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu.$$

The right-hand side of the above equation contains the expected value of the terms evaluated at the successor nodes $\nu \in S(n^t)$. We know that

$$x^C(\nu_1) = x^C(\nu_2); \quad \forall \nu_1, \nu_2 \in S(n^t),$$

and

$$\sum_{\nu \in S(n^t)} \pi^\nu x^C(\nu) = x^C(\nu).$$

Since the matrix $(I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)$ is assumed to be invertible, we have

$$x^C(\nu) = (I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu),$$

and

$$\begin{aligned} \lambda^C(\nu) &= k^\nu x^C(\nu) + \alpha^\nu \\ &= k^\nu (I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu) + \alpha^\nu. \end{aligned} \tag{35}$$

From the optimality conditions given in (25), we have

$$-\lambda^C(n^t) = \frac{\partial \mathcal{L}^C}{\partial x^{n^t}} = \pi^{n^t} \sum_{j=1}^{2} (Q_j^{n^t} x^{n^t} + p_j^{n^t}) + A^{n^t} \lambda^C(S(n^t)) - \pi^{n^t} \lambda^C(n^t).$$

Substituting (35) in the above equation, yields the following equation:

$$\begin{aligned} (1 + \pi^{n^t})(k^{n^t} x^C(n^t) + \alpha^{n^t}) &= \pi^{n^t} \Big((Q_1^{n^t} + Q_2^{n^t}) x^C(n^t) + (p_1^{n^t} + p_2^{n^t}) \Big) \\ &\quad + A^{n^t} \Big(k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu) + \alpha^\nu \Big) \\ &= \Big(\pi^{n^t} (Q_1^{n^t} + Q_2^{n^t}) + A^{n^t} k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} A^{n^t} \Big) x^C(n^t) \\ &\quad + \pi^{n^t} (p_1^{n^t} + p_2^{n^t}) + A^{n^t} \Big(\alpha^\nu - k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu \Big). \end{aligned}$$

Collecting the coefficients of $x^C(n^t)$, the relations in (28) follow. The remaining statements follow from using the terminal conditions.

Proof of Proposition 5. Using the optimality conditions defined in (30)

$$\begin{aligned} x^C(\nu) &= A^{n^t} x^C(n^t) \\ &- B_1^{n^t}\Big\{\frac{1}{\pi^{n^t}}(R_{11}^{n^t})^{-1}(B_1^{n^t}+B_2^{n^t}\frac{\partial\psi_2}{\partial u_1})\lambda_1(S(n^t)) - (R_{11}^{n^t})^{-1}\psi_2'(u_1(n^t))\frac{\partial\psi_2}{\partial u_1}R_{12}^{n^t}\Big\} \\ &- B_2^{n^t}\Big\{\frac{1}{\pi^{n^t}}(R_{22}^{n^t})^{-1}(B_2^{n^t}+B_1^{n^t}\frac{\partial\psi_1}{\partial u_2})\lambda_2(S(n^t)) - (R_{22}^{n^t})^{-1}\psi_1'(u_2(n^t))\frac{\partial\psi_1}{\partial u_2}R_{21}^{n^t}\Big\}. \end{aligned}$$

Using the definitions in (32), we can simplify the above relation as follows:

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{i=1}^{2}\Big((S_i^{n^t}+l_i^{n^t})\lambda_i(S(n^t)) + m_i^{n^t}\Big).$$

Now define

$$\lambda_i(n^t) = k_i^{n^t} x^C(n^t) + \alpha_i^{n^t}; \quad n^t \in \mathcal{N}^t; \forall t,$$

which leads to

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{\nu\in S(n^t)} \pi^\nu \sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k_i^\nu x^C(\nu) - \sum_{\nu\in S(n^t)} \pi^\nu \sum_{i=1}^{2}\Big((S_i^{n^t}+l_i^{n^t})\alpha_i^\nu + m_i^{n^t}\Big).$$

Since $x^C(\nu_1) = x^C(\nu_2);\ \ \forall \nu_1,\nu_2 \in S(n^t)$ and $\sum_{\nu\in S(n^t)}\pi^\nu x^C(\nu) = x^C(\nu)$, and the matrix $(I+\sum_{i=1}^2(S_i^{n^t}+l_i^{n^t})k^\nu)$ is assumed to be invertible, we have

$$x^C(\nu) = (I+\sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k^\nu)^{-1}(A^{n^t}x^C(n^t) - \sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})\alpha^\nu + m_i^{n^t}),$$

and

$$\begin{aligned} \lambda_i(\nu) &= k_i^\nu x^C(\nu) + \alpha_i^\nu \\ &= k_i^\nu (I+\sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k^\nu)^{-1}\left(A^{n^t}x^C(n^t) - \sum_{i=1}^{2}\Big((S_i^{n^t}+l_i^{n^t})\alpha_i^\nu + m_i^{n^t}\Big)\right) + \alpha_i^\nu. \end{aligned} \tag{36}$$

From the optimality conditions given in (30), we have

$$-\lambda_i(n^t) = \frac{\partial\mathcal{L}_i}{\partial x^{n^t}} = \pi^{n^t}\sum_{i=1}^{2}(Q_i^{n^t}x(n^t)+p_i^{n^t}) + A^{n^t}\lambda_i(S(n^t)) - \pi^{n^t}\lambda_i(n^t).$$

Substituting (36) in the above equation, we obtain the following equation:

$$\begin{aligned} (1+\pi^{n^t})(k_i^{n^t}x^C(n^t)+\alpha_i^{n^t}) &= \pi^{n^t}\Big(Q_i^{n^t}x(n^t)+p_i^{n^t}\Big) \\ &+ A^{n^t}\left\{k_i^\nu(I+\sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k_i^\nu)^{-1}\left(A^{n^t}x^C(n^t) - \sum_{i=1}^{2}\Big((S_i^{n^t}+l_i^{n^t})\alpha_i^\nu + m_i^{n^t}\Big)\right) + \alpha_i^\nu\right\} \\ &= \Big(\pi^{n^t}Q_i^{n^t} + A^{n^t}k_i^\nu(I+\sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k_i^\nu)^{-1}A^{n^t}\Big)x^C(n^t) \\ &+ \pi^{n^t}p_i^{n^t} + A^{n^t}\Big\{\alpha_i^\nu - k_i^\nu(I+\sum_{i=1}^{2}(S_i^{n^t}+l_i^{n^t})k_i^\nu)^{-1}\sum_{i=1}^{2}\Big((S_i^{n^t}+l_i^{n^t})\alpha_i^\nu + m_i^{n^t}\Big)\Big\}. \end{aligned}$$

Collecting the coefficients of $x^C(n^t)$ leads to the relations in (31). The remaining statements directly follow from using the terminal conditions.

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