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Abstract: In this paper, we introduce the randomized Condorcet voting system. Our main contribution is to present it as a natural extension of Condorcet's ideas, hence giving it solid philosophical grounds. Namely, the randomized Condorcet voting system elects the essentially unique Condorcet winner of lotteries of candidates. Moreover, we prove three major results regarding Condorcet compatibility, that is, the fact that a voting system elects Condorcet winners when they exist and is incentive-compatible. First, we show that there is no strongly Condorcet-compatible voting system. Second, we show that the randomized Condorcet voting system is the unique dominant-strategy Condorcet-compatible voting system, in a large class of voting systems. Third, we prove that, as opposed to other known methods, the randomized Condorcet voting system is strongly incentive-compatible when alternatives range on a left-right axis. Eventually, these fundamental properties of the randomized Condorcet voting system lead us to strongly recommend its use in practice, especially when deterministic Condorcet winners are likely to exist.

Key Words: Voting system, social choice, Condorcet winner, incentive-compatibility.

Résumé :   Dans cet article, nous introduisons le scrutin de Condorcet randomisé. Notre principale contribution est de le présenter en tant qu'extension naturelle des idées de Condorcet, ce qui permet de justifier sa légitimité d'un point de vue philosophique. En effet, le scrutin de Condorcet randomisé élit le vainqueur de Condorcet des loteries des candidats, dont l'existence et l'unicité sont garanties en pratique. De plus, nous prouvons trois résultats majeurs concernant la Condorcet-compatibilité. De façon grossière, un scrutin est Condorcet-compatible s'il élit les vainqueurs de Condorcet lorsqu'ils existent, et si les électeurs ont alors intérêt à voter conformément à leurs vraies préférences. Premièrement, nous montrons qu'il n'existe pas de scrutin fortement Condorcet-compatible. Ensuite, nous prouvons que le scrutin de Condorcet randomisé est l'unique scrutin Condorcet-compatible à stratégies dominantes, parmi une large classe de scrutins. Enfin, nous montrons que, contrairement à d'autres méthodes connues, le scrutin de Condorcet randomisé est fortement compatible avec les incitatifs lorsque les alternatives se placent sur un axe gauche-droite. Au final, toutes ces propriétés fondamentales du scrutin de Condorcet randomisé nous amènent à fortement recommander son utilisation en pratique, surtout lorsqu'il y a de bonnes chances qu'il existe un vainqueur de Condorcet déterministe.

"Qu'importe que tout soit bien, pourvu que nous fassions en sorte que tout soit mieux qu'il n'était avant nous." Marquis de Condorcet

1 Introduction

Social choice theory consists in choosing an alternative for a group of people whose individual preferences may greatly differ from one another. The first mathematician to address this question was Condorcet (1785). Condorcet introduced the idea that an alternative which is preferred to any other by the majority should be the one chosen for the group. Such an alternative is now known as a *Condorcet winner*.

Unfortunately, Condorcet went on proving that a Condorcet winner does not necessarily exist. Indeed, if a third of the people prefers x to y to z, another third prefers y to z to x, and the last third prefers z to x to y, then a majority of 2/3 prefers x to y, while another one of 2/3 prefers y to z, and a third majority still of 2/3 prefers z to x. This example is now known as a Condorcet paradox. It has been the essence of many impossibility theorems in more recent years. For instance, Arrow (1951) famously derived the impossibility of a "fair" aggregation of the preferences of the individuals into a preference of the group.

Much progress has been made in the understanding of the Condorcet paradox. Mainly, McGarvey (1953) introduced the concept of tournament, which can be regarded as a directed graph of alternatives. More precisely, an arc is drawn from alternative x to alternative y if the majority prefers x to y. By assuming no draw between alternatives, the tournament is then a complete antisymmetric directed graph. A Condorcet winner is a node with no incoming arc. It is easy to see that, if it exists, the Condorcet winner is unique. When no Condorcet winner exists, several sets have been defined to consider all the nearly Condorcet winners of the tournament. For instance, the top cycle contains all alternatives from which a path leads to any other alternative. This top cycle has strong connections with binary agendas, which consist in sequentially removing one of two alternatives. A more exhaustive survey of these sets, including the uncovered set and the Banks set, appears in Myerson (1996).

The main contribution of this paper is notice that there is a natural way to extend Condorcet's ideas to cases where no Condorcet winner exists. To do so, we need to think in terms of probability distributions over alternatives, known as *lotteries*. For instance, electing x with probability 2/3, y with probability 1/6 and z with probability 1/6 is a lottery. Now, the preferences of the majority over alternatives can be naturally extended to preferences over lotteries. Interestingly, when there is no draw between alternatives, these preferences over lotteries always yield a unique Condorcet winner, that is, there is one and only one lottery that the majority prefers to any other. It is this Condorcet winner that we propose to elect through the *randomized Condorcet voting system*. While this voting system is hinted at in Myerson (1996) and studied by Peyre (2012c), we are the first to present it as a natural extension of Condorcet's ideas, hence providing firm ground for its legitimacy. We give its formal definition in Section 2.

Historically, de Borda (1781) proposed a voting system which consisted in having voters marking the alternatives. Condorcet criticized this method. He claimed that it gave incentives to voters not to reveal their preferences truthfully. In modern terms, the Borda voting system is not *incentive-compatible*. The trouble with non-incentive-compatible voting systems is that we have no reason to trust the meaningfulness of the ballots of the people. As a result, we may end up making decisions which are completely irrelevant as they are based on erroneous intelligence. For this reason, incentive-compatibility, which is sometimes also called strategy-proofness, has been an essential concept of social choice theory and mechanism design. Loosely, it corresponds to truthfulness being people's best strategy. The famous Gibbard (1973) - Satterthwaite (1975) impossibility theorem and the Gibbard (1978) theorem have shown how restrictive the impossibility requirements are, as they assert that the only incentive-compatible voting systems are mixtures of referendums and (stochastic) dictatorships.

In addition to its naturalness, the randomized Condorcet voting system also benefits from enviable incentive-compatibility properties. Now, the Gibbard (1978) theorem immediate proves that the randomized Condorcet voting system is not incentive-compatible in Gibbard's sense. But it is noteworthy that his the-

orem assumes preferences to satisfy the axioms by Von Neumann and Morgenstern (1945).[1] These axioms make incentive-compatibility very constraining. For this reason, in this paper, we will restrict ourselves to a simpler extension of preferences over alternatives to preferences over lotteries. Namely, we will consider that a lottery $\tilde{x}$ is preferred to $\tilde{y}$ if the choice of $\tilde{x}$ is more often preferred to that of $\tilde{y}$ rather than the other way around. Interestingly, as a result, preferences over lotteries are fully determined by the ordering of alternatives.

In Section 3, using these preferences over lotteries, we define the concept of *Condorcet compatibility*. Namely, a voting system is Condorcet-compatible if, when a unique Condorcet winner exists, it elects this Condorcet winner and is incentive-compatible. In our analysis, incentive-compatibility comes in two flavors. First is the strong incentive-compatibility, which requires truthfulness to be a strong Nash equilibrium. This property is often also known as coalitional strategy-proofness as it asserts that no coalition has incentive to deviate altogether from truthfulness. However, strong incentive-compatibility is often regarded as too constraining, and we will confirm this by proving that there is no strongly Condorcet-compatible voting system.

One way to overcome this new impossibility theorem is to weaken the concept of incentive-compatibility. A classical approach to do so is to require truthfulness to be a Nash equilibrium. However, because deviations from truthfulness by a single voter usually does not affect the status of Condorcet winner of an alternative, most ballots are in fact Nash equilibria. Another more restrictive approach widely used in social choice theory is dominant strategy incentive-compatibility. This is the one we shall use, albeit we will slightly adapt it to our setting. Namely, a voting system shall be called dominant strategy incentive-compatible if no group of similarly minded conspirators ever has incentive to deviate from truthfulness. This leads us to define the dominant strategy Condorcet compatibility (DSCC). Importantly, the randomized Condorcet voting system is DSCC.

In fact, in Section 4, we show a near-uniqueness of the DSCC voting system. Obviously, whenever the ballots yield a Condorcet winner, any Condorcet-compatible voting system elects it. But in addition, we show that if the pairwise comparisons of the alternatives by the majority are close to equality, any DSCC voting system which is defined based on these pairwise comparisons must agree with the randomized Condorcet voting system. A major corollary is that the randomized Condorcet voting system is the only DSCC voting system that is defined on the tournament of the ballots.

One reason why we find Condorcet-compatibility very relevant is the existence of Condorcet winners in practice in many cases. An explanation for this is the one-dimensionality of alternatives. Typically, in politics, candidates usually range on a left-right axis, which is reflected by the preferences the people may have. Two distinct characterizations of this phenomenon have appeared in the literature. First is *single-peakedness*, introduced by Black (1958), which asserts that each voter has a preferred alternative, and that, the further an alternative is from the preferred alternative, the less it is appreciated by the voter. However, Roberts (1977) argues that the popularity of an alternative does not only depend on its positioning on the left-right line. Typically, an unknown centered alternative may not be popular among voters, despite its privileged positioning. This has led him to introduce *single-crossing*, which is based on a left-right line-up of the voters as well. Then, if a left voter prefers the right alternative to a left alternative, a right voter must then agree with the left voter.

In any of these two cases, it has been shown (Black (1958); Roberts (1977); Rothstein (1990, 1991); Gans and Smart (1996)), that a Condorcet winner exists, and is the median voter's favorite. This led Moulin (1980) to design the median social rule, which consists in choosing the median voter's favorite alternative. He shows that this is a dominant strategy incentive-compatible when preferences are single-peaked. This voting system has been generalized by Saporiti (2009) who also prove that any deterministic strongly incentive-compatible voting system for single-crossing preferences must have a similar form. However, these median social choice rules require taking advantage of a known left-right line on which alternatives range. In practice though, in many cases such as politics, while the left-right structure exists informally, it is not official and can therefore not be used to design the voting system. Plus, in these settings (Moulin (1980); Saporiti (2009)), voters are

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[1]These axioms assume that preferences over lotteries are complete, continuous, independent and transitive.

constrained to choose ballots which only belong to a subset of all orderings. In fact, Penn et al. (2011) proved that, when preferences are single-peaked but ballots are not restricted to such single-peaked orderings of the alternatives, a deterministic strongly incentive-compatible voting system must be dictatorial.

In Section 5, we detail single-peakedness and single-crossing, and we prove that, when preferences satisfy one of these criteria, the randomized Condorcet voting system is strongly incentive-compatible, without any restriction on the ballots voters are allowed to choose. Finally, Section 6 will conclude.

2   The randomized Condorcet voting system

We consider an election with a set X of alternatives. We denote $\mathcal{O}$ the set of total order relations[2] on X. We consider that each voter has a preference $\theta \in \mathcal{O}$ which is such an ordering of alternatives. We denote $\theta : x \succ y$ the fact that θ ranks x ahead of y. The relation $\theta : x \succeq y$ then corresponds to $\theta : x \succ y$ or $x = y$. A preference is fully defined by the ordering of all alternatives. For instance, if $X = \{x, y, z\}$, an example of preference is $\theta : y \succ x \succ z$.

To determine an outcome from a vote, we first need to aggregate votes. This can be done by computing the preference profile $(\theta_1, \ldots, \theta_n) \in \mathcal{O}^n$. However, if we require the voting system to be anonymous (that is, independent from players' labeling), then it suffices to count the frequencies of each preference of $\mathcal{O}$. These frequencies are equivalently described by a probability distribution $\tilde{\theta} \in \Delta(\mathcal{O})$. The fact that we can restrict ourselves to focusing on $\tilde{\theta}$ rather than the preference profile is guaranteed by the following theorem, which we merely loosely state for the sake of exposition.

Theorem 1 *Anonymous voting systems are restrictions of voting systems with inputs in $\Delta(\mathcal{O})$.*

Proof. See appendix for the proof, as well as for a more formal statement of the theorem.   □

Importantly, because of this theorem, we will consider that the only information retrieved by the preferences (or the ballots) of the people is the probability distribution $\tilde{\theta} \in \Delta(\mathcal{O})$. Note that this modeling can also describe ballots where some voters are given greater weights than others.

Remark 1 *We identify canonically $\Delta(\mathcal{O})$ with the simplex of the vector space $\mathbb{R}^{\mathcal{O}}$. In particular, this defines convex combinations over $\Delta(\mathcal{O})$. Plus, for simplicity of notations, the canonical injection $\mathcal{O} \to \Delta(\mathcal{O})$, which maps a preference $\theta \in \mathcal{O}$ to the Dirac distribution δ_θ that chooses θ with probability 1, leads us to identify each preference $\theta \in \mathcal{O}$ with its image $\delta_\theta \in \Delta(\mathcal{O})$. Thus, $\mathcal{O} \subset \Delta(\mathcal{O})$.*

2.1   Tournaments

For preferences $\tilde{\theta} \in \Delta(\mathcal{O})$, we define the relative surplus $\mathsf{referendum}(\tilde{\theta}, x, y)$ of voters who prefer x to y by

$$\mathsf{referendum}(\tilde{\theta}, x, y) = \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : x \succ y\right] - \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : x \prec y\right], \tag{1}$$

where we assimilated the probability distribution $\tilde{\theta}$ with a corresponding random variable. Note that the probability operators should rather be read as the ratios of citizens who prefer x to y, and y to x. Now, the fact that a majority of the people prefers alternative x to y then corresponds to the inequality $\mathsf{referendum}(\tilde{\theta}, x, y) > 0$. In other words, $\mathsf{referendum}(\tilde{\theta}, x, y)$ precisely describes the result of a referendum between x and y.

This remark leads us to map any preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the people to a weighted directed graph. The nodes are the alternatives $x \in X$, and the graph has an arc $x \gg y$ if $\mathsf{referendum}(\tilde{\theta}, x, y) > 0$. This arc is then given the weight $\mathsf{referendum}(\tilde{\theta}, x, y)$. An arc from x to y is thus to be read as the fact that x beats y in a

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[2]A relation $\succeq$ on X is a total order if it satisfies all following properties:
   Antisymmetry: If $x \succeq y$ and $y \succeq x$, then $x = y$.
   Transitivity: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
   Totality: We have $x \succeq y$ or $y \succeq x$.

referendum. Note that the weighted directed graph we obtained is antisymmetric. This means that if there is an arc $x \gg y$, then $y \gg x$ is not an arc.

Weighted directed antisymmetric graphs are called weighted tournaments. We denote $\mathcal{WT}$ the set of weighted tournaments. The construction above yields a natural mapping $WT : \Delta(\mathcal{O}) \to \mathcal{WT}$, which defines the weighted tournament $WT(\tilde{\theta})$ of the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the people.

Example 1 *Consider $X = \{x, y, z\}$ and preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the following form:*

$$\mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : x \succ y \succ z\right] = 27/100, \quad \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : y \succ z \succ x\right] = 21/100, \quad \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : z \succ x \succ y\right] = 20/100,$$

$$\mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : x \succ z \succ y\right] = 8/100, \quad \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : y \succ x \succ z\right] = 14/100, \quad \mathbb{P}_{\tilde{\theta}}\left[\tilde{\theta} : z \succ y \succ x\right] = 10/100,$$

Given these preferences, we have the following surplus:

$$100 \text{ referendum}(\tilde{\theta}, x, y) = 27 + 8 - 21 - 14 + 20 - 10 = 10 > 0, \tag{2}$$

$$100 \text{ referendum}(\tilde{\theta}, y, z) = 27 - 8 + 21 + 14 - 20 - 10 = 24 > 0, \tag{3}$$

$$100 \text{ referendum}(\tilde{\theta}, z, x) = -27 - 8 + 21 - 14 + 20 + 10 = 2 > 0. \tag{4}$$

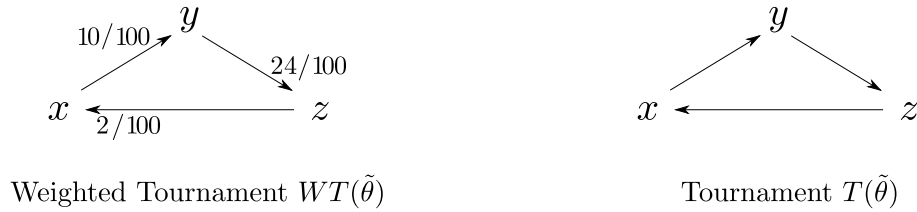*The corresponding tournaments are depicted in Figure 1.*

Figure 1: Tournaments illustrating a Condorcet paradox

If we drop the weights on the arcs, we obtain a merely asymmetric graph. Such a graph is called a tournament.[3] The dropping of weights gives us a canonical surjection $\mathcal{WT} \to \mathcal{T}$. This creates a natural map $T : \Delta(\mathcal{O}) \to \mathcal{T}$ that defines the tournament $T(\tilde{\theta})$ of the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the people. In this setting, $\text{referendum}(\tilde{\theta}, x, y) > 0$ if and only if $x \gg y$ is an arc in $T(\tilde{\theta})$, which we denote $T(\tilde{\theta}) : x \gg y$.

Given a tournament $T \in \mathcal{T}$, we denote $T : x \gneqq y$ the fact that $y \gg x$ is not an arc of T. In other words, we have $T : x \gneqq y$ when y does not beat x in T. Naturally, if T is a complete tournament, then $T : x \gneqq y$ is equivalent to $T : x \gg y$ or $x = y$. However, this equivalence no longer holds for incomplete tournaments.

This leads us to the well-known concept of Condorcet winner, which we slightly adapted for our purposes.[4]

Definition 1 *A Condorcet winner of a tournament $T \in \mathcal{T}$ is an alternative $x \in X$ that no other alternative beats, i.e. $T : x \gneqq y$ for all $y \in X$. Moreover, a Condorcet winner of the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ is an alternative that is Condorcet winner of the tournament $T(\tilde{\theta})$.*

As Condorcet asserted it himself, a good voting system must elect the Condorcet winner if it exists and is unique. After all, what better alternative is there than the one that is preferred to any other alternative by the majority? Another argument to defend Condorcet's ideas is to note that if another alternative were to be elected, then the majority of the people would want to have him replaced by the Condorcet winner.

A well-known result is the uniqueness of a Condorcet winner in complete tournaments. Unfortunately though, the Condorcet paradox (see Figure 1) shows that a Condorcet winner does not necessarily exist.

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[3]In the literature, tournaments are usually rather defined as complete antisymmetric graphs, for which we also know that $x \gg y$ or $y \gg x$ is an arc. However, we will show the relevancy of including incomplete graphs as we will introduce randomized tournaments.

[4]The literature often defines a Condorcet winner as an alternative that beats all other alternatives. Evidently, for complete tournaments, the two definitions coincide. But, for incomplete tournaments, they do not.

This is due to the fact that a tournament $T(\tilde{\theta}) \in \mathcal{T}$ of the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ is no longer necessarily a transitive graph. In particular, the relation $\gg$ on X is not an order relation.

Interestingly, in politics, there usually is a Condorcet winner. The explanation lies in the fact that alternatives often range on a one-dimensional left-right line. In such a setting, the median voter theorems (Black (1958), Roberts (1977)) ensure that there always is a Condorcet winner. We shall discuss this case in more details in Section 5. Still, even then, while the preferences of the people usually yield a Condorcet winner, voters may still have incentives not to reveal their preferences truthfully. Recall that this is the essence of the issues underlined by the Gibbard (1973) - Satterthwaite (1975) theorem. Thus, ballots may well not have a Condorcet winner even though preferences do, which means that we still have to address the case where a Condorcet winner does not exist, even when alternatives yield a left-right structure.

2.2 Randomized tournaments

To face the case where ballots do not yield a Condorcet winner, similarly to Gibbard (1978), we introduce randomization on the set of alternatives. In particular, we extend the relation $\gg$ defined on the set X of alternatives by a tournament T to the set $\Delta(X)$ of lotteries. The extended relation will be called the randomized tournament.

To define this randomized tournament, let us introduce the quantity $\mathsf{gain}(T, \tilde{x}, \tilde{y})$ that counts the frequency at which $\tilde{x}$ beats rather than is beaten by $\tilde{y}$ in tournament T, i.e.

$$\mathsf{gain}(T, \tilde{x}, \tilde{y}) = \mathbb{P}_{\tilde{x}, \tilde{y}} \left[T : \tilde{x} \gg \tilde{y} \right] - \mathbb{P}_{\tilde{x}, \tilde{y}} \left[T : \tilde{x} \ll \tilde{y} \right]. \tag{5}$$

We can now define the randomized tournament.

Definition 2 *The randomized tournament* $\mathsf{Rand}(T)$ *of a tournament* $T \in \mathcal{T}$ *is the tournament whose nodes are lotteries and where* $\mathsf{Rand}(T) : \tilde{x} \gg \tilde{y}$ *is an arc if and only if the majority more often prefers the choice of* $\tilde{x}$ *to that of* $\tilde{y}$ *than the other way around. Equivalently,* $\mathsf{Rand}(T) : \tilde{x} \gg \tilde{y}$ *if and only if* $\mathsf{gain}(T, \tilde{x}, \tilde{y}) > 0$. *Plus, a Condorcet winner of the randomized tournament* $\mathsf{Rand}(T)$ *shall be called a randomized Condorcet winner of tournament* T.

It is immediate to see that the tournament T is an induced subgraph of the randomized tournament $\mathsf{Rand}(T)$. Now, instead of choosing a Condorcet winner of T that may not exist, the voting system we shall propose consists of choosing a randomized Condorcet winner. Crucially, this randomized Condorcet winner does exist and will be unique in practice. Plus, naturally, if T does yield a unique Condorcet winner, then this Condorcet winner will also be the unique randomized Condorcet winner.

Now, the existence and uniqueness of the Condorcet winner of the randomized tournament are asserted by the following theorem. This theorem can be regarded as the main contribution of this paper, as it is key to the definition of the randomized Condorcet voting system.

Theorem 2 *Any finite tournament* $T \in \mathcal{T}$ *has a randomized Condorcet winner. Plus, if* T *is complete, then the randomized Condorcet winner is unique.*

To prove Theorem 2, we shall reformulate it as a known theorem of existence and uniqueness of a Nash equilibrium of a game. This game is the so-called tournament game. It is a 2-player zero-sum symmetric game, where actions are alternatives and where player 1 wins 1 if his action is preferred to player 2's by T. More explicitly, this gain is defined by

$$\begin{cases} +1 & \text{if } T : x \gg y, \\ -1 & \text{if } T : y \gg x, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It is straightforward that the payment matrix is antisymmetric. In fact, this matrix payment is $\mathsf{gain}(T) = \{\mathsf{gain}(T, x, y)\}_{(x,y) \in X^2}$. We then have the following immediate lemma:

Lemma 1 *The expected gain of player 1 with strategy $\tilde{x}$ against player 2 with strategy $\tilde{y}$ in the tournament game of tournament T equals $\mathsf{gain}(T, \tilde{x}, \tilde{y})$.*

Proof. This expected gain is computed by

$$\mathbb{E}_{\tilde{x},\tilde{y}}\left[\mathbf{1}_{T:\tilde{x}\gg\tilde{y}} - \mathbf{1}_{T:\tilde{y}\gg\tilde{x}}\right] \;=\; \mathbb{E}_{\tilde{x},\tilde{y}}\left[\mathbf{1}_{T:\tilde{x}\gg\tilde{y}}\right] \;-\; \mathbb{E}_{\tilde{x},\tilde{y}}\left[\mathbf{1}_{T:\tilde{y}\gg\tilde{x}}\right] \;=\; \mathbb{P}_{\tilde{x},\tilde{y}}\left[T:\tilde{x}\gg\tilde{y}\right] \;-\; \mathbb{P}_{\tilde{x},\tilde{y}}\left[T:\tilde{y}\gg\tilde{x}\right], \quad (7)$$

which is exactly the expression of $\mathsf{gain}(T, \tilde{x}, \tilde{y})$ and proves the lemma. $\qquad\square$

Importantly, the lemma implies the following corollary which connects randomized Condorcet winners to Nash equilibria of the tournament game.

Corollary 1 *A lottery is a randomized Condorcet winner of a tournament if and only if it is a Nash equilibrium of the tournament game.*

Proof. Because of the minimax theorem (von Neumann (1928)) and symmetry, the value of the tournament game is necessarily 0. Thus, $\tilde{x}$ is a Nash equilibrium if and only if $\tilde{x}$ does not lose to any strategy $\tilde{y}$. According to Lemma 1, this corresponds to $\mathsf{gain}(T, \tilde{x}, \tilde{y}) \geq 0$. By definition of the randomized tournament, this is equivalent to $\mathsf{Rand}(T) : \tilde{x} \ggeq \tilde{y}$ for all $\tilde{y} \in \Delta(\mathcal{O})$. Yet, this is equivalent to saying that $\tilde{x}$ is a Condorcet winner of the randomized tournament. $\qquad\square$

It is easy to see that the tournament game of a finite tournament is a finite game. Therefore, von Neumann (1928)'s minimax theorem and the Nash (1951) theorem ensure that a Nash equilibrium exists in mixed strategies. Using the previous corollary, we thus know that every finite tournament T has a randomized Condorcet winner. This proves the first part of Theorem 2. The second part is given by the following theorem from the literature.

Theorem 3 (Fisher and Ryan (1992); Laffond et al. (1993)) *Assume the tournament $T \in \mathcal{T}$ complete. Then, its tournament game has a unique equilibrium. Plus, every best-reply to the equilibrium is played with a positive probability by the equilibrium.*

We refer to the original papers and to Myerson (1996) for a proof of the theorem. What is important for our purpose is that the theorem guarantees the uniqueness of a randomized Condorcet winner when T is complete. Now, a better understanding of this equilibrium is given by the following corollary.

Corollary 2 *Assume the tournament $T \in \mathcal{T}$ complete. If a probability distribution $\tilde{x}$ is the randomized Condorcet winner, then, for any $x \in X$, we either have $\mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\gg x\right] = \mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\ll x\right]$ or $\mathbb{P}_{\tilde{x}}\left[\tilde{x} = x\right] = 0$.*

Proof. If $\tilde{x}$ is a Condorcet winner of the randomized tournament, then it beats any alternative $x \in X$. This means that $\mathsf{gain}(T, \tilde{x}, x) \geq 0$, which corresponds to $\mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\gg x\right] \geq \mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\ll x\right]$. Yet, Theorem 3 ensures that all best-reply to $\tilde{x}$ are played with positive probability, which means that $\mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\gg x\right] = \mathbb{P}_{\tilde{x}}\left[T:\tilde{x}\ll x\right]$ if and only if $\mathbb{P}_{\tilde{x}}\left[\tilde{x} = x\right] > 0$. This concludes the proof. $\qquad\square$

It can be shown that the reciprocal is not true. Indeed, if $x, y, z \in X$ form a cycle, as well as w, y, z, and if $T : x \gg w$, then the uniform distribution on w, y, z satisfy the properties of the corollary but is not Condorcet winner, as it is beaten by x. Now, in the case where T is not complete, we do not necessarily have the uniqueness of the randomized Condorcet winner. This is a degenerate case, for which we still have the following theorem.

Theorem 4 *The set of randomized Condorcet winners is a non-empty polyhedron.*

Proof. A randomized Condorcet winner is a lottery $\tilde{x} \in \Delta(X)$ described by a vector $p = (\mathbb{P}[\tilde{x} = x])_{x \in X} \in \mathbb{R}^X$ satisfying the three following linear constraints:

$$\mathsf{gain}(T)p \geq 0 \qquad \text{and} \qquad e^T p = 1 \qquad \text{and} \qquad p \geq 0, \tag{8}$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^X$. Therefore, the set of randomized Condorcet winners is a polyhedron. Plus, Theorem 2 implies that this polyhedron is not empty. $\square$

In practice though, when there is an odd number of voters, or when the number of voters is sufficiently large, then T is complete and we do not have to involve the case where there are draws between alternatives.

2.3 The randomized Condorcet voting system

Finally, we can introduce the main contribution of this paper, which is the randomized Condorcet voting system. This voting system strongly relies on Theorem 2. It requires voters to choose ballots $a \in \mathcal{O}$ that order all alternatives. Similarly to preferences, we then define the ballots $\tilde{a} \in \Delta(\mathcal{O})$ of the people as the probability distribution that maps total orders to the ratio of people who voted these orders. The tournament $T(\tilde{a})$ is then the tournament of the ballots. If it is complete, then it has a unique randomized Condorcet winner, which the randomized Condorcet voting system will elect.

While this tournament is almost always complete, let us propose a single tractable procedure that always yields a randomized Condorcet winner of $T(\tilde{a})$, even when it is not unique. To do so, we draw a direction $\zeta \in \mathbb{R}^X$ from a standard normal distribution (or, equivalently, any isotropic distribution[5]). The choice of the randomized Condorcet voting system is then the optimal solution of the following linear program:

$$\begin{array}{ll} \underset{p \in \mathbb{R}^X}{\text{Maximize}} & \zeta^T p \\ \text{Subject to :} & \mathsf{gain}(T(\tilde{a}))p \geq 0, \\ & e^T p = 1, \\ & p \geq 0. \end{array} \tag{9}$$

Interestingly, this is a simple linear program with $O(|X|)$ of variables and constraints. It can thus be solved efficiently. Let $p^*(\zeta, T(\tilde{a}))$ its optimum. It is uniquely defined with probability 1. In particular, $\mathbb{E}_\zeta[p^*(\zeta, T(\tilde{a}))]$ is well-defined and belongs to the simplex of $\mathbb{R}^{\mathcal{O}}$. Finally, we can define the randomized Condorcet voting system.

Definition 3 *The randomized Condorcet voting system* $\mathscr{C} : \Delta(\mathcal{O}) \to \Delta(X)$ *elects the lottery* $\mathbb{E}_\zeta[p^*(\zeta, T(\tilde{a}))] \in \Delta(\mathcal{O})$ *when given the ballots* $\tilde{a} \in \Delta(\mathcal{O})$ *of the people.*

Importantly, the randomized Condorcet voting system satisfies the following property. We state it as a proposition for its importance, even though it is immediate.

Proposition 1 *For all preferences* $\tilde{\theta} \in \Delta(\mathcal{O})$ *and any lottery* $\tilde{x} \in \Delta(X)$, *the majority likes the alternative drawn by* $\mathscr{C}(\tilde{\theta})$ *at least more often to the one drawn by* $\tilde{x}$ *than the other way around. In other words, we always have* $\mathsf{Rand}(T(\tilde{\theta})) : \mathscr{C}(\tilde{\theta}) \gqq \tilde{x}$, *or, equivalently,* $\mathsf{gain}(T(\tilde{\theta}), \mathscr{C}(\tilde{\theta}), \tilde{x}) \geq 0$.

Proof. This is by the definition of the randomized Condorcet voting system and of the randomized Condorcet winner, combined with Lemma 1. $\square$

Finally, we conclude this section by highlighting tricky aspects of the function gain.

Remark 2 *The quantity* $\mathsf{gain}(T(\tilde{\theta}), \tilde{x}, \tilde{y})$ *is not to be confused with the difference of the probabilities that a voter chosen randomly will prefer a random choice of* $\tilde{x}$ *to a random choice of* $\tilde{y}$ *rather than the other way*

---

[5]By isotropic, we mean invariant by rotations of $\mathbb{R}^X$.

around. Nor should it be confused with the difference of probabilities that a voter chosen randomly will prefer the probability $\tilde{x}$ to $\tilde{y}$. In other words, in general, the following quantities are not equal:

$$\mathsf{gain}(T(\tilde{\theta}), \tilde{x}, \tilde{y}) \neq \mathbb{P}_{\tilde{\theta}, \tilde{x}, \tilde{y}}\left[\tilde{\theta} : \tilde{x} \succ \tilde{y}\right] - \mathbb{P}_{\tilde{\theta}, \tilde{x}, \tilde{y}}\left[\tilde{\theta} : \tilde{x} \prec \tilde{y}\right] = \mathbb{E}_{\tilde{x}, \tilde{y}}\left[\mathsf{referendum}(\tilde{\theta}, \tilde{x}, \tilde{y})\right], \tag{10}$$

$$\mathsf{gain}(T(\tilde{\theta}), \tilde{x}, \tilde{y}) \neq \mathbb{P}_{\tilde{\theta}}\left[\mathsf{Rand}(T(\tilde{\theta})) : \tilde{x} \gg \tilde{y}\right] - \mathbb{P}_{\tilde{\theta}}\left[\mathsf{Rand}(T(\tilde{\theta})) : \tilde{x} \ll \tilde{y}\right]. \tag{11}$$

What $\mathsf{gain}(T(\tilde{\theta}), \tilde{x}, \tilde{y})$ counts is rather the surpluses of times that the choice of $\tilde{x}$ will be preferred to that of $\tilde{y}$ by the majority of the people. In other words, it really represents the phrase "majority rule".

3   Incentive and Condorcet-compatibility

The revelation principle is a well-known result of mechanism design. Loosely, it asserts that, when we aim at designing a decision procedure with players' preferences that we do not know, we can, without loss of generality, restrict ourselves to procedures that ask players to reveal their preferences to make a decision. Such decision procedures are called direct mechanisms. More precisely, in our case, the revelation principle guarantees that any voting system is perfectly equivalent to a direct mechanism. This means that we can merely focus on voting systems where a ballot is a ranking $a \in \mathcal{O}$. Accordingly to definitions of Section 2.3, we also define the ballots $\tilde{a} \in \Delta(\mathcal{O})$ of the people. A (randomized) voting system is then a function $\mathcal{V} : \Delta(\mathcal{O}) \to \Delta(X)$.

Now, an important concept of mechanism design and social choice theory is *incentive-compatibility*. Roughly, incentive-compatibility requires truthfulness to be voters' best strategies. In this section, we will formalize this concept. Section 3.1 introduce further modelings necessary to define incentive-compatibility. Then, Section 3.2 defines strong Condorcet-compatibility, which is a variant of incentive-compatibility combined with a property related to Condorcet winners. We prove that there exists no strongly Condorcet-compatible voting system. Finally, Section 3.3 introduces dominant strategy Condorcet-compatibility (DSCC) and proves that the randomized Condorcet voting system is DSCC.

3.1   Preferences and strategies

To still be able to talk about incentive-compatibility in a randomized setting, we now need to extend the definition domain of preferences $\theta \in \mathcal{O}$ to compare independent lotteries $\tilde{x}$ and $\tilde{y} \in \Delta(X)$. To do so, we will consider $\theta : \tilde{x} \succeq \tilde{y}$ when θ more often prefers the alternative drawn from $\tilde{x}$ to the one drawn from $\tilde{y}$ than the other way around, i.e.

$$\mathbb{P}_{\tilde{x}, \tilde{y}}(\theta : \tilde{x} \succ \tilde{y}) \geq \mathbb{P}_{\tilde{x}, \tilde{y}}(\theta : \tilde{y} \succ \tilde{x}). \tag{12}$$

When the inequality is strict, we denote $\theta : \tilde{x} \succ \tilde{y}$.

Remark 3 *This extension to probability distribution is not compatible with Von Neumann and Morgenstern (1945) preferences. In particular, while θ is transitive on X, it is no longer transitive on $\Delta(X)$, as illustrated by the examples of nontransitive dices (see Grime (2010)). However, it has the advantage of giving a canonical ordering over $\Delta(X)$ given an ordering over X.*

Intuitively, Von Neumann and Morgenstern (1945) preferences are richer than the order relation we have here as they also describe by how much more x is preferred to y, than v is preferred to z. More precisely, there are many Von Neumann and Morgenstern (1945) preferences that match a certain ordering of alternatives. This indicates that using our order relation for lotteries leaves more room for incentive-compatible voting system. In particular, the Gibbard (1978) theorem no longer applies to our setting.

Next, to define incentive-compatibility, we need to introduce strategies. A strategy is a mapping $s : \mathcal{O} \to \Delta(\mathcal{O})$, where $s(\theta) \in \Delta(\mathcal{O})$ is the mix of ballots chosen by voters of preference θ. A more relevant way to interpret this mix of ballots is to regard it as the way these voters spread their votes among the different possible ballots. We denote S the set of strategies. What is more, we extend the domain of s to the set $\Delta(\mathcal{O})$

by $s(\tilde{\theta}) = \mathbb{E}[s(\tilde{\theta})]$. Then, if $\tilde{\theta}$ are the preferences of the people, then $s(\tilde{\theta})$ are the ballots of the people when they follow strategy s.

Given a strategy $s \in S$, we define the two subsets $\mathsf{Truthful}(s)$ and $\mathsf{Conspirator}(s)$ of $\mathcal{O}$ defined by

$$\mathsf{Truthful}(s) = \{\theta \in \mathcal{O} \mid s(\theta) = \theta\} \quad \text{and} \quad \mathsf{Conspirator}(s) = \{\theta \in \mathcal{O} \mid s(\theta) \neq \theta\} \tag{13}$$

The two sets $\mathsf{Truthful}(s)$ and $\mathsf{Conspirator}(s)$ obviously form a partition of the set $\mathcal{O}$ of preferences. Now, the truthful strategy s^{truth} is defined as the identity function of $\mathcal{O}$, i.e. $s^{truth}(\theta) = \theta$ for all $\theta \in \mathcal{O}$. Equivalently, a strategy s is truthful if $\mathsf{Truthful}(s) = \mathcal{O}$.

Also, we denote $S^* = S - \{s^{truth}\}$ the set of untruthful strategies. Equivalently, it is the set of strategies s such that $\mathsf{Conspirator}(s)$ is not empty.

Lemma 2 *Let the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$, the ballots $\tilde{a} \in \Delta(\mathcal{O})$ and a subset $\subset \mathcal{O}$ of conspirators. Then, there exists a strategy $s \in S$ such that $s(\tilde{\theta}) = \tilde{a}$ and $\mathsf{Conspirator}(s) = \mathcal{C}$ if and only if $\mathbb{P}_{\tilde{\theta}}[\mathcal{D}] \leq \mathbb{P}_{\tilde{a}}[\mathcal{D}]$ for all supset $\mathcal{D} \supset \mathcal{O} - \mathcal{C}$. In particular, if X is finite, this condition amounts to $\mathbb{P}_{\tilde{\theta}}(\tilde{\theta} = a) \leq \mathbb{P}_{\tilde{a}}(\tilde{a} = a)$ for all $a \notin \mathcal{C}$.*

The proof is given in the appendix. The idea is that inequalities hold if and only if we can distribute the ballots of conspirators so that we can obtain the probability distribution $\tilde{a}$.

3.2   Strong Condorcet-compatibility

Now, note that, in general, each voter's individual action does not affect the outcome at all. Thus, nearly all ballots of the people is a Nash equilibrium. This leads us to modify the usual concept of incentive-compatibility to adapt it for masses of voters rather than each of them individually. This idea is quite realistic, as guidelines given by leaders to their followers yield such a deviation of a mass of voters.

One incentive-compatibility concept we define for the setting of voting systems rely on the notion of strong Nash equilibrium.

Definition 4 *A voting system $\mathscr{V}$ is strongly incentive-compatible if no set of conspirators has strict incentives to deviate collectively from truthfulness. This means that if a set of conspirators deviate, at least one conspirator does not gain strictly, i.e.*

$$\forall \tilde{\theta} \in \Delta(\mathcal{O}), \forall s \in S^*, \exists \theta \in \mathsf{Conspirator}(s), \quad \theta : \mathscr{V}(s(\tilde{\theta})) \preceq \mathscr{V}(\tilde{\theta}). \tag{14}$$

Strong incentive-compatibility is equivalent to saying that truthfulness is always a strong Nash equilibrium.

Strong incentive-compatibility is called coalitional strategy-proofness in Penn et al. (2011) and group strategy-proofness in Saporiti (2009). They introduced this concept by defining its opposite, which is what they called coalitional manipulability. This corresponds to saying that $\mathscr{V}$ is not strongly incentive-compatible if

$$\exists \tilde{\theta} \in \Delta(\mathcal{O}), \exists s \in S^*, \forall \theta \in \mathsf{Conspirator}(s), \quad \theta : \mathscr{V}(s(\tilde{\theta})) \succ \mathscr{V}(\tilde{\theta}). \tag{15}$$

This concept leads us to define the strong Condorcet-compatibility.

Definition 5 *A voting system $\mathscr{V}$ is strongly Condorcet-compatible if, for all the preferences $\tilde{\theta}$ of the people that yield a Condorcet winner x, $\mathscr{V}$ is strongly incentive-compatible and elects x.*

Strong incentive-compatibility is often regarded as too strong. We prove this point with the following theorem, which is one of the main contributions of this paper.

Theorem 5 *There is no strongly Condorcet-compatible voting system.*

Proof. The proof only requires 4 alternatives and 7 voters, among whom only 2 need to be assumed to be conspirators. Let $X = \{w, x, y, z\}$ and the ballots:

$$a_1 : x \succ y \succ z \succ w, \qquad a_2 : z \succ x \succ y \succ w \tag{16}$$

$$a_3 : x \succ w \succ y \succ z, \qquad a_4 : y \succ w \succ z \succ x, \qquad a_5 : z \succ w \succ x \succ y. \tag{17}$$

We define the ballots of the people by

$$7\tilde{a} = a_1 + a_2 + a_3 + 2a_4 + 2a_5. \tag{18}$$

The weighted tournament $WT(\tilde{a})$ is pictured in Figure 2, where weights have to be divided by 7.
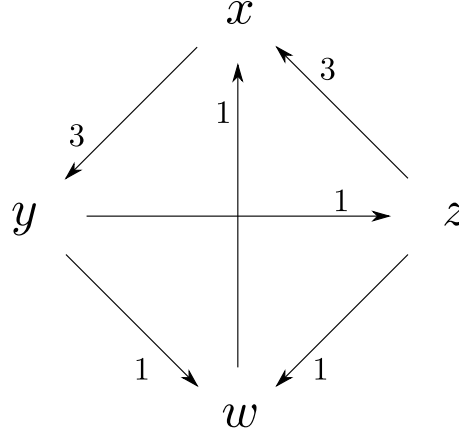

Figure 2: Weighted tournament $WT(\tilde{a})$

Let $\mathscr{V}$ a strongly Condorcet-compatible voting system. We will show that no choice of $\mathscr{V}(\tilde{a}) \in \Delta(X)$ can be made without leading to the possibility that $\tilde{a}$ is the ballot produced by conspirators who had incentives not to reveal their preferences truthfully. This will show that $\mathscr{V}$ cannot exist. To do so, let us denote $p_v = \mathbb{P}_{\mathscr{V}(\tilde{a})}[\mathscr{V}(\tilde{a}) = v]$ for all $v \in X$.

Now, consider $\theta_y : z \succ w \succ y \succ x$, and the preferences $\tilde{\theta}_y \in \Delta(\mathcal{O})$ of the people defined by

$$7\tilde{\theta}_y = a_1 + a_2 + a_3 + 2a_4 + 2\theta_y. \tag{19}$$

In other words, we are investigating the case, where the two θ_y were conspirators and voted a_5 instead of θ_y. We have $7\mathsf{referendum}(\tilde{\theta}_y, y, x) = 1 > 0$, yielding $T(\tilde{\theta}_y) : y \gg x$. Plus, as for $\tilde{a}$, we have $T(\tilde{\theta}_y) : y \gg z$ and $T(\tilde{\theta}_y) : y \gg w$. Thus, y is Condorcet winner of $\tilde{\theta}_y$. To make sure that the two θ_y do not have incentives to conspire, we cannot allow to have $\theta_y : \mathscr{V}(\tilde{a}) \gg y$. This means that $\mathsf{gain}(T(\theta_y), \mathscr{V}(\tilde{a}), y) \leq 0$, and corresponds to $\mathbb{P}_{\mathscr{V}(\tilde{a})}[\mathscr{V}(\tilde{a}) \in \{w, z\}] \leq \mathbb{P}_{\mathscr{V}(\tilde{a})}[\mathscr{V}(\tilde{a}) = x]$. This can be written $p_w + p_z \leq p_x$.

We then investigate a similar manipulation by $\theta_z : x \succ w \succ z \succ y$ when the preferences of the people are $\tilde{\theta}_z$ defined by

$$7\tilde{\theta}_z = a_1 + a_2 + \theta_z + 2a_4 + 2a_5. \tag{20}$$

Similarly to above, we verify that z is the Condorcet winner of $\tilde{\theta}_z$. The strong incentive-compatibility of $\mathscr{V}$ then implies that $p_w + p_x \leq p_y$.

Now consider $\theta_x : y \succ x \succ w \succ z$ and $\tilde{\theta}_x$ defined by

$$7\tilde{\theta}_x = a_1 + a_2 + a_3 + 2\theta_x + 2a_5. \tag{21}$$

Alternative x is the Condorcet winner of $\tilde{\theta}_x$, implying $p_y \leq p_w + p_z$.

Before going further, let us find out the implication of the three inequalities we have seen so far. Using successively the second, third and first inequalities yields:

$$p_w + p_x \leq p_y \leq p_w + p_z \leq p_x. \tag{22}$$

This leads to $p_w \leq 0$, and, since probabilities are non-negative, $p_w = 0$. It then follows that $p_x = p_y = p_z = 1/3$. This is a condition $\mathscr{V}(\tilde{a})$ must satisfy to guarantee DSIC (which we shall define soon). Interestingly, this is precisely the lottery prescribed by the randomized Condorcet voting system.

Back to the proof, let us now consider $\theta_w^1 : x \succ y \succ w \succ z$ and $\theta_w^2 : z \succ x \succ w \succ y$ and $\tilde{\theta}_w$ defined by

$$7\tilde{\theta}_w = \theta_w^1 + \theta_w^2 + a_3 + 2a_4 + 2a_5. \tag{23}$$

Now, w is the Condorcet winner of $\tilde{\theta}_w$. Conspirators θ_w^1 and θ_w^2 both have incentive to conspire if $p_x + p_y > p_z$ and $p_x + p_z > p_y$. Since both do hold, we have a contradiction. This proves that a strongly Condorcet-compatible voting system cannot exist.   □

3.3 Dominant-strategy Condorcet-compatibility

This impossibility theorem leads us to restrict ourselves to a weaker concept of incentive-compatibility. We will use the well-known concept of dominant-strategy incentive-compatibility, albeit we slightly adapt it to our setting.

Definition 6 *A voting system $\mathscr{V}$ is dominant-strategy incentive-compatible (DSIC) if the group of voters of preference $\theta \in \Theta$ never have incentives to conspire, i.e.*

$$\forall \tilde{\theta} \in \Delta(\mathcal{O}), \forall \theta \in \Theta, \forall s \in S, \quad \mathsf{Conspirator}(s) = \{\theta\} \implies \theta : \mathscr{V}(s(\tilde{\theta})) \preceq \mathscr{V}(\tilde{\theta}). \tag{24}$$

Plus, we say that a voting system is dominant-strategy Condorcet-compatible (DSCC) if, whenever the preferences of the people yield a Condorcet winner, it elects it and is DSIC.

Note that DSIC corresponds to the strong incentive-compatibility when sets $\mathsf{Conspirator}(s)$ of conspirators are reduced to singletons. Therefore, a strongly incentive-compatible voting system is DSIC. Similarly, strong Condorcet compatibility implies DSCC.

In the literature, the usual definition of DSIC differs with ours, in the sense that it only applies to one individual rather than a group of like-minded conspirators. However, as we have already pointed it out through the issue with Nash equilibria, deviations of a single person usually have no effect on the outcome of the vote. This is why the usual concept of DSIC is not relevant in our setting.

Another way to interpret our definition of DSIC is to consider the case where there is merely a small number of voters, but these voters are given weights in their votes. This can happen, for instance, in a European Union vote where Germany's ballot counts more than Luxembourg's, because its population is of much greater size. In such a case, it is not hard to see that the usual concept of DSIC coincides with ours.

A major result regarding the randomized Condorcet voting system is the following result.

Theorem 6 (Peyre (2012c)) *The randomized Condorcet voting system $\mathscr{C}$ is DSCC.*

Proof. Evidently, $\mathscr{C}$ elects the Condorcet winner when it exists. Let now $\tilde{\theta} \in \Delta(\mathcal{O})$ the preferences of the people, which we assume to yield a Condorcet winner $x \in X$. Let $\theta \in \Theta$ the preference of conspirators, and s their strategy. Let $s(\tilde{\theta}) = \tilde{a}$. We will show that θ must in fact prefer x to $\mathscr{C}(\tilde{a})$, which will prove that conspirators did not incentive to conspire.

By definition of the voting system $\mathscr{C}$, we must have $T(\tilde{a}) : \mathscr{C}(\tilde{a}) \gg x$. Using Lemma 1, this corresponds to $\mathsf{gain}(T(\tilde{a}), \mathscr{C}(\tilde{a}), x) \geq 0$. Yet, for any $y \in X$, we know that $T(\tilde{\theta}) : x \gg y$, since x is the Condorcet winner. Thus, if y beats x in the ballots, it must be because some conspirators inverted x and y in the ballots, hence favoring y over x. More precisely, if $T(\tilde{a}) : y \gg x$, then $\theta : x \succ y$. Thus,

$$\{y \in X \mid T(\tilde{a}) : y \gg x\} \subset \{y \in X \mid \theta : x \succ y\}. \tag{25}$$

As a result, we obtain the following inequality:

$$0 \leq \mathsf{gain}(T(\tilde{a}), \mathscr{C}(\tilde{a}), x) \leq \mathsf{gain}(T(\theta), x, \mathscr{C}(\tilde{a})). \tag{26}$$

This shows that $\theta : x \succeq \mathscr{C}(\tilde{a})$, and proves that conspirators did not have incentive to deviate from truthfulness. This is what we had to prove. □

4 Uniqueness

Interestingly, the randomized Condorcet voting system is nearly the unique DSCC voting system. Before better formalizing and proving this idea, let us start by providing the intuitive argument for uniqueness.

4.1 Intuition

The intuition of the uniqueness was given to me by Rémi Peyre in informal discussions. The idea is that, provided there are enough of them, conspirators can invert arcs of the tournament of preferences they agree with. Indeed, if $\theta : x \succ y$, we have $\mathsf{referendum}(s(\theta), y, x) \geq -1 = \mathsf{referendum}(\theta, y, x)$ for any strategies $s \in S$. So, by conspiring, θ may increase the ratios of ballots favoring y over x and invert an arc $T(\tilde{\theta}) : x \gg y$ to $T(s(\tilde{\theta})) : x \ll y$. However, if $T(\tilde{\theta}) : y \gg x$, a conspirator θ cannot invert the arrow he does not agree with.

Now, imagine that a voting system $\mathscr{V}$ differs with the randomized Condorcet voting system for certain ballots $\tilde{a}$ of the people. Then, there must be some alternative $x \in X$ such that $\mathsf{gain}(T(\tilde{a}), \mathscr{V}(\tilde{a}), x) < 0$, i.e.

$$\mathbb{P}_{\mathscr{V}(\tilde{a})}\left[T(\tilde{a}) : \mathscr{V}(\tilde{a}) \gg x\right] < \mathbb{P}_{\mathscr{V}(\tilde{a})}\left[T(\tilde{a}) : \mathscr{V}(\tilde{a}) \ll x\right]. \tag{27}$$

Let us take an example to refine our intuition. Consider $X = \{v, w, x, y, z\}$ and the tournament $T(\tilde{a})$ depicted in Figure 3. Equation (27) can be restated in this example as

$$\mathbb{P}_{\mathscr{V}(\tilde{a})}\left[\mathscr{V}(\tilde{a}) \in \{y, z\}\right] < \mathbb{P}_{\mathscr{V}(\tilde{a})}\left[\mathscr{V}(\tilde{a}) \in \{v, w\}\right]. \tag{28}$$

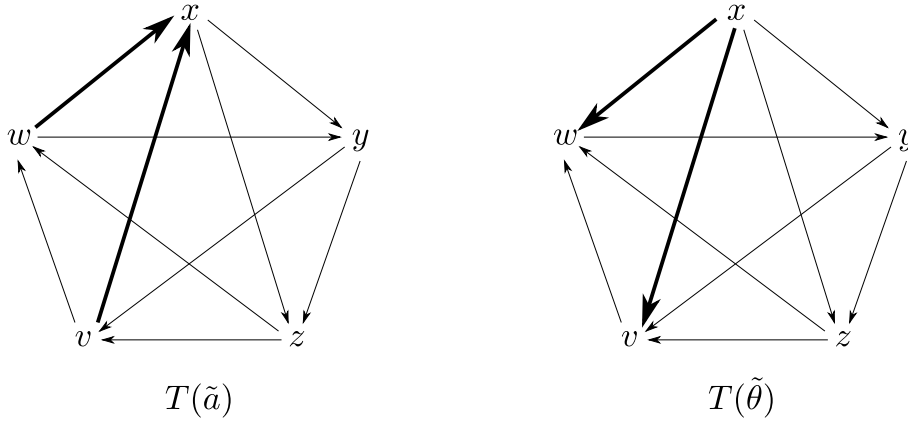$$T(\tilde{a}) \qquad\qquad\qquad T(\tilde{\theta})$$

Figure 3: Inversion of arcs

Now, consider $\theta : v \succ w \succ x \succ y \succ z$. The key idea of our proof of uniqueness lies in the idea that the arcs $T(\tilde{a}) : v \gg x$ and $T(\tilde{a}) : w \gg x$ have been inverted by the conspiring strategy of θ, from a tournament $T(\tilde{\theta})$ where x was the Condorcet winner. Plus, equation (28) proves that $\theta : \mathscr{V}(\tilde{a}) \succ x$, as θ more often prefers the alternative picked by $\mathscr{V}(\tilde{a})$ to x than the other way around. This shows that θ indeed has incentive to deviate from truthfulness to find itself in the case of $\mathscr{V}(\tilde{a})$.

The difficulty though is to guarantee that there indeed exist some initial preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ for which x is Condorcet winner and such that conspirators θ are sufficiently numerous such that they can invert arcs from x to v and w and reach exactly the ballots $\tilde{a}$. In the sequel, we show that when $\tilde{a}$ is in a neighborhood of the uniform distribution $\tilde{u}$, and if the voting system is defined on the weighted graph, then a DSCC voting system must coincide with the randomized Condorcet voting system.

4.2 Preliminaries

To better formalize this idea and to prove it, we need to better understand the source of non-transitivity in tournaments of preferences of the people. This is provided by the following lemma and its two corollaries. In essence, they prove that all tournaments can be written $T(\tilde{\theta})$ for some preferences $\tilde{\theta} \in \Delta(\mathcal{O})$.

The mapping $\mathsf{referendum} : \Delta(\mathcal{O}) \times \Delta(X) \times \Delta(X) \to \mathbb{R}$ can be regarded as a function $\Delta(\mathcal{O}) \to \mathbb{R}^{X \times X}$ that maps preferences of the people to square matrices of entries $\mathsf{referendum}(\tilde{\theta}, x, y)$, for $x, y \in X$. Now, recall that $\Delta(\mathcal{O})$ can be regarded as the simplex of the vector space $\mathbb{R}^{\mathcal{O}}$. Therefore, the function $\mathsf{referendum}$ can be uniquely extended to a linear map $\mathcal{R} : \mathbb{R}^{\mathcal{O}} \to \mathbb{R}^{X \times X}$.

Lemma 3 *The image of $\mathcal{R}$ coincides with the space of antisymmetric matrices.*[6]

Proof. It is straightforwards to see that the image of $\mathsf{referendum}$ is included in the space of antisymmetric matrices. Reciprocally, to show the equality, we need only show that any element of the canonical basis of antisymmetric matrices is in the image of $\mathsf{referendum}$. Such an element $R \in \mathbb{R}^{X \times X}$ is of the form $R_{xy} = -R_{yx} = 1$ for some two different alternatives $x, y \in X$, and $R_{vw} = 0$ in any other case. Denote $v_1, \ldots, v_n$ the other alternatives. We define $\tilde{\theta} \in \mathbb{R}^{\mathcal{O}}$ by $\tilde{\theta} = (\theta_1 + \theta_2)/2$, where

$$\theta_1 : y \succ x \succ v_1 \succ \ldots \succ v_n \quad \text{and} \quad \theta_2 : v_n \succ \ldots \succ v_1 \succ y \succ x. \tag{29}$$

We then have $\mathsf{referendum}(\tilde{\theta}, v, w) = 0$ if $(v, w) \notin \{(x, y), (y, x)\}$ and $\mathsf{referendum}(\tilde{\theta}, x, y) = -\mathsf{referendum}(\tilde{\theta}, y, x) = -1$, which proves that $R = \mathcal{R}(\tilde{\theta})$. □

We define the taxicab norm of a matrix as the sum of the absolute values of its entries.

Corollary 3 *All weighted tournaments with a sum of all weights of at most 1 are obtained from some ballots $\tilde{a} \in \Delta(\mathcal{O})$.*

Proof. In the proof of Lemma 3, we showed that each element of the canonical basis is the image $\mathsf{referendum}(\tilde{\theta})$ of some preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the people. Yet, such elements of the canonical basis are of taxicab norm 2. Since $\mathcal{R}$ is linear and $\Delta(\mathcal{O})$ convex, $\mathsf{referendum}(\Delta(\mathcal{O}))$ therefore contains all convex combinations of the elements of the canonical basis of antisymmetric matrices. In particular, all antisymmetric matrices of taxicab norm 2 are images by $\mathsf{referendum}$ of some preferences of $\Delta(\mathcal{O})$. Plus, the uniform distribution $\tilde{u} \in \Delta(\mathcal{O})$ on all ballots is trivially in the kernel of $\mathsf{referendum}$. A convex combination involving $\tilde{u}$ then enables to obtain any antisymmetric matrix of taxicab norm at most 2. Such antisymmetric matrices correspond to weighted directed antisymmetric graphs whose sum of weights is at most 1. This proves the corollary. □

Corollary 4 *All directed graphs can be generated by some ballots $\tilde{a} \in \Delta(\mathcal{O})$.*

Proof. It suffices to put weights $\dfrac{2}{|X|(|X| - 1)}$ on all arcs and apply the previous theorem. □

The two corollaries enable us to exhibit ballots which produce certain kinds of weighted tournaments.

4.3 Proof of near uniqueness

This enables us to prove the uniqueness of the DSCC voting system for a large class of voting systems, and for an important type of ballots.

Theorem 7 *A DSCC voting system based on weighted tournaments must agree with the randomized Condorcet voting system whenever the weighted tournament has a sum of weights less than 1.*

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[6]An antisymmetric matrix R is a matrix whose transposition R^T equals its opposite $-R$, i.e. $R^T = -R$.

Proof. By contradiction, assume that $\mathscr{V}$ disagrees with the randomized Condorcet voting system $\mathscr{C}$ for some weighted tournament WT whose sum of weights is at most 1. We denote T the tournament derived from WT. Let us consider ballots $a_{vw}^+, a_{vw}^- \in \mathcal{O}$ defined by

$$a_{vw}^+ : v \succ w \succ x_1 \succ \ldots \succ x_n \quad \text{and} \quad a_{vw}^- : x_n \succ \ldots \succ x_1 \succ v \succ w. \tag{30}$$

Like in the proof of Lemma 3, the choices of $x_1, \ldots, x_n$ are irrelevant. Denoting $\lambda_{vw} = \mathsf{gain}(T, v, w)$, the weighted tournament WT is generated by ballots $\tilde{a}$ (that is, $WT = WT(\tilde{a})$), where

$$\tilde{a} = \frac{1}{2} \sum_{T:v \gg w} \lambda_{vw} a_{vw}^+ + \frac{1}{2} \sum_{T:v \gg w} \lambda_{vw} a_{vw}^- + \delta \tilde{u}, \tag{31}$$

where $\delta = 1 - \sum_{T:v \gg w} \lambda_{vw}$ is 1 minus the sum of all weights of arcs of WT. By assumption on WT, we have $\delta > 0$.

Since $\mathscr{V}$ disagrees with $\mathscr{C}$, there must be some alternative $x \in X$ which beats $\mathscr{V}(\tilde{a})$ in the tournament. Denoting $Z = \{z_1, \ldots, z_n\}$ and $Y = \{y_1, \ldots, y_m\}$ the sets of alternatives which respectively are beaten and beat x, this means that $\mathbb{P}[\mathscr{V}(\tilde{a}) \in Z] > \mathbb{P}[\mathscr{V}(\tilde{a}) \in Y]$. Evidently, by definitions of sets Y and Z, we have $\lambda_{yx} > 0$ and $\lambda_{xz} > 0$ for all $y \in Y$ and $z \in Z$.

We then define the preference $\theta \in \mathcal{O}$ of conspirators by

$$\theta : z_1 \succ \ldots \succ z_n \succ x \succ y_1 \succ \ldots \succ y_m. \tag{32}$$

Notice that we have $\theta : \mathscr{V}(\tilde{a}) \succ x$. We can now define the preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ of the people by

$$\tilde{\theta} = \frac{1}{2} \sum_{\substack{T:v \gg w \\ (v,w) \notin Y \times \{x\}}} \lambda_{vw}(a_{vw}^+ + a_{vw}^-) + \frac{1}{2} \sum_{y \in Y} \lambda_{yx} a_{yx}^+ + \left(\epsilon + \frac{1}{2} \sum_{y \in Y} \lambda_{yx} \right) \theta + (\delta - \epsilon)\tilde{u}, \tag{33}$$

with $0 < \epsilon < \delta$. Importantly, any ballot $a \in \mathcal{O}$ except θ is more frequent in $\tilde{a}$ than in $\tilde{\theta}$. Thus, by Lemma 2, conspirators θ can have produced the ballots $\tilde{a}$. Moreover, we have:

$$\forall z \in Z, \quad \mathsf{referendum}(\tilde{\theta}, x, z) = \lambda_{xz} + \frac{1}{2} \sum_{y \in Y} \lambda_{yx} - \epsilon - \frac{1}{2} \sum_{y \in Y} \lambda_{yx} = \lambda_{xz} - \epsilon,$$

$$\forall y' \in Y, \quad \mathsf{referendum}(\tilde{\theta}, x, y') = -\frac{1}{2} \sum_{y \in Y} \lambda_{yx} + \epsilon + \frac{1}{2} \sum_{y \in Y} \lambda_{yx} = +\epsilon.$$

Recall that $\lambda_{xz} > 0$ for all $z \in Z$, hence we can choose ϵ smaller than all λ_{xz}. By doing so, we guarantee that $\mathsf{referendum}(\tilde{\theta}, x, z)$ and $\mathsf{referendum}(\tilde{\theta}, x, y)$ are positive for all $z \in Z$ and $y \in Y$. Thus, x is a Condorcet winner for $\tilde{\theta}$. Yet, by creating ballots $\tilde{a}$, conspirators θ have obtained strictly better, as we have seen that $\theta : \mathscr{V}(\tilde{a}) \succ x$. This shows that $\mathscr{V}$ is not DSIC, and concludes the proof. $\qquad \square$

Corollary 5 *The randomized Condorcet voting system is the only DSCC voting system based on the tournament of the ballots.*

Proof. This is immediately deduced from the previous theorem, by adding weights of $2/|X|^2$ to the arcs of the graph of the tournament. $\qquad \square$

While the theorems here indicate that a sort of uniqueness of DSCC voting systems, at least in a neighborhood of the uniform distribution, I have not succeeded in characterizing these DSCC voting systems. I suspect them not to be unique though. My intuition is that some ballots are so extreme that there are not many preferences that could have led to them, which makes incentive-compatibility not restrictive enough to impose uniqueness.

5 Median voter

In practice, alternatives and voters in politics range on a left-right line. This structure has led to many median voter theorem (Black (1958); Roberts (1977); Rothstein (1990, 1991); Gans and Smart (1996)). Formally, there are two unrelated structures which fit the median voter theorem. In both cases, a total left-right order relation on alternatives is given. We denote $x < y$ the fact that alternative x is strictly on the left of y. Plus, accordingly to classical notations for integers, we denote intervals of alternatives as follows:

$$(x, \infty) = \{y \in X \mid x < y\}, \qquad [x, y] = \{z \in X \mid x \leq z \leq y\}. \tag{34}$$

Similarly, we define intervals $(-\infty, x)$, (x, y), $[x, y)$...

5.1 Single-peakedness and single-crossing

Denote $x^*(\theta) \in X$ the preferred alternative of $\theta \in \mathcal{O}$, also known as its *ideal point*. The set $\mathcal{O}^{SP}$ of single-peaked preferences is the set of preferences $\theta \in \mathcal{O}$ which prefer alternatives closer to their ideal point $x^*(\theta)$ to extreme alternatives. Formally, *single-peakedness* requires that if $y < x < x^*(\theta)$ or $x^*(\theta) > x > y$, then $\theta : x^*(\theta) \succ x \succ y$. In other words, the further away an alternative is from θ's ideal point, the less it is appreciated by θ. A partial order of preferences $\theta \in \mathcal{O}^{SP}$ can then be defined accordingly to their ideal points. A median voter is then a voter whose ideal point is a median of the ideal points of all voters. In fact, in this setting, the concept of median voter rather refers to the median of ideal points. While the existence of a median voter is not guaranteed a priori, it is satisfied whenever there is an odd number of voters or nearly always when the number of voters is large enough.

In contrast, *single-crossing* requires the existence of a subset $\mathcal{O}^{SC} \subset \mathcal{O}$ all preferences belong to, and on which another total left-right order relation is defined. Similarly to the left-right order of alternatives, we denote $\theta_1 < \theta_2$ for $\theta_1, \theta_2 \in \mathcal{O}^{SC}$ the fact that θ_1 is on the left of θ_2. Plus, single-crossing requires that, if $x < y$ and $\theta_1 < \theta_2$, we have

$$\big(\theta_1 : y \succ x \Rightarrow \theta_2 : y \succ x\big) \qquad \text{and} \qquad \big(\theta_2 : x \succ y \Rightarrow \theta_1 : x \succ y\big). \tag{35}$$

This criterion means that all voters preferring x to y are on the left of the others. Once again, if there is an odd number of voters, or in most cases when there is a large number of voters, the preferences $\tilde{\theta} \in \Delta(\mathcal{O}^{SC})$ of the people yield a median voter.

Single-crossing and single-peakedness are two unrelated assumptions. Neither is implied by the other. Also, note that, while the set $\mathcal{O}^{SP}$ of single-peaked preferences is uniquely defined, the set $\mathcal{O}^{SC}$ of single-crossing preferences is not. Here are examples to better understand what is meant by these concepts.

Example 2 *Consider $w < x < y < z$ four alternatives. The set $\mathcal{O}^{SP}$ of single-peaked preferences is given by*

$$\mathcal{O}^{SP} = \left\{ \begin{array}{llll} w \succ x \succ y \succ z, & x \succ w \succ y \succ z, & x \succ y \succ w \succ z, & x \succ y \succ z \succ w, \\ y \succ x \succ w \succ z, & y \succ x \succ z \succ w, & y \succ z \succ x \succ w, & z \succ y \succ x \succ w \end{array} \right\}. \tag{36}$$

Let us show that $\mathcal{O}^{SP}$ is not single-crossing. Denote $\theta_1 : x \succ y \succ z \succ w$ and $\theta_2 : y \succ x \succ w \succ z$. Since $x < y$, $\theta_1 : x \succ y$ and $\theta_2 : y \succ x$, we must have $\theta_1 < \theta_2$. However, since $w < z$, $\theta_2 : w \succ z$ and $\theta_1 : z \succ w$, we must also have $\theta_2 < \theta_1$. Thus, it is not possible to order preferences of $\mathcal{O}^{SP}$. In particular, θ_1 and θ_2 are not single-crossing.

By opposition, imagine that the alternative y, despite being at the center, is not very attractive to voters, while w is more charismatic than x. We may then obtain the following set $\mathcal{O}^{SC}$ of single-crossing preferences:

$$\mathcal{O}^{SC} = \{w \succ x \succ z \succ y, \quad w \succ z \succ x \succ y, \quad z \succ w \succ x \succ y, \quad z \succ x \succ w \succ y, \quad z \succ x \succ y \succ w\}. \tag{37}$$

The preferences of $\mathcal{O}^{SC}$ are related to one another by swaps of two alternatives which are consistent with the left-right orders of the alternatives. It is noteworthy that several of these preferences are not single-peaked.

Also, note that, because the order relation on single-crossing preferences is total, any single-crossing subset of preferences has a smallest and a largest elements. These correspond to the most leftist and the most rightist preferences of the subset. This remark will be useful for Theorem 9.

Now, the well-known median voter which works in both cases is the following result.

Theorem 8 (Black (1958); Roberts (1977); Rothstein (1990, 1991); Gans and Smart (1996)) *If preferences are single-peaked or single-crossing and yield a median voter, then the median voter's favorite alternative is the Condorcet winner.*

We refer to original papers or Myerson (1996) for a proof of this theorem. Importantly, this theorem shows that the left-right line assumption greatly simplifies the setting, and that ballots in practice are in fact much simpler than the general setting we have been dealing with so far.

5.2 Manipulable voting systems

However, as pointed out by Penn et al. (2011), this apparent simplicity vanishes as we involve incentive-compatibility. In particular, Penn et al. (2011) prove that a deterministic strongly incentive-compatible voting system must be dictatorial even when preferences are assumed single-peaked.

In the last decades, Schulze (2011) introduced a seductive deterministic voting system which elects Condorcet winners when they exist. When they do not, Schulze proposes to remove the arcs of the weighted tournament which have the smallest weights until the tournament yields a Condorcet winner. Unfortunately, we show that the Schulze method fails to be incentive-compatible even in the simplified setting of preferences assumed both single-crossing and single-peaked, and with the weaker version of incentive-compatibility.

Proposition 2 *Even for preferences both single-peaked and single-crossing with a unique median voter, the Schulze method is not DSIC.*

Proof. Let $x < y < z$ three alternatives and preferences both single-crossing and single-peaked defined by

$$\theta_1 : x \succ y \succ z, \qquad \theta_2 : y \succ x \succ z, \qquad \theta_3 : y \succ z \succ x, \qquad \theta_4 : z \succ y \succ x. \tag{38}$$

Consider 15 voters whose preferences $\tilde{\theta} \in \Delta(\mathcal{O})$ are

$$15\,\tilde{\theta} = 7\theta_1 + 3\theta_2 + 3\theta_3 + 2\theta_4. \tag{39}$$

Then, $15\ \mathsf{referendum}(\tilde{\theta}, y, x) = -7 + 3 + 3 + 2 = 1 > 0$ and $15\ \mathsf{referendum}(\tilde{\theta}, y, z) = 7 + 3 + 3 - 2 = 11$. Thus, y is the Condorcet winner of $\tilde{\theta}$, and the one that the Schulze method elects. But now consider that conspirators θ_1 choose ballot $s(\theta_1) = a : x \succ z \succ y$, and that other voters are truthful (i.e. $\mathsf{Conspriator}(s) = \{\theta_1\}$). Then,

$$15\,s(\tilde{\theta}) = 7a + 3\theta_2 + 3\theta_3 + 2\theta_4. \tag{40}$$

We still have $15\ \mathsf{referendum}(s(\tilde{\theta}), y, x) = 1$ and $15\ \mathsf{referendum}(s(\tilde{\theta}), x, z) = 7 + 3 - 3 - 2 = 5$. But now, $15\ \mathsf{referendum}(s(\tilde{\theta}), z, y) = 7 - 3 - 3 + 2 = 3$. Thus, we now have a Condorcet paradox $T(s(\tilde{\theta})) : y \gg x \gg z \gg y$. This is illustrated in Figure 4.
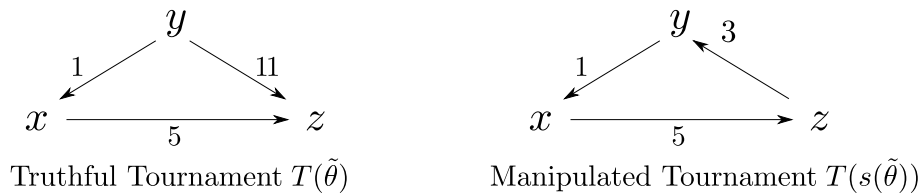

Figure 4: On the left is the tournament $T(\tilde{\theta})$, where y is Condorcet winner, while on the right is the tournament $T(s(\tilde{\theta}))$ for which the Schulze method elects x

At this point, the Schulze method consists in removing the arc with the lowest weight, which is $T(s(\tilde{\theta}))$: $y \gg x$, leading to the election of x. Since conspirators θ_1 indeed prefer x to y, they benefit from conspiring, which proves that the Schulze method is not DSIC. □

Another recently proposed voting system is majority judgment introduced by Balinski and Laraki (2010). In this setting, voters are assumed to have more cardinal-like preferences. They are asked to note each alternative on a certain scale. Then, for each alternative, we compute the median of his notes. The elected alternative is then the one with the highest median note. Unfortunately, once again, this voting system is still not incentive-compatible. For instance, the median voter can easily manipulate the notes. Granted, this sounds unlikely that this unknown median voter does so in practice, notably because he cannot possibly know that *a priori*. However, by considering a collective deviation, this sounds more likely.

Proposition 3 *Even for preferences both single-peaked and single-crossing with a unique median voter, the majority judgment is not DSIC.*

Proof. The proof uses a similar example to the previous proof. Consider the preferences of Table 1.

Table 1: Example of majority judgment

Population	Preference	x	y	z
7	θ_1	4	3	0
3	θ_2	2	4	0
3	θ_3	0	4	2
2	θ_4	0	1	4

As proved earlier, the preferences are both single-peaked and single-crossing with a unique median voter. The median notes are 2 for x, 3 for y and 0 for z. The winner is thus the Condorcet winner y. However, if conspirators θ_1 vote $(4, 1, 0)$ for x, y and z, then x becomes the winner. □

These propositions confirm the indications given by Penn et al. (2011) of the limits of the known voting systems. Now, Moulin (1980) proposed a dominant-strategy incentive compatible voting system for single-peaked preferences. Saporiti (2009) goes on proving its strong incentive-compatibility for single-crossing preferences. This method is based on the median rule and its variants, which consist in electing the winner accordingly to the median voter. However, this method requires the knowledge of the position of alternatives on the left-right line, and uses this knowledge to restrict the set of ballots voters can choose from. Yet, although the left-right line structure may well exist informally, it may not be official, and hence it may be unacceptable to use it to design the voting system. This would prevent us from using the left-right line structure to define the voting system, which make the approaches by Moulin (1980) and Saporiti (2009) inapplicable in practice.

5.3   Median voter and the randomized Condorcet voting system

Before studying how the randomized Condorcet voting system handles single-peaked or single-crossing preferences, let us prove a general fact about the conspirators in this left-right line setting.

Lemma 4 *When preferences are single-peaked or single-crossing with a unique median voter, conspirators must either be all strictly on the left or all strictly on the right of the median voter.*

Proof. Let $\mathcal{V}$ a voting system. Denote x the Condorcet winner of the single-peaked preferences $\tilde{\theta} \in \Delta(\mathcal{O})$. Denote $Z = (-\infty, x)$ and $Y = (x, \infty)$. Now, consider a strategy s. We denote $s(\tilde{\theta}) = \tilde{a}$. The strict incentive to conspire means that

$$\forall \theta \in \mathsf{Conspirator}(s), \quad \theta : \mathcal{V}(\tilde{a}) \succ x. \tag{41}$$

If preferences are single-peaked or single-crossing and $\theta \in \mathsf{Conspirator}(s)$ is on the right of the median voter, we know that $\theta : x \succ z$ for all $z \in Z$. Indeed, if preferences are single-peaked, this is due to θ's ideal point

being on the right of x. And if preferences are single-crossing, this is because x cannot be switched with a left alternative as we look preferences on the right of the median voter. Since the median voter ranked x better than any $z \in Z$, all voters on its right must do so too.

Now that we know that $\{z \in X \mid \theta : x \succ z\} \supset Z$, we necessarily have

$$0 < \mathsf{gain}(\theta, \mathscr{V}(\tilde{a}), x) = \mathbb{P}_{\mathscr{V}(\tilde{a})}\left[\theta : \mathscr{V}(\tilde{a}) \succ x\right] - \mathbb{P}_{\mathscr{V}(\tilde{a})}\left[\theta : \mathscr{V}(\tilde{a}) \prec x\right] \le \mathbb{P}_{\mathscr{V}(\tilde{a})}[Y] - \mathbb{P}_{\mathscr{V}(\tilde{a})}[Z]. \qquad (42)$$

Therefore, we have $\mathbb{P}_{\mathscr{V}(\tilde{a})}[Y] > \mathbb{P}_{\mathscr{V}(\tilde{a})}[Z]$. But if $\theta \in \mathsf{Conspirator}(s)$ is on the left of the median voter, we must have the opposite inequality. Both cases cannot occur simultaneously, which proves that all conspirators must be on the same side of the left-right spectrum with regards to the median voter. This proves the lemma. $\square$

Finally, we show that the randomized Condorcet voting system behaves very well in the left-right line setting. More precisely, whether preferences are single-peaked or single-crossing, the randomized Condorcet voting system is strongly incentive-compatible.

Theorem 9 *The randomized Condorcet voting system is strongly incentive-compatible when preferences are single-peaked or single-crossing with a unique median voter.*

Proof. The proofs in the two cases of single-peakedness and single-crossing are slightly different, and are given in the appendix. To avoid confusion, we wrote them in two different blocks. $\square$

This theorem ensures that in many practical applications, the randomized Condorcet voting system behaves exactly as Condorcet would have wanted voting systems to behave. Namely, it guarantees the election of Condorcet winners when they exist, even when voters try to conspire.

6 Conclusion

In this article, we have introduced the randomized Condorcet voting system. By defining it as the choice of the Condorcet winner of the randomized tournament, we have shown that it is a natural generalization of Condorcet's insights to preferences which yield a Condorcet paradox. As often in mathematics, natural structures have mesmerizing properties, and this is the case of the randomized Condorcet voting system. Most importantly, it is DSCC, and we have given strong indications that it is essentially unique. In addition, in many cases in practice, the structure of the preferences of the people, which often corresponds to a mixture of single-peakedness and single-crossing, even guarantees the strong incentive-compatibility. These fascinating mathematical properties of the randomized Condorcet voting system lead us to strongly recommend its implementation in practice.

This is why we end this article with remarks on this implementation. First, note that if voters do not want to bother ranking all the alternatives they do not need to. In fact, any partial (and not even necessarily transitive!) ordering of the candidates will do, as what we need for the randomized Condorcet voting system is pairwise comparisons of alternatives. Typically, a voter may rank his three favorite alternatives, leaves a blank and add his least favorite one at the bottom of his ballot. Now, evidently, if ballots are to be read by hand, it may take a while to add up all the pairwise comparisons. But since no more than $|X|(|X| - 1)/2$ pairwise comparisons need to be tracked, these can be easily computed using computers. The linear program proposed in equation (9) then only features about $|X|$ variables and constraints. It can thus be solved nearly instantaneously for reasonable sizes of X. Plus, interestingly, because the result of the vote only depends on the tournament, a graphical representation of the induced subgraph of the main alternatives can then be displayed to explain and analyze the outcome of the votes. For these reasons, we end this paper by recommending the use of the randomized Condorcet voting system in practice.

Appendices

A Proof and formal statement of Theorem 1

Let Σ_n the set permutations of $\{1, \ldots, n\}$. Any permutation $\sigma \in \Sigma_n$ acts on $\mathcal{O}^n$ by $(\sigma \cdot (\theta_1, \ldots, \theta_n))_k = \theta_{\sigma^{-1}(k)}$. In other words, σ permutes the labels of voters. We also define the equivalence relation $\sim$ on $\mathcal{O}^n$ by $\theta \sim \theta'$ if there exists $\sigma \in \Sigma_n$ such that $\sigma \cdot \theta = \theta'$. In other words, two preference profiles are equivalent if they are the same up to labeling.

Lemma 5 *For all integers $n \geq 1$, there is a canonical injection $\iota_n : (\mathcal{O}^n / \sim) \to \Delta(\mathcal{O})$ that counts frequencies of preferences.*

Proof. We define the function ι_n by $\mathbb{P}_{\iota_n(a)}[\iota_n(a) = \theta] = \dfrac{1}{n} |\{k \mid a_k = \theta\}|$ for all $\theta \in \mathcal{O}$. It is straightforward to see that if $\sigma \cdot a = b$, then $\mathbb{P}_{\iota_n(b)}[\iota_n(b) = \theta] = \mathbb{P}_{\iota_n(a)}[\iota_n(a) = \theta]$ for all $\theta \in \mathcal{O}$. In other words, if $a \sim b$, then $\iota_n(b) = \iota_n(a)$, which proves that ι_n is well-defined.

Finally, we need to prove that ι_n is injective. But this follows from the fact that two sets $\{k \mid a_k = \theta\}$ and $\{k \mid b_k = \theta\}$ are in bijection if and only if they have the same cardinals. So, if $\iota_n(a) \neq \iota_n(b)$, there cannot be a permutation $\sigma \in \Sigma_n$ such that $\sigma \cdot a = b$. $\qquad \square$

Now, formally, what we meant in Theorem 1 was that, for all integers $n \geq 1$, there is a canonical surjection from the set of functions $\Delta(\mathcal{O}) \to Z$ onto the set of anonymous functions $\mathcal{O}^n \to Z$.

Proof. We first notice the trivial bijection f between the set of functions $(\mathcal{O}^n / \sim) \to Z$ and the set of anonymous functions $\mathcal{O}^n \to Z$. Indeed, the fact that the anonymity of a function $\mathcal{O}^n \to Z$ is exactly the fact that the function $(\mathcal{O}^n / \sim) \to Z$ is well-defined. We can then use the canonical injection ι_n to define the canonical surjection required for Theorem 1. Let $\mathscr{V} : \Delta(\mathcal{O}) \to Z$. Then, we define $\kappa_n(\mathscr{V}) : (\mathcal{O}^n / \sim) \to Z$ by $\kappa_n(\mathscr{V})(a) = \mathscr{V}(\iota_n(a))$. It is straightforward to see that κ_n is a surjection. Composing the bijection f with the surjection κ_n yields the surjection $f \circ \kappa_n$ from the set of functions $\Delta(\mathcal{O}) \to Z$ onto the set of anonymous functions $\mathcal{O}^n \to Z$. $\qquad \square$

B Proof of Lemma 2

Proof. First notice that $\tilde{\theta}$ could have produced $\tilde{a}$ with a set $\mathcal{C}$ of conspirators if and only if there exists a strategy s such that $\mathsf{Conspirator}(s) = \mathcal{C}$ and $s(\tilde{\theta}) = \tilde{a}$. Consider that we indeed have $s(\tilde{\theta}) = \tilde{a}$, and let us prove the direct implication of the lemma. Let $\mathcal{D} \subset \mathcal{O} - \mathcal{C}$. Then,

$$\mathbb{P}_{\tilde{a}}[\mathcal{D}] = \mathbb{E}_{\tilde{\theta}}\left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}]\right] = \mathbb{E}_{\tilde{\theta}}\left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}]\Big|\tilde{\theta} \in \mathcal{D}\right]\mathbb{P}_{\tilde{\theta}}[\mathcal{D}] + \mathbb{E}_{\tilde{\theta}}\left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}]\Big|\tilde{\theta} \notin \mathcal{D}\right]\mathbb{P}_{\tilde{\theta}}[\mathcal{O} - \mathcal{D}]. \qquad (43)$$

But since $\mathcal{D} \cap \mathcal{C} = \emptyset$, for all $\theta \in \mathcal{D}$, we have $\theta \notin \mathcal{C}$. Thus, $s(\theta) = s^{truth}(\theta) = \theta$. Thus, $\mathbb{P}_{s(\theta)}[\mathcal{D}] = 1$. Thus, the expression above simplifies to

$$\mathbb{P}_{\tilde{a}}[\mathcal{D}] = \mathbb{P}_{\tilde{\theta}}[\mathcal{D}] + \mathbb{E}_{\tilde{\theta}}\left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}]\Big|\tilde{\theta} \notin \mathcal{D}\right]\mathbb{P}_{\tilde{\theta}}[\mathcal{O} - \mathcal{D}]. \qquad (44)$$

Therefore, $\mathbb{P}_{\tilde{a}}[\mathcal{D}] \geq \mathbb{P}_{\tilde{\theta}}[\tilde{\theta} \in \mathcal{D}]$, hence proving the direct implication.

Reciprocally, if $\mathbb{P}_{\tilde{\theta}}[\mathcal{C}] = 0$, then the inequality $\mathbb{P}_{\tilde{\theta}}[\mathcal{D}] \geq \mathbb{P}_{\tilde{a}}[\mathcal{D}]$ implies $\tilde{\theta} = \tilde{a}$. Thus, $s^{truth}(\tilde{\theta}) = \tilde{a}$, which proves that $\tilde{\theta}$ could have produced $\tilde{a}$ with the set $\mathcal{C}$ of conspirators. Otherwise, $\mathbb{P}_{\tilde{\theta}}[\mathcal{C}] \neq 0$. We define $s : \mathcal{C} \to \Delta(\mathcal{O})$ by $\mathbb{P}_{s(\theta)}[\mathcal{C} - \{\theta\}] = 0$,

1. $\forall \mathcal{E} \subset \mathcal{C}, \mathbb{P}_{s(\theta)}[\mathcal{E}] = \mathbb{P}_{\tilde{a}}[\mathcal{E}]/\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]$.
2. $\forall \mathcal{D} \subset \mathcal{O} - \mathcal{C}, \mathbb{P}_{s(\theta)}[\mathcal{D}] = \left(\mathbb{P}_{\tilde{a}}[\mathcal{D}] - \mathbb{P}_{\tilde{\theta}}[\mathcal{D}]\right)/\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]$.

Assuming $\mathbb{P}_{\tilde{\theta}}[\mathcal{D}] \leq \mathbb{P}_{\tilde{a}}[\mathcal{D}]$ for all supsets $\mathcal{D} \supset \mathcal{O} - \mathcal{C}$, the probabilities we have defined here are all non-negative. It is straightforward to see that the additivity of the probability is satisfied. Plus,

$$\mathbb{P}_{\tilde{\theta}}[\mathcal{O}] = \mathbb{P}_{\tilde{\theta}}[\mathcal{C}] + \mathbb{P}_{\tilde{\theta}}[\mathcal{O} - \mathcal{C}] = \frac{\mathbb{P}_{\tilde{a}}[\mathcal{C}]}{\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]} + \frac{\mathbb{P}_{\tilde{a}}[\mathcal{O} - \mathcal{C}] - \mathbb{P}_{\tilde{\theta}}[\mathcal{O} - \mathcal{C}]}{\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]} = 1. \tag{45}$$

Therefore, $s(\theta)$ is a well-defined probability, and s a well-defined strategy of support $\mathcal{C}$. Plus, for $\mathcal{D} \subset \mathcal{O} - \mathcal{C}$,

$$\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}] = \mathbb{P}_{\tilde{\theta}}[\mathcal{D}] + \mathbb{E}_{\tilde{\theta}} \left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}] \Big| \tilde{\theta} \in \mathcal{C} \right] \mathbb{P}_{\tilde{\theta}}[\mathcal{C}] + \mathbb{E}_{\tilde{\theta}} \left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{D}] \Big| \tilde{\theta} \notin \mathcal{C} \cup \mathcal{D} \right] \mathbb{P}_{\tilde{\theta}}[\mathcal{O} - \mathcal{C} \cup \mathcal{D}] \tag{46}$$

$$= \mathbb{P}_{\tilde{\theta}}[\mathcal{D}] + \frac{\mathbb{P}_{\tilde{a}}[\mathcal{D}] - \mathbb{P}_{\tilde{\theta}}[\mathcal{D}]}{\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]} \mathbb{P}_{\tilde{\theta}}[\mathcal{C}] = \mathbb{P}_{\tilde{a}}[\mathcal{D}], \tag{47}$$

and, similarly for any $\mathcal{E} \subset \mathcal{C}$, we have

$$\mathbb{P}_{s(\tilde{\theta})}[\mathcal{E}] = \mathbb{E}_{\tilde{\theta}} \left[\mathbb{P}_{s(\tilde{\theta})}[\mathcal{E}] \Big| \tilde{\theta} \in \mathcal{C} \right] \mathbb{P}_{\tilde{\theta}}[\mathcal{C}] = \frac{\mathbb{P}_{\tilde{a}}[\mathcal{E}]}{\mathbb{P}_{\tilde{\theta}}[\mathcal{C}]} \mathbb{P}_{\tilde{\theta}}[\mathcal{C}] = \mathbb{P}_{\tilde{a}}[\mathcal{E}]. \tag{48}$$

These two equalities prove that $s(\tilde{\theta}) = \tilde{a}$, which is what we had to prove. $\qquad\square$

C   Proof of Theorem 9 for single-peakedness

Proof. Let $\mathscr{C}$ the randomized Condorcet voting system. We use the same notations $x, \tilde{\theta}, Y, Z, s$ and $\tilde{a}$ as in the proof of Lemma 4. Without loss of generality, we can assume that conspirators are all strictly on the right of the median voter.

Let $z \in Z$. As we have seen in the previous proof, we have $\theta : x \succ z$ for all $\theta \in \mathsf{Conspirator}(s)$. Since all conspirators agree with arrows $T(\tilde{\theta}) : x \gg z$ for $z \in Z$, they cannot invert these. Therefore, $T(\tilde{a}) : x \gg z$. Yet, for conspirators to gain by conspiring, $\mathscr{C}(\tilde{a})$ must differ from x, which means that there must be some $y \in Y$ such that $T(\tilde{a}) : y \gg x$. Let y^* the most leftist alternative which beats x, i.e.

$$y^* = \min\{y \in Y \mid T(\tilde{a}) : y \gg x\}. \tag{49}$$

We denote $Y_- = (x, y^*)$ and $Y_+ = [y^*, \infty)$.

Since x is Condorcet winner of $\tilde{\theta}$, we know that $T(\tilde{\theta}) : x \gg y^*$. Thus, conspirators must have inverted the arc from x to y^*. Since conspirators can only invert arcs they agree with, this means that there must be a conspirator $\theta \in \mathsf{Conspirator}(s)$ who agrees with arc $T(\tilde{\theta}) : x \gg y^*$. This conspirator thus thinks $\theta : x \succ y^*$. We will show that assuming that he had incentive to conspire leads to a contradiction.

Now, by definition of y^*, we have $T(\tilde{a}) : x \gg y$ for all $y \in Y_-$, i.e.

$$Y_- \subset \left\{ w \in X \mid T(\tilde{a}) : x \gg w \right\} \quad \text{and} \quad \left\{ w \in X \mid T(\tilde{a}) : w \gg x \right\} \subset Z \cup Y_+. \tag{50}$$

Combining this with the property $T(\tilde{a}) : \mathscr{C}(\tilde{a}) \geqq x$ satisfied by the randomized Condorcet voting system yields

$$\mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Y_-] \leq \mathbb{P}_{\mathscr{C}(\tilde{a})} \left[T(\tilde{a}) : x \gg \mathscr{C}(\tilde{a}) \right] \leq \mathbb{P}_{\mathscr{C}(\tilde{a})} \left[T(\tilde{a}) : \mathscr{C}(\tilde{a}) \gg x \right] \leq \mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Z \cup Y_+]. \tag{51}$$

Now, strict incentives to conspire for θ imply that

$$\mathbb{P}_{\mathscr{C}(\tilde{a})} \left[\theta : \mathscr{C}(\tilde{a}) \succ x \right] > \mathbb{P}_{\mathscr{C}(\tilde{a})} \left[\theta : x \succ \mathscr{C}(\tilde{a}) \right]. \tag{52}$$

Since $\theta : x \succ y^*$, we know that the ideal point of θ is necessarily on the left of y^*. As a result, for $y \in Y_+$, we have $\theta : x \succ y^* \succ y$. Therefore,

$$Y_- \supset \left\{ w \in X \mid \theta : w \succ x \right\} \quad \text{and} \quad \left\{ w \in X \mid \theta : x \succ w \right\} \supset Z \cup Y_+. \tag{53}$$

which leads to $\mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Y_-] > \mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Z \cup Y_+]$. This contradicts equation (51), and proves the theorem for single-peaked preferences. $\qquad\square$

D   Proof of Theorem 9 for single-crossing

Proof. We reuse the same notations $x, \tilde{\theta}, Y, Z, s$ and $\tilde{a}$ as in the proof of Theorem 9, except that we now consider $\tilde{\theta}$ single-crossing. Let θ_m the median voter. Lemma 4 allows us to assume without loss of generality that $\mathsf{Conspirator}(s) \subset (\theta_m, \infty)$. Then, we have, once again, $T(\tilde{a}) : x \gg z$ for all $z \in Z$.

Let now $\theta = \min \mathsf{Conspirator}(s)$ the most leftist conspirator. Denote Y^+ and Y^- defined by

$$Y^+ = \{y \in Y \mid \theta : y \succ x\} \quad \text{and} \quad Y^- = \{y \in Y \mid \theta : x \succ y\}. \tag{54}$$

Contrary to the proof of Theorem 9, Y^+ now corresponds to the alternatives some conspirators prefer to x, as the sign " $+$ " now refers to θ's preference rather than the left-right line of alternatives.

Let $y^+ \in Y^+$. Since for any $\theta' \in \mathsf{Conspirator}(s)$, we have $\theta < \theta'$, single-crossing implies that $\theta' : y^+ \succ x$. Therefore, conspirators all disagree with arcs $T(\tilde{\theta}) : x \gg y^+$, and hence cannot invert them. Therefore, $T(\tilde{a}) : x \gg y^+$. Since this holds for all $y^+ \in Y^+$, we have

$$Y^+ \subset \{w \in X \mid T(\tilde{a}) : x \gg w\} \quad \text{and} \quad \{w \in X \mid T(\tilde{a}) : w \gg x\} \subset Z \cup Y^-. \tag{55}$$

Yet, the fundamental property of the randomized Condorcet voting system applied to x then implies

$$\mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Y^+] \le \mathbb{P}_{\mathscr{C}(\tilde{a})}\left[T(\tilde{a}) : x \gg \mathscr{C}(\tilde{a})\right] \le \mathbb{P}_{\mathscr{C}(\tilde{a})}\left[T(\tilde{a}) : \mathscr{C}(\tilde{a}) \gg x\right] \le \mathbb{P}_{\mathscr{C}(\tilde{a})}\left[\mathscr{C}(\tilde{a}) \in Z \cup Y^-\right]. \tag{56}$$

This contradicts the strict incentives for θ to conspire, i.e.

$$\mathbb{P}_{\mathscr{C}(\tilde{a})}[\mathscr{C}(\tilde{a}) \in Y^+] > \mathbb{P}_{\mathscr{C}(\tilde{a})}\left[\mathscr{C}(\tilde{a}) \in Z \cup Y^-\right]. \tag{57}$$

Thus, we reach the same conclusion for single-crossing preferences. $\square$

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