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# On the distance signless Laplacian of a graph

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**Abstract:** The distance signless Laplacian of a connected graph $G$ is defined by $\mathcal{D}^{\mathcal{Q}} = Diag(Tr) + \mathcal{D}$, where $\mathcal{D}$ is the distance matrix of $G$, and $Diag(Tr)$ is the diagonal matrix whose main entries are the vertex transmissions in $G$. The spectrum of $\mathcal{D}^{\mathcal{Q}}$ is called the distance signless Laplacian spectrum of $G$. In the present paper, we study some properties of the distance signless Laplacian eigenvalues. Among other results, we show that the complete graph is the unique graph with only two distinct distance signless Laplacian eigenvalues. We prove several bounds on $\mathcal{D}^{\mathcal{Q}}$ eigenvalues and establish a relationship between $n-2$ being a distance signless Laplacian eigenvalue of $G$ and $\overline{G}$ containing a bipartite component.

**Key Words:** Distance matrix, eigenvalues, Laplacian, signless Laplacian, spectral radius.

**Résumé :** Le laplacien sans signe des distances d'un graphe connexe $G$ est défini par $\mathcal{D}^{\mathcal{Q}} = Diag(Tr) + \mathcal{D}$, où $\mathcal{D}$ est la matrice des distances de $G$ et $Diag(Tr)$ est la matrice diagonale dont les principaux éléments sont les transmissions des sommets de $G$. Le spectre de $\mathcal{D}^{\mathcal{Q}}$ est appelé le spectre du laplacien sans signe des distances de $G$. Dans le présent article, nous étudions les propriétés des valeurs propres du laplacien sans signe des distances. Entre autres résultats, nous montrons que seul le graphe complet admet exactement deux valeurs propres du laplacien sans signe, distinctes. Nous prouvons plusieurs bornes sur les valeurs propres de $\mathcal{D}^{\mathcal{Q}}$, et établissons une relation entre le fait que $n-2$ soit une valeur propre du laplacien sans signe des distances de $G$ et l'existence de composantes biparties dans $\overline{G}$.

**Mots clés :** Matrice des distance, valeurs propres, laplacien, laplacien sans signe, rayon spectral.

# 1 Introduction

In the present paper, we consider only simple and finite graphs, *i.e*, graphs on a finite number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = G(V, E)$, where $V$ is its vertex set and $E$ its edge set. The *order* of $G$ is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges.

As usual, we denote by $P_n$ the path, by $C_n$ the cycle, by $S_n$ the star, by $S_n^+$ the unicyclic graph obtained from sthe star $S_n$ by adding an edge, by $K_{a,n-a}$ the complete bipartite graph and by $K_n$ the complete graph, each on $n$ vertices. A *kite* $Ki_{n,\omega}$ is the graph obtained from a clique $K_\omega$ and a path $P_{n-\omega}$ by adding an edge between an endpoint of the path and a vertex from the clique.

The *adjacency matrix* $A$ of $G$ is a 0–1 $n \times n$–matrix indexed by the vertices of $G$ and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the $A$–spectrum of $G$, *i.e.*, the spectrum of the adjacency matrix of $G$, and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix $L = Diag(Deg) - A$, where $Diag(Deg)$ is the diagonal matrix whose diagonal entries are the degrees in $G$, is called *the Laplacian* of $G$. Denote by $(\mu_1, \mu_2, \ldots, \mu_n)$ the $L$–spectrum of $G$, *i.e.*, the spectrum of the Laplacian of $G$, and assume that the eigenvalues are labeled such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The matrix $Q = Diag(Deg) + A$ is called *the signless Laplacian* of $G$. Denote by $(q_1, q_2, \ldots, q_n)$ the $Q$–spectrum of $G$, *i.e.*, the spectrum of the signless Laplacian of $G$, and assume that the eigenvalues are labeled such that $q_1 \geq q_2 \geq \cdots \geq q_n$.

Given two vertices $u$ and $v$ in a connected graph $G$, $d(u, v) = d_G(u, v)$ denotes the *distance* (the length of a shortest path) between $u$ and $v$. The *Wiener index* $W(G)$ of a connected graph $G$ is defined to be the sum of all distances in $G$, *i.e.*,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

The *transmission* $Tr(v)$ of a vertex $v$ is defined to be the sum of the distances from $v$ to all other vertices in $G$, *i.e.*,

$$Tr(v) = \sum_{u \in V} d(u, v).$$

A connected graph $G = (V, E)$ is said to be $k$*–transmission regular* if $Tr(v) = k$ for every vertex $v \in V$.

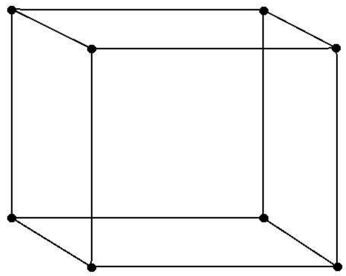
The *distance matrix* $\mathcal{D}$ of a connected graph $G$ is the matrix indexed by the vertices of $G$ where $\mathcal{D}_{i,j} = d(v_i, v_j)$ and $d(v_i, v_j)$ denotes the distance between the vertices $v_i$ and $v_j$. Let $(\partial_1, \partial_2, \ldots, \partial_n)$ denote the spectrum of $\mathcal{D}$ and assume that the eigenvalues are labeled such that $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. It is called the distance spectrum of the graph $G$.

Similarly to the (adjacency) Laplacian $L = Diag(Deg) - A$, we defined in [4] the *distance Laplacian* of a connected graph $G$ as the matrix $\mathcal{D}^L = Diag(Tr) - \mathcal{D}$, where $Diag(Tr)$ denotes the diagonal matrix of the vertex transmissions in $G$. Let $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$ denote the spectrum of $\mathcal{D}^L$ and assume that the eigenvalues are labeled such that $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L$. We call it the *distance Laplacian spectrum* of the graph $G$. Some properties of the distance Laplacian eigenvalues are discussed in [3]. In [14], Nath and Paul studied the second smallest distance Laplacian eigenvalue $\partial_{n-1}^L$ and characterized some families of graphs for which $\partial_{n-1}^L = n + 1$. They [14] also studied the distance Laplacian spectrum of the path $P_n$.

Also in [4], and similarly to the (adjacency) signless Laplacian $L = Diag(Deg) + A$, we introduced the *distance signless Laplacian* of a connected graph $G$ to be $\mathcal{D}^Q = Diag(Tr) + \mathcal{D}$. Let $(\partial_1^Q, \partial_2^Q, \ldots, \partial_n^Q)$ denote the spectrum of $\mathcal{D}^Q$ and assume that the eigenvalues are labeled such that $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$. We call it the *distance signless Laplacian spectrum* of the graph $G$.

In Figure 1, we give the cube graph with its different spectra.

For a connected graph $G$, let $P_{\mathcal{D}}^G(t)$, $P_{\mathcal{L}}^G(t)$ and $P_{\mathcal{Q}}^G(t)$ denote the distance, the distance Laplacian and the distance signless Laplacian characteristic polynomials respectively.



| | |
|---|---|
| $A$–spectrum | $(3, 1, 1, 1, -1, -1, -1, -3)$ |
| $L$–spectrum | $(6, 4, 4, 4, 2, 2, 2, 0)$ |
| $Q$–spectrum | $(6, 4, 4, 4, 2, 2, 2, 0)$ |
| $\mathcal{D}$–spectrum | $(12, 0, 0, 0, 0, -4, -4, -4)$ |
| $\mathcal{D}^L$–spectrum | $(16, 16, 16, 12, 12, 12, 12, 0)$ |
| $\mathcal{D}^Q$–spectrum | $(24, 12, 12, 12, 12, 8, 8, 8)$ |

Figure 1: The cube graph with its different spectra.

For the complete graph $K_n$, we have

$$\begin{aligned} P_{\mathcal{D}}^{K_n}(t) &= (t-n+1)(t+1)^{n-1}; \\ P_{\mathcal{L}}^{K_n}(t) &= t(t-n)^{n-1}; \\ P_{\mathcal{Q}}^{K_n}(t) &= (t-2n+2)(t-n+2)^{n-1}. \end{aligned}$$

For the complete bipartite graph $K_{a,b}$, we have

$$\begin{aligned} P_{\mathcal{D}}^{K_{a,b}}(t) &= \left(t-n+2-\sqrt{a^2-ab+b^2}\right)\left(t-n+2+\sqrt{a^2-ab+b^2}\right)(t+2)^{n-2}; \\ P_{\mathcal{L}}^{K_{a,b}}(t) &= t(t-n)(t-(2n-a))^{b-1}(t-(2n-b))^{a-1}; \\ P_{\mathcal{Q}}^{K_{a,b}}(t) &= \left(t-\frac{5n-8+\sqrt{9(a-b)^2+4ab}}{2}\right)\left(t-\frac{5n-8-\sqrt{9(a-b)^2+4ab}}{2}\right) \\ &\quad \times (t-2n+b+4)^{a-1}(t-2n+a+4)^{b-1}. \end{aligned}$$

From the above polynomials, one can easily derive those corresponding to the star $S_n$, *i.e.*, $a = n-1$ and $b = 1$.

Distance, distance Laplacian and distance signless Laplacian spectra of some common families of graphs can be found in [3, 4].

In [8, 9, 10], Cvetković and Simić studied the spectral graph theory based on the signless Laplacian matrix. Among other results, they showed equivalence between the spectrum of the signless Laplacian and

- the adjacency spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of bipartite graphs.

In [4], we showed equivalence between the distance Laplacian spectrum and

- the distance spectrum among the class of transmission regular graphs;
- the distance signless Laplacian spectrum among the class of transmission regular graphs;
- the Laplacian spectrum among the class of graphs with diameter at most two.

The rest of the paper is organized as follows. In Section 2, we discuss some local properties of the distance signless Laplacian spectrum. In Section 3, we prove a series of bounds on the eigenvalues of $\mathcal{D}^Q$, in particular the largest and the smallest of them. We also establish a relationship between the smallest eigenvalue of $\mathcal{D}^Q$ of a connected graph $G$ and the existence of a bipartite component in the complement $\overline{G}$. Finally, we list some open conjectures in Section 4.

# 2 Local properties

Some regularities in graphs are useful in calculating certain eigenvalues of the matrices related to these graphs. It is the case, for instance, for the largest eigenvalue of the adjacency matrix or the signless Laplacian whenever the graph is degree regular. The same is true for the largest eigenvalue of the distance Laplacian, and of the distance signless Laplacian, whenever the graph is transmission regular. Sometimes, a local regularity in a graph suffices to determine some eigenvalue. We prove below that it is possible to know a distance signless Laplacian eigenvalue of a graph if it contains a clique or an independent set whose vertices share the same transmission.

**Theorem 2.1** *Let $G$ be a connected graph on $n$ vertices. If $S = \{v_1, v_2, \dots v_p\}$ is an independent set of $G$ such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, then $\tau = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\tau - 2$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}$ with multiplicity at least $p - 1$.*

**Proof.** Since the vertices in $S$ share the same neighborhood, any vertex in $V - S$ is at the same distance from all vertices in $S$. Each vertex of the independent set $S$ is at distance 2 from any other vertex in $S$. Thus all vertices in $S$ have the same transmission, say $\tau$.

To show that $\tau - 2$ is a distance Laplacian eigenvalue with multiplicity $p - 1$, it suffices to observe that the matrix $(\tau - 2)I_n - \mathcal{D}^{\mathcal{Q}}$ contains $p$ identical rows (columns). □

**Theorem 2.2** *Let $G$ be a connected graph on $n$ vertices. If $K = \{v_1, v_2, \dots v_p\}$ is a clique of $G$ such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \dots, p\}$, then $\tau = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\tau - 1$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}$ with multiplicity at least $p - 1$.*

The proof of this theorem is similar to that of the previous one and is omitted here.

Note that results similar to Theorem 2.1 and Theorem 2.2 are proved in [3] for the distance Laplacian spectrum.

# 3 Bounds on the eigenvalues

In this section, we prove some bounds on the eigenvalues of the distance signless Laplacian of a connected graph.

First, recall the following result proved in [4].

**Proposition 3.1** *If $G$ is a connected graph on $n \geq 3$ vertices, then $\partial_i^{\mathcal{Q}}(G) \geq \partial_i^{\mathcal{Q}}(K_n) = n-2$, for all $2 \leq i \leq n$. Moreover, $\partial_2^{\mathcal{Q}}(G) = \partial_2^{\mathcal{Q}}(K_n) = n - 2$ if and only if $G$ is the complete graph $K_n$.*

The next proposition gives a sharp upper bound on the index of $\mathcal{D}^{\mathcal{Q}}$ in terms of the Wiener index and the order of the graph.

**Proposition 3.2** *Let $G$ be a connected graph on $n \geq 2$ vertices with Wiener index $W$, then $\partial_1^{\mathcal{Q}}(G) \leq 2W - (n-1)(n-2)$ with equality if and only if $G$ is the complete graph $K_n$.*

**Proof.** From spectral theory, we have

$$\partial_1^{\mathcal{Q}}(G) + \partial_2^{\mathcal{Q}}(G) + \cdots + \partial_n^{\mathcal{Q}}(G) = Tr_1 + Tr_2 + \cdots + Tr_n = 2W.$$

Then

$$\partial_1^{\mathcal{Q}}(G) = 2W - \partial_2^{\mathcal{Q}}(G) - \cdots - \partial_n^{\mathcal{Q}}(G).$$

We conclude using Proposition 3.1. □

Note that the gap between $\partial_1^{\mathcal{Q}}(G)$ and $2W - (n-1)(n-2)$ may be arbitrarily large when the graph is not dense. To illustrate, the gap for an even cycle on $n$ vertices is exactly $n^2(n-2)/4$.

To prove the next theorem, we need the following well-known result from matrix theory.

**Lemma 3.3 (Gershgorin Theorem, [13])** *Let $M = (m_{ij})$ be a complex $n \times n$–matrix and denote by $\lambda_1, \lambda_2, \ldots \lambda_p$ its distinct eigenvalues. Then*

$$\{\lambda_1, \lambda_2, \ldots \lambda_p\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}| \right\}.$$

We now give sharp bounds on $\partial_1^{\mathcal{Q}}$ in terms of minimum, average and maximum transmissions.

**Theorem 3.4** *Let $G$ be a connected graph with minimum, average and maximum transmissions $Tr_{min}$, $\overline{Tr}$ and $Tr_{max}$ respectively. Then*

$$2Tr_{min} \leq 2\overline{Tr} \leq \partial_1^{\mathcal{Q}}(G) \leq 2Tr_{max}$$

*with equalities if and only if $G$ is a transmission regular graph.*

**Proof.** Using the Rayleigh's quotient, we have

$$\partial_1^{\mathcal{Q}}(G) = \max_{X \neq 0} R(X) = \max_{X \neq 0} \frac{X^t \mathcal{D}^{\mathcal{Q}} X}{X^t X}.$$

If we take $X = \mathbb{1}$, the all 1's vector, we get $R(\mathbb{1}) = 2\overline{Tr}$ and then $\partial_1^{\mathcal{Q}}(G) \geq 2\overline{Tr} \geq 2Tr_{min}$.

The upper bound follows immediately from Lemma 3.3.

It is easy to see that equalities hold if and only if $Tr_{min} = \overline{Tr} = Tr_{max}$ and $\mathbb{1}$ is an eigenvector belonging to the largest eigenvalue $\partial_1^{\mathcal{Q}}(G)$. □

Combining the above theorem and Proposition 3.1, we easily get the following corollary.

**Corollary 3.5** *If $G$ is a connected graph on $n \geq 2$ vertices, then $\partial_1^{\mathcal{Q}}(G) \geq \partial_1^{\mathcal{Q}}(K_n) = 2n - 2$ with equality if and only if $G$ is the complete graph $K_n$.*

**Proposition 3.6** *Let $G = (V, E)$ be a connected graph on $n \geq 2$ vertices and $k$ an integer such that $1 \leq k \leq n$. Denote by $\mathcal{P}_k(V)$ the family of subsets of $V$ with cardinality $k$. Then*

$$\partial_1(G) \geq \max_{S \in \mathcal{P}_k(V)} \left\{ \frac{1}{k} \sum_{u \in S} Tr(u) + \frac{1}{k} \sum_{u,v \in S} d(u,v) \right\} \quad \text{and} \quad \partial_n(G) \leq \min_{S \in \mathcal{P}_k(V)} \left\{ \frac{1}{k} \sum_{u \in S} Tr(u) + \frac{1}{k} \sum_{u,v \in S} d(u,v) \right\}.$$

**Proof.** Using Rayleigh's quotient, we have

$$\partial_1^{\mathcal{Q}}(G) = \max_{X \neq 0} R(X) = \max_{X \neq 0} \frac{X^t \mathcal{D}^{\mathcal{Q}} X}{X^t X} \quad \text{and} \quad \partial_n^{\mathcal{Q}}(G) = \min_{X \neq 0} R(X) = \min_{X \neq 0} \frac{X^t \mathcal{D}^{\mathcal{Q}} X}{X^t X}.$$

Thus, to be done, it suffices to take $X = [x_1, x_2, \ldots x_n]^t$ with $x_i = 1$ if $u_i \in S$ and 0 otherwise. □

We next establish some interconnections, as inequalities, between the distance signless Laplacian spectrum of a connected graph $G$ and the signless Laplacian spectrum of its complement $\overline{G}$. First, recall the following well-known result from matrix theory.

**Lemma 3.7 (Courant–Weyl inequalities, [6])** *For a real symmetric matrix $M$ of order $n$, let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ denote its eigenvalues. If $N_1$ and $N_2$ are two real symmetric matrices of order $n$ and if $N = N_1 + N_2$, then for every $i = 1, \ldots, n$, we have*

$$\lambda_i(N_1) + \lambda_1(N_2) \geq \lambda_i(N) \geq \lambda_i(N_1) + \lambda_n(N_2).$$

**Theorem 3.8** *Let $G$ be a connected graph on $n \geq 3$ vertices with diameter $D$. Let $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$ and $\overline{q}_1 \geq \overline{q}_2 \geq \cdots \geq \overline{q}_n$ be the distance signless Laplacian of $G$ and the signless Laplacian of the complement $\overline{G}$ of $G$.*

*(1) If $D = 2$, then*

$$n - 2 + \overline{q}_i \leq \partial_i^Q \leq 2n - 2 + \overline{q}_i \qquad \text{for every } 1 \leq i \leq n; \tag{1}$$

$$n - 2 + \overline{q}_n \leq \partial_i^Q \leq n - 2 + \overline{q}_1 \qquad \text{for every } 1 \leq i \leq n-1; \tag{2}$$

$$2n - 2 + \overline{q}_n \leq \partial_1^Q \leq 2n - 2 + \overline{q}_1. \tag{3}$$

*(2) If $D \geq 3$, then*

$$\partial_i^Q \geq n - 2 + \overline{q}_i \qquad \text{for every } 1 \leq i \leq n. \tag{4}$$

**Proof.**

(1) In a connected graph with diameter 2, we have $Tr(v) = d(v) + 2(n - d(v) - 1) = 2n - 2 - d(v)$, and therefore $Diag(Tr) = (2n-2)I - Diag(Deg)$. Moreover, the distance between two vertices is 1 if they are neighbors and 2 otherwise. Thus the distance matrix can be written as $\mathcal{D} = A + 2\overline{A}$, where $A$ and $\overline{A}$ denote the adjacency matrices of $G$ and its complement $\overline{G}$ respectively. Now, if we denote by $\overline{Q}$ and $\overline{Deg}$ the signless Laplacian matrix and the degree vector of $\overline{G}$, the distance signless Laplacian of $G$ can be written as

$$\begin{aligned} \mathcal{D}^Q &= \mathcal{D} + Diag(Tr) \\ &= A + 2\overline{A} + (2n-2)I - Diag(Deg) \\ &= A + \overline{A} + (n-1)I + \big((n-1)I - Diag(Deg) + \overline{A}\big) \\ &= J + (n-2)I + Diag\left(\overline{Deg}\right) + \overline{A} \\ &= J + (n-2)I + \overline{Q}, \end{aligned}$$

where $J$ is the all ones $n \times n$ matrix, whose eigenvalues are 0 with multiplicity $n-1$ and $n$ with multiplicity 1.
Applying Lemma 3.7 with $N_1 = (n-2)I + \overline{Q}$ and $N_2 = J$, we get (1), and with $N_1 = J$ and $N_2 = (n-2)I + \overline{Q}$, we get (2) and (3).

(2) Consider the $n \times n$ matrix $M = (m_{,j})$ defined by $m_{i,j} = \max\{0, d_{i,j} - 2\}$ for $1 \leq i, j \leq n$, where $\mathcal{D} = (d_{i,j})$ denotes the distance matrix of $G$. For a vertex $i$ in $G$, we write its transmission as $Tr_i = d_i + 2\overline{d}_i + Tr'_i$, where $\overline{d}_i$ denotes the degree of $i$ in $\overline{G}$. Using this notation, we have

$$\begin{aligned} \mathcal{D}^Q &= Diag(Tr) + \mathcal{D} \\ &= Diag(Deg) + Diag(\overline{Deg}) + Diag(Tr') + A + 2\overline{A} + M \\ &= \big(A + \overline{A} + Diag(Deg) + Diag\left(\overline{Deg}\right)\big) + \big(\overline{A} + Diag\left(\overline{Deg}\right)\big) + (Diag(Tr') + M) \\ &= Q(K_n) + \overline{Q} + M', \end{aligned}$$

where $M' = Diag(Tr') + M$. It is easy to see that $M'$ is diagonally dominant, and then, its least eigenvalue is nonnegative. Now, applying twice Lemma 3.7 (with $N_1 = Q(K_n)$ and $N_2 = \overline{Q} + M'$ and then with $N_1 = \overline{Q}$ and $N_2 = M'$), we get $\partial_i^Q \geq n - 2 + \overline{q}_i$, for $1 \leq i \leq n$.

$\square$

As a corollary of the above theorem, we establish a relationship between the fact that $n-2$ is a distance signless Laplacian eigenvalue of a connected graph $G$ and the existence of a bipartite component or an isolated vertex in the complement $\overline{G}$.

**Corollary 3.9** *Let $G$ be a connected graph on $n$ vertices. If $\partial^{\mathcal{Q}} = n-2$ is a distance signless Laplacian eigenvalue with multiplicity $\mu$, then the complement $\overline{G}$ of $G$ contains at least $\mu$ components, each of which is bipartite or an isolated vertex.*

**Proof.** From (1) of Theorem 3.8, if $n-2$ is a distance signless Laplacian eigenvalue, then 0 is a signless Laplacian eigenvalue at least as many times as $n-2$ for $\mathcal{D}^{\mathcal{Q}}$. To complete the proof, we use the fact (see [7, 11]) that 0 is a $Q$–eigenvalue of a graph $G$ if and only if $G$ contains a bipartite component or an isolated vertex, and in that case, the multiplicity of 0 is at most equal to the number of bipartite components plus the number of isolated vertices. □

Note that there exist graphs with bipartite complements with $\partial_n > n-2$. For instance, if $G$ is the complement of the path on 7 vertices, *i.e.* $G = \overline{P}_7$, we have $\partial_7^{\mathcal{Q}}(G) \simeq 5.042816 > 5$ while $\overline{G} = P_7$ is bipartite. Another example is illustrated on Figure 2.

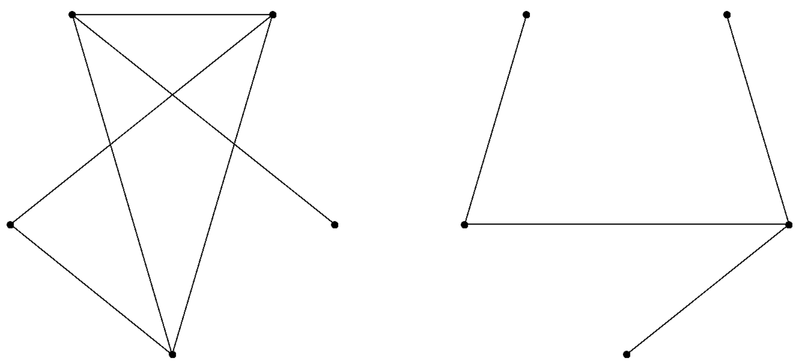


Figure 2: A graph $G$ (left) on 5 vertices with $\partial_5^{\mathcal{Q}} \simeq 3.050286 > 3$ and a bipartite complement (right).

**Corollary 3.10** *Let $G$ be a connected graph on $n$ vertices. Let $\mu$ be the multiplicity of $\partial^{\mathcal{Q}} = n-2$ as a distance signless Laplacian eigenvalue, then $\mu \leq n-1$ with equality if and only if $G \cong K_n$.*

**Corollary 3.11** *Let $G$ be a connected graph on $n$ vertices with diameter $D$. If $D \geq 4$, then $\partial_n^{\mathcal{Q}} > n-2$.*

**Proof.** Since $D \geq 4$, $\overline{G}$ is connected and contains at least a triangle (a cycle on 3 vertices). Thus $\overline{G}$ is not bipartite, and therefore $\overline{q}_i \geq \overline{q}_n > 0$. The results follows from (2) of Theorem 3.8. □

Another consequence of Theorem 3.8 is that, for a given order $n \geq 3$, the bipartite graphs with $\partial_n^{\mathcal{Q}} = n-2$ are entirely characterized.

**Corollary 3.12** *Let $G$ be a bipartite graph on $n \geq 3$ vertices, then $\partial_n^{\mathcal{Q}}(G) = n-2$ if and only if $G$ is the path $P_4$ or the complete bipartite graph $K_{n-2,2}$.*

**Proof.** If $G$ is the star $S_n$ with $n \geq 3$, then $\partial_n^{\mathcal{Q}}(G) > n-2$ except for $S_3 = K_{1,2}$.

If $n = 4$, the only bipartite graphs are $S_4$, $P_4$ and $K_{2,2}$, for which $\partial_n^{\mathcal{Q}}(S_n) > 2$ and $\partial_n^{\mathcal{Q}}(P_n) = \partial_n^{\mathcal{Q}}(K_{2,2}) = 2$. If $n \geq 5$ and $G \ncong S_n$, then the bipartition of the vertex set of $G$ defines two independent sets $V_1$ and $V_2$, each of which induces a clique in $\overline{G}$. By Theorem 3.8, $G$ contains at least a bipartite component. To be done, it suffices to note that $\overline{G}$ contains a bipartite component if and only if $G$ is a complete bipartite graph and $\min\{|V_1|, |V_2|\} = 2$. □

# 4 Some conjectures

In this section, we list a series of conjectures about some particular distance Laplacian eigenvalues of a connected graph. These conjectures, as well as some of the results proved in this paper, were obtained using the AutoGraphiX system [1, 2, 5] devoted to conjecture–making in graph theory.

First, we conjecture about an upper bound on the largest distance Laplacian eigenvalue over the class of all connected graphs with a given order $n$.

**Conjecture 4.1** *Let $G$ be connected graph on $n$ vertices. Then*

$$\partial_1^{\mathcal{Q}}(G) \leq \partial_1^{\mathcal{Q}}(P_n)$$

*with equality if and only if $G$ is the path $P_n$.*

Since a path is a tree, the above conjecture can be stated also for the set of trees. A general lower bound on $\partial_1^{\mathcal{Q}}$ is given in Corollary 3.5, if we assume that the graph is a tree, the bound is no more valid since $K_n$ is not a tree for $n \geq 3$. We next conjecture a lower bound on $\partial_1^{\mathcal{Q}}$ over the set of trees.

**Conjecture 4.2** *Let $T$ be a tree on $n$ vertices. Then*

$$\partial_1^{\mathcal{T}}(G) \geq \partial_1^{\mathcal{Q}}(S_n) = \frac{5n-8+\sqrt{9n^2-32n+32}}{2}$$

*with equality if and only if $T$ is the star $S_n$.*

For the class of unicyclic graphs, we conjecture a lower and an upper bound as well as a characterization of the extremal graphs for each bound.

**Conjecture 4.3** *Let $G$ be a connected unicyclic graph on $n \geq 6$ vertices. Then*

$$\partial_1^{\mathcal{Q}}(S_n^+) \leq \partial_1^{\mathcal{Q}}(G) \leq \partial_1^{\mathcal{Q}}(Ki_{n,3})$$

*with equality for the lower (resp. upper) bound if and only if $G$ is the graph $S_n^+$ (resp. the long kite $Ki_{n,3}$).*

Before stating the next conjecture, we need to define the *Soltés graph* [15]. Let $u$ be an isolated vertex or one endpoint of a path. Let us join $u$ with at least one vertex of a clique. The graph so obtained is the *Soltés graph* $PK_{n,m}$, also called the *path-complete* graph, where $n$ is its order and $m$ its size. There is exactly one $PK_{n,m}$ for given $n$ and $m$ such that $1 \leq n-1 \leq m \leq n(n-1)/2$. The kite $Ki_{n,\omega}$, defined in the introduction, is a particular path-complete graph with $m = \omega(\omega-1)/2 + n - \omega$.

For given $n$ and $m$ such that $1 \leq n-1 \leq m \leq n(n-1)/2$, $PK_{n,m}$ maximizes (non uniquely) the diameter $D$ [12] and (uniquely) the average distance $\overline{l}$ [15].

**Conjecture 4.4** *Let $n$ and $m$ be integers such that $2 \leq n-1 \leq m$. The path-complete (Soltés) graph $PK_{n,m}$ maximizes $\partial_1^{\mathcal{Q}}(G)$ over all connected graphs with order $n$ and size $m$.*

The next three conjectures are about the second largest distance signless Laplacian eigenvalue. First, we conjecture an upper bound on $\partial_2^{\mathcal{Q}}$, as well as a characterization of the corresponding extremal graphs, over all the connected graphs on $n$ vertices.

**Conjecture 4.5** *Let $G$ be connected graph on $n$ vertices. Then*

$$\partial_2^{\mathcal{Q}}(G) \leq \partial_2^{\mathcal{Q}}(P_n)$$

*with equality if and only if $G$ is the path $P_n$.*

We proved in Proposition 3.1 that, among the class of connected graphs on $n$ vertices, $\partial_2^{\mathcal{Q}}$ is minimum for the complete graph $K_n$. If we consider only the class of trees, the minimum of $\partial_2^{\mathcal{Q}}$ seems to be reached for the star $S_n$.

**Conjecture 4.6** *Let $T$ be a tree on $n \geq 4$ vertices. Then*

$$\partial_2^Q(T) \geq \partial_2^Q(S_n) = 2n - 5.$$

*with equality if and only if $T$ is the star $S_n$.*

For the class of unicyclic graphs, we conjecture a lower and an upper bound as well as a characterization of the extremal graphs for each bound.

**Conjecture 4.7** *Let $G$ be a connected unicyclic graph on $n \geq 5$ vertices. Then*

$$2n - 5 = \partial_2^Q(S_n^+) \leq \partial_1^Q(G) \leq \partial_1^Q(Ki_{n,3})$$

*with equality for the lower (resp. upper) bound if and only if $G$ is the graph $S_n^+$ (resp. the long kite $Ki_{n,3}$).*

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