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Classification of Idempotent Semi-Modules with Strongly Independent Basis

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Abstract: We show here that every m-dimensional semiring module M over an idempotent semiring S with strongly independent basis can be embedded into S^m, and provide an algebraic invariant – the Λ-matrix – which characterises the isomorphy class of M. The strong independence condition also yields a significant improvement to the Whitney embedding for tropical torsion modules published earlier [28]. We also show that the strong independence of the basis of M is equivalent to the unique representation of elements of M. Numerous examples illustrate our results, and a fast test for strong independence of the columns of a matrix is provided.

Résumé : On montre ici que tout semimodule M de dimension m sur un anneau idempotent S ayant une base fortement indépendante peut être plongé dans S^m et nous proposons un invariant algébrique – la matrice Λ – qui caractérise la classe d'isomorphie de M. La condition d'indépendance au sens fort fournit aussi une amélioration importante du Théorème de Whitney pour les modules tropicaux publiée dans [28]. On montre également que notre condition d'indépendance au sens fort est équivalente à la condition de représentation unique des éléments de M. De nombreux exemples illustrent nos résultats et nous proposons un test simple et rapide pour l'indépendance forte des colonnes d'une matrice.

1 Introduction

Idempotent mathematics arose from applications. Basically, we could say from the modelling and analysis of man-made systems – which include in particular computers, and production systems – and from mathematical physics. After the cerebrated paper by Kleene [14], many authors used idempotent mathematics: semigroups in language theory [21], semirings in network routing problems [8]. From the mathematical point of view, these idempotent structures have been widely investigated by Cuninghame-Green [10], and applications to control and optimization of production systems have been developed [1, 9], to mention only a few.

In mathematical physics, the dequantization point of view on idempotent mathematics was founded in the 1980's by V.P. Maslov and his school. This approach consists in an asymptotic view of traditional mathematics over the numerical fields making the Planck constant $\hbar$ tend to zero, taking imaginary values (cf [16]).

Once introduced, the topic has been found intrinsically interesting and arouse the interest of a large number of scientists (again without any pretention to completeness) in the automatic control [2, 3, 13] and mathematical communities [4, 5, 6, 7, 12].

As a result of the Maslov dequantization of real algebraic geometry, O. Viro [23] constructed a piecewise linear geometry of a special kind of polyhedra in finite dimensional Euclidean space. Subsequently, the tropical approach arouse an increased interest in the algebraic geometry community [11, 17, 20, 22]. A more complete list of references can be found in [15] and [16].

The classification of modules over a principal ideal domain is given by their decomposition into a direct sum of free and torsion modules. No such result exists for tropical modules. This is essentially due to the fact that he direct sum decomposition of tropical modules is trivial, on the one hand, and that this classification problem received scant attention in the other. In a previous approach, we shoved that although the direct sum decomposition misses the target, we can introduce the weaker concept of semi-direct sum [27] which is more closely related to the algebraic structure of tropical modules, which are to idempotent abelian monoïds (i.e. semilattices) what modules are to abelian groups. Also in [27], we show that every general tropical module may be decomposed into a semi-direct sum of four sub-semimodules: a free, Boolean, semi-Boolean, and torsion tropical module, respectively.

The aim of this paper is to prove a classification result for idempotent semiring modules. Our main result (Theorem 1) shows that this problem can be completely solved for such modules whose basis satisfy a strong independence condition. To make it short, the aim of the paper is to fill the gap in the table below.

ALGEBRAIC INVARIANTS

category	specify	char.
vector space	field F, n	F^n
module	PID	free $\oplus$ torsion.
idempotent semimodule	?	?

The paper is organised as follows. In Section 2 below, we recall the basic properties of tropical modules, and torsion. In Section 3, we state and prove the classification theorem for idempotent semiring modules. Section 4, is dedicated to the analysis of some examples. The completed table of algebraic invariants then concludes the paper.

2 Iidempotent semiring and semiring modules

For any set $S, (S, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is a semiring if $(S, \vee, \underline{\mathbf{0}})$ is a commutative monoid. Also, $\cdot$ distributes over $\vee$, and $\underline{\mathbf{0}}$ is the neutral element for $\vee$, which is also absorbing for $\cdot$, i.e. $\forall \sigma \in S,\ \underline{\mathbf{0}} \cdot \sigma = \sigma \cdot \underline{\mathbf{0}} = \underline{\mathbf{0}}$, and $\mathbb{1}$ is the neutral element for $\cdot$. $(S, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is an idempotent semiring or a dioïd if $\vee$ is idempotent, i.e. $\forall \sigma \in S,\ \sigma \vee \sigma = \sigma$. $(S, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is a semifield (resp. idempotent semifield) iff it is a semiring (resp.

idempotent semiring) s.t. $(S \setminus \{\underline{\mathbf{0}}\}, \cdot, \mathbb{1})$ is a group, i.e. $(S \setminus \{\underline{\mathbf{0}}\}, \cdot, \mathbb{1})$ is a monoid such that every element is invertible $(\forall \sigma \in S,\ \exists \sigma^{-1}\ :; \sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = \mathbb{1})$.
$(S, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is said to be an abelian (idempotent) semiring or semifield if $(S, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is a (idempotent) semiring or semifield such that $\cdot$ is commutative.

Note that S is endowed with an order relation defined by $\sigma \leq \mu \iff \sigma \vee \mu = \mu$. Since $\underline{\mathbf{0}}$ is the neutral element of $\vee$, it follows that $\underline{\mathbf{0}}$ is the bottom element of S, i.e $\forall \sigma \in S, \underline{\mathbf{0}} \leq \sigma$.

Dually, we define the semiring $(S, \wedge, \cdot, \overline{\mathbf{0}}, \mathbb{1})$, with top element $\overline{\mathbf{0}}$ as neutral for $\wedge$. We will also consider the extended (idempotent) semiring with bottom $\underline{\mathbf{0}}$ $(\underline{\mathbf{0}} \leq \sigma)$, and top $\overline{\mathbf{0}}$ $(\sigma \leq \overline{\mathbf{0}})$ for all $\sigma \in S$).

By abuse of language, the structure $(S, \vee, \wedge, \underline{\mathbf{0}}\ ,\ \overline{\mathbf{0}}\ ,\ \cdot \mathbb{1})$ will also be called a semiring (or semifield, or dioïd).

In the sequel we will assume that both $\vee$ and $\wedge$ are idempotent.

Also, the following convention will be used:

C1: $\underline{\mathbf{0}} \cdot \overline{\mathbf{0}} = \underline{\mathbf{0}}$,
C2: $\overline{\mathbf{0}} \cdot \underline{\mathbf{0}} = \overline{\mathbf{0}}$.
C3: $(\underline{\mathbf{0}})^- = \overline{\mathbf{0}}$, and $(\overline{\mathbf{0}})^- = \underline{\mathbf{0}}$.

2.1 Notation

In the literature on semirings and semiring modules, the notation $+$ or $\oplus$ is often used for either max or min composition laws. As we claim that idempotent semirings are at the intersection of linear algebra and ordered structures, there is as much justification for the use of the lattice and ordered structures notation (i.e. $\vee$ for max and $\wedge$ forr min) as for the use of the linear algebra notation (either $+$ or $\oplus$). Moreover, as we will see in the sequel, we will often need the use of both $\vee$ and $\wedge$, and to keep the $+$ or $\oplus$ notation would soon become a bit awkward.

Note also that, unless necessary, the notation $\cdot$ will usually be omitted.

Matrix multiplication: Let A, B be two matrices of appropriate sizes with entries $(A)_{ik}$ – written a_{ik} – (resp $(B)_{kj}$ –written b_{kj}–) in S.

Define $(A \cdot B)_{ij} = \bigvee\limits_k a_{ik} b_{kj}$, and $(A \star B)_{ij} = \bigwedge\limits_k a_{ik} b_{kj}$.

Also, we write A^t for the transpose of A , A^- for the matrix with entries a_{ij}^{-1}, and A^{-t} for $(A^t)^- = (A^-)^t$, where a^{-1} is the multiplicative inverse of $a \in S \setminus \{\underline{\mathbf{0}}, \overline{\mathbf{0}}\}$, with the convention **C3**.

2.2 Semimodules over an idempotent semiring

Left (right) $\vee$-semimodule over a semiring is defined similarly as module over a ring:

1. $(M, \vee)$ is a monoid with neutral $\underline{\mathbf{0}}$
2. There is a map $S \times M\ \rightarrow M$ called exterior multiplication, satisfying: $(\sigma\ ,\ x) \mapsto \sigma x$.
   i) $(\sigma \vee \mu,\ x) = (\sigma\, x \vee \mu\, x)$,
   ii) $(\sigma\ ,\ x \vee y) = (\sigma\, x \vee \sigma\, y)$
   iii) $(\underline{\mathbf{0}},\ x) = (\sigma\ ,\ \underline{\mathbf{0}}) = \underline{\mathbf{0}}$.

If the semiring (semifield) is idempotent, then so is the semimodule, since $x \vee x = \mathbb{1}x \vee \mathbb{1}x = (\mathbb{1} \vee \mathbb{1})\ x = \mathbb{1}\ x = x$ (and similarly for $\wedge$).

The first composition laws $\vee$ and $\wedge$ in S extend to vector and matrices in a natural way. Also exterior multiplication by a scalar $\lambda \in S$ is defined componentwise (resp. entrywise) for vectors (matrices). This makes S^n and the set of matrices with entries in S, left (or right) $\vee$-semimodules over S.

Notwithstanding the fact that we consider here $\vee$-semimodules, ($\wedge$-semimodules can be defined similarly), we will however use the $\wedge$ composition whenever required by the developments of the theory.

We will further assume that S is a totally ordered, and conditionally complete semifield.

2.3 Independence

Let M be a S semimodule, and $X = (x_i)_{i\in I} \subset M$. We say that $M_X = \{\bigvee_{i\in I} \lambda_i x_i | x_i \in X\,,\ \lambda_i \in S\,,\ \lambda_i = \underline{\mathbf{0}}$ except for a finite number of them$\}$ is the semimodule generated by X, and that X is the set of generators of M.

In [24], we considered the following concepts of independence for $X \subset S^n$.

1. $\forall Y, Z \subset X\ M_X \bigcap M_Y = M_{X\cap Y}$
2. $\forall Y, Z \subset X\,,\ Y \bigcap Z = \emptyset \Rightarrow M_Y \bigcap M_Z = \{\underline{\mathbf{0}}\}$
3. $\forall x \in X\,,\ x \notin M_{X\setminus\{x\}}$.

Note that $1 \Rightarrow 2 \Rightarrow 3$, while the converse does not hold, although they are equivalent in vector spaces.

In [24] (see also [18]), the proof that every finitely generated semimodule has generating set satisfying 3, and that this set is unique up to a homothetic transformation $x_i \mapsto \lambda_i x_i\,,\ x_i \in X\,,\ \lambda_i \in S$ is given.

Let $A \in \mathrm{Hom}(\mathrm{S^m}, \mathrm{S^n})$, i.e. A is a rectangular matrix of size $n \times m$ with entries in S. Clearly, the columns of A generate a finite dimensional semimodule over S. We write M_A for this subsemimodule of S^n. Also, if the columns of A are independent in the sense of 3 above, then $\dim M_A = m$. From the existence and uniqueness theorem mentioned above, follows that for any diagonal and permutation matrices of appropriate sizes $D_1, D_2, P_1, P_2\,,\ A$ and $B = D_1 P_1 A P_2 D_2$ generate isomorphic semimodules. We write in this case $A \sim B$.

The problem we address in this paper is twofold. First, is there an algebraic invariant which characterises the isomorphy class of M_A? Second, what is the minimal p such that M_A is isomorphic to a subsemimodukle of S^p? In [28], we addressed this problem for semimodules over $S = \mathbb{R}_{\max}$ with finite entries (i.e. $\neq \underline{\mathbf{0}}$), where $\mathbb{R}_{\max}$ is the tropical semifield of reals endowed with "$\vee$" ($=\max$) and "$\cdot$" ($=+$) operations [29].

2.4 The standard/canonical Λ-matrix of a semimodule

Inspired by the torsion in modules (or abelian groups), in [25], we defined the (slightly different) concept of **torsion** in semiring modules as follows.

Consider the congruence relation in M defined by $x \sim y$ iff $x \prec y$ and $y \prec x$ where $x \prec y$ iff $\exists\, \xi \in S$ s.t. $x \leq \xi y$. For any basis X, we consider the semilattice generated by $X : X^* = \{\bigvee_{i\in I} \xi_i x_i | x_i \in X\ \xi_i \in \{\underline{\mathbf{0}}\, \mathbb{1}\}$, $\xi_i = \underline{\mathbf{0}}$ except for a finite number of them. Since $X^+ \subset M$, the congruence $\sim$ is well-defined on X^+, and the map $\pi\colon X^+ \to X^+|_\sim$ is an epimorphism of semilattices (Lemma 4.1 of [25]).

Semi-Boolean semimodules have been defined in [27] by the condition that $\pi\colon X^+ \to X^+|_\sim$ is an isomorphism. We may say that the generators of a semi-Boolean semimodule have at least one entry $\underline{\mathbf{0}}$, as opposed to a pure **torsion** semimodule whose generators have no $\underline{\mathbf{0}}$ entry. A general semimodule is semi-direct sum of Semi-Boolean and torsion semimodules. Roughly speaking, the distinction between direct and semi-direct sum of (say two) components is that in the latter we will have some order relation between the element of the first and the second components. It is beyond the scope of this paper to recall how we can further distinguish the free and Boolean parts of an idempotent semi-Boolean semimodule. The interested reader may find such details in [27].

Let X be the basis of M, given by the (independent) columns of a matrix A of size $n \times m$. In [28] we showed how to construct the Λ matrix of a torsion matrix written in canonical form. This "canonical" form has been later (cf [29]) renamed "standard". One significant outcome of our classification theorem below is to bring some clarification in this terminology (cf 3.1).

The method used in [28] to define the canonical/standard form of a tropical matrix A can be generalised to the case considered in this paper, provided we specify that the generators of the semi-Boolean part of M, stand first, followed by the generators of the torsion part. For the generators of the semi-Boolean part, we first write the $a_{ij} \neq \underline{\mathbf{0}}$ on the first rows. Then, by left and right multiplication of A by appropriate diagonal and permutation matrices, we get a standard matrix $B \sim A$ such that $\Lambda_B = B^t \cdot B^-$, with the condition that

$$\lambda_{i,i+1} = \begin{cases} \mathbb{1}, & \exists\, \xi \text{ s.t. } b_{\cdot,i} \leq \xi b_{\cdot,i+1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

3 The classification theorem

In this section, we prove and state the main result of this paper. Let $A \in \mathrm{Hom}(\mathrm{S^m}, \mathrm{S^n})$, and M_A the S-semimodule generated by the columns of A. It is well-known in residuation theory, that the inequation $A \cdot X \leq B$ has a maximal solution $A \backslash B$, called the right residuate of A by B, and we have $A \backslash B = A^{-t} \star B$ (cf [3], eq 4.82, [19], or [30]). In particular, for $B = A$, the matrix $A \backslash A$ has been defined in [29] as the Γ-matrix of A, written Γ^A.

It is not difficult to see that, although in [3] the entries in A lie in the semifield $\mathbb{R}_{max}$, the statement still holds for more general idempotent semimodules considered in this paper, and with the conventions **C1**, **C2**, **C3** above.

For any matrix $A \in \mathrm{Hom}(S^m, S^n)$, we write $x \overset{A}{\sim} y \iff Ax = Ay$, and, for every $x \in S^m$: $\overline{x} = \bigvee \{y \in S^m |\ y \overset{A}{\sim} x\}$. In [29], we defined $\mathrm{DOMINJ}_A = \{\overline{x} | x \in S^m\}$. Since $\overset{A}{\sim}$ is a congruence, DOMINJ_A is a semimodule, which is isomorphic to $S^m|_\sim$, thus, $\mathrm{DOMINJ}_A \simeq M_A$.

In [28], we defined, for a square tropical torsion matrix A of size n : $\mathrm{INJ}_A = \{\xi \in \mathbb{R}^n | \exists \sigma \in \mathcal{S}_n$ such that $\forall\, k,\ \bigvee_{j=1, j\neq k}^{n} a_{\sigma(k)j}\xi_j \leq a_{\sigma(k)k}\xi_k\}$. We proved that, when such a permutation exists, then it is unique, A is injective on INJ_A, and the columns of $\tilde{A}^*$ generate INJ_A, where $\tilde{A} \sim A$ is such that the permutation in $\mathrm{INJ}_{\tilde{A}}$ is the identity permutation, and $A^* = I \vee A \vee A^2 \vee \ldots$ is the Kleene star of A (cf [14]). Hence $\tilde{A}x = x$. It is well-known in this case that the columns of $\tilde{A}^*$ generate $\mathrm{INJ}_{\tilde{A}}$.

In order to see that $\mathrm{INJ}_A = \mathrm{INJ}_{\tilde{A}}$, note that $\tilde{A} = DPA$ for some diagonal and permutation matrices D and P. Hence $Ax = Ay \ \Leftrightarrow\ DPAx = DPAy \ \Leftrightarrow\ \tilde{A}x = \tilde{A}y$. It is not difficult to see that these results also hold if $\mathbb{R}$ is replaced by S.

ERRATUM (cf [28])

In Corollary to Theorem 1 of [28], we stated that $\mathrm{INJ}_A \simeq M_A$, which fails to be true. In the following counterexample, we show that the torsion coefficients of A and INJ_A are not equal, which is an obstruction to the existence of an isomorphism $\mathrm{INJ}_A \simeq M_A$. This raises the question of the conditions for the isomorphism $INJ_A \simeq \mathrm{DOMINJ}_A$.

Example 3.1 $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 12 \\ \mathbb{1} & 4 & 4 \\ \mathbb{1} & 6 & 11 \end{bmatrix}$

Writing c_j for column j of A, we have $12c_1 \vee 9c_2 = (12\ 13\ 15)^t = 9c_2 \vee c_3 \in M_{c_1,c_2} \bigcap M_{c_2,c_3}$. But $(12\ 13\ 15)^t \notin M_{c_2}$. Hence $M_{c_1,c_2} \bigcap M_{c_2,c_3} \neq M_{c_2}$, and $\{c_1, c_2, c_3\}$ is not strongly independent.

The torsion coefficients of A : $\tau_{12}, \tau_{13}, \tau_{23}$ are equal to $6, 8, 12$ respectively (the λ_{ij}'s are given by the matrix $\Lambda_A = A^t \cdot A^-$ [28]). Let $P = \begin{bmatrix} \underline{\mathbf{0}} & \mathbb{1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \mathbb{1} \\ \mathbb{1} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix}$, and $D = diag(\mathbb{1}\ 6^{-1}\ 12^{-1})$, then $A \sim DPA = \begin{bmatrix} \mathbb{1} & 4 & 4 \\ 6^{-1} & \mathbb{1} & 5 \\ 12^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix} = \tilde{A}$. It is easy to see that $\tilde{A}^* = \tilde{A}^2 = \begin{bmatrix} \mathbb{1} & 4 & 9 \\ 6^{-1} & \mathbb{1} & 5 \\ 12^{-1} & 8^{-1} & \mathbb{1} \end{bmatrix}$. Hence INJ_A is generated by $\begin{bmatrix} \mathbb{1} \\ 6^{-1} \\ 12^{-1} \end{bmatrix}, \begin{bmatrix} 4 \\ \mathbb{1} \\ 8^{-1} \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ \mathbb{1} \end{bmatrix}$. Now, $\Lambda_{\mathrm{INJ}_A} = \begin{bmatrix} \mathbb{1} & 4^{-1} & 9^{-1} \\ 6 & \mathbb{1} & 5^{-1} \\ 12 & 8 & \mathbb{1} \end{bmatrix}$, and the τ_{ij} of INJ_A are given by $2, 3, 3$ respectively. It follows that $\mathrm{INJ}_A \not\simeq M_A$ (hence also $\mathrm{INJ}_A \not\simeq \mathrm{DOMINJ}_A$).

It is not difficult to see that the following statements (Propositions 5.1–5.4, as well as Theorem 1) in [29], stated for tropical torsion matrices extend to the case considered in this paper. Matrices are always assumed to have independent columns (in the sense of 3 above).

Proposition 3.1 *For an arbitrary matrix A, we have* $\Gamma^A = \Lambda_A^-$

Proposition 3.2 *For an arbitrary matrix A, we have* $\Lambda_{\Gamma^A} = \Lambda_A$.

Proposition 3.3 *For any square matrix A,* $I \vee A^2 = A \iff A^* = A$.

Proposition 3.4 *For an arbitrary matrix A, we have* $\Gamma^{\Gamma^A} = \Gamma^A$.

Proposition 3.5 *(Theorem 1 of [29])* $\mathrm{INJ}_{\Gamma^A} = M_{\Gamma^A}$

We have the following statement.

Theorem 1 *A is injective on M_{Γ^A} $\iff$ its columns are strongly independent.*

Proof. The sufficient condition. Assume the columns of A are strongly independent. We show that A is injective on M_{Γ^A}. Let $x = \bigvee_{\gamma_{\cdot,j}\in X} \xi_j \gamma_{\cdot,j}\,,\ y = \bigvee_{\gamma_{\cdot,k}\in Y} \lambda_k \gamma_{\cdot,k} \in M_{\Gamma^A}$ be such that $Ax = Ay$.

Then $Ax = \bigvee_{a_{\cdot,j}\in X} \xi_j a_{\cdot,j}\,,\ Ay = \bigvee_{a_{\cdot,k}\in Y} \lambda_k a_{\cdot,k}$. Since the columns of A are strongly independent, we must have $Y = X$, and we may write $Ax = \bigvee_{j\in J} \xi_j a_{\cdot,j} = \bigvee_{j\in J} \lambda_j a_{\cdot,j}$.

Assume by contradiciton that $\exists\ \ell$ s.t. $\xi_\ell < \lambda_\ell$. Clearly $\exists i$ s.t. $(Ax)_i = \xi_\ell a_{i\ell}$, for if not, then $Ax = \bigvee_{j\in J, j\neq \ell} \xi_j a_{\cdot,j}$, which would contradict the strong independence assumption (i.e. we would have $X \neq Y$). But $(Ay)_i = \lambda_\ell a_{i\ell} \vee \bigvee_{j\in J,j\neq\ell} \lambda_j a_{ij} = \xi_\ell a_{i\ell} \Rightarrow \lambda_\ell \leq \xi_\ell$, which contradicts our assumption. Hence we must have $\forall j \in J\,,\ \lambda_j \leq \xi_j$. Similarly, the assumption $\exists\, \ell \in J$ s.t. $\lambda_\ell < \xi_\ell$ yields $\forall j \in J\,, \xi_j \leq \lambda_j$. It follows that $\xi_j = \lambda_j\, \forall j \in J$, hence $x = y$.

The necessary condition. If the columns of A are not strongly independent, then, as shown by Example 3.1 above, A fails to be injective on M_{Γ^A}. □

Remark 3.1 *For any X s.t. $AX = A$, the map $A\colon M_X \to M_A$ is surjective: $\forall u \in M_A\,,\ u = \bigvee_{j=1}^{m} \xi_j a_{\cdot,j} = A\xi$. Let $y = X\xi$, then $Ay = AX\xi = A\xi = u$.*

Theorem 2 *Let $A \in \mathrm{Hom}(S^m, S^n)$. The following are equivalent:*

i) *The columns of A are strongly independent.*
ii) $\mathrm{INJ}_A = M_{\Gamma^A}$.
iii) $\mathrm{INJ}_A \simeq \mathrm{DOMINJ}_A$.
iv) *The representation of any $x \in M_A$ is unique.*
v) *M_A is characterised by the $(m-1)^2$ entries $\lambda_{ij}\,,\ j \neq i, i+1$ of Λ_A.*
vi) $\Lambda_A < \Lambda_A^2$.

Proof. The equivalence $i) \iff ii) \iff iii)$ is straightforward, by Theorem 1 and Remark 3.1.

The implication $i) \Rightarrow iv)$ has been proved in the proof of Theorem 1. Assume then that iv) holds. By Remark 3.1 A is surjective on M_{Γ^A}. It remains to show that A is injective on M_{Γ^A}. As above, let $x = \bigvee_{\gamma_{\cdot,j}\in X} \xi_j \gamma_{\cdot,j}\,,\ y = \bigvee_{\gamma_{\cdot,k}\in Y} \lambda_k \gamma_{\cdot,k} \in M_{\Gamma^A}$. Then by iv), for $z = Ax = \bigvee_{a_{\cdot,j}\in X} \xi_j a_{\cdot,j} = \bigvee_{a_{\cdot,j}\in X} \lambda_j a_{\cdot,j} = Ay$, we must have $X = Y$, and $\xi_j = \lambda_j\, \forall j\, :\, z = \bigvee_{a_{\cdot,j}\in X} \xi_j a_{\cdot,j}$. Thus $y = x$, and $M_A \simeq M_{\Gamma^A}$.

The proofs of v) and vi) are easy (although somewhat tricky) and are left to the reader. □

Remark 3.2 *Note that vi) provides a* **fast test for strong independence** *of the columns of a matrix.*

Corollary 3.1 *The "canonical" form of A defined in [27] is indeed* **canonical** *iff the columns of A are strongly independent.*

Proof. This statement is a straightforward consequence of iv) in the theorem. However, as there are misprints in Example 4.2 of [29], we provide below the correct entries of the matrix. □

Example 3.2 *We have* $A = \begin{bmatrix} 2 & \mathbb{1} & 4 & \mathbb{1} \\ 1 & 1 & 2 & \mathbb{1} \\ 3 & 2 & 3 & \mathbb{1} \\ 4 & 3 & 5 & \mathbb{1} \end{bmatrix} \sim \begin{bmatrix} \mathbb{1} & \mathbb{1} & 3 & 4 \\ \mathbb{1} & 1 & 1 & 3 \\ \mathbb{1} & 1 & 1 & 2 \\ \mathbb{1} & 2 & 2 & 5 \end{bmatrix}$, *with* $\Lambda_A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 1^{-1} & 2^{-1} \\ 2 & \mathbb{1} & \mathbb{1} & 1^{-1} \\ 3 & 3 & \mathbb{1} & \mathbb{1} \\ 5 & 4 & 3 & \mathbb{1} \end{bmatrix}$, *which yields* $\left\{ \begin{smallmatrix} \tau_{12}=\tau_{13}=2\,,\ \tau_{14}= \\ \tau_{24}=\tau_{23}=\tau_{34}=3 \end{smallmatrix} \right.$.

Permutation of columns 2 and 3 of A yields the equivalent matrix:
$B = \begin{bmatrix} 2 & 4 & \mathbb{1} & \mathbb{1} \\ 1 & 2 & 1 & \mathbb{1} \\ 3 & 3 & 2 & \mathbb{1} \\ 4 & 5 & 3 & \mathbb{1} \end{bmatrix} \sim \begin{bmatrix} \mathbb{1} & \mathbb{1} & 3 & 4 \\ \mathbb{1} & 1 & 4 & 6 \\ \mathbb{1} & 1 & 3 & 3 \\ \mathbb{1} & 2 & 2 & 5 \end{bmatrix}$, *with* $\Lambda_B = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 2^{-1} & 3^{-1} \\ 2 & \mathbb{1} & \mathbb{1} & 2^{-1} \\ 4 & 3 & \mathbb{1} & \mathbb{1} \\ 6 & 5 & 3 & \mathbb{1} \end{bmatrix}$ *and the same τ_{ij}'s, although the $\lambda_{ij}(A)$ may differ from the $\lambda_{ij}(B)$*

This shows that the standard form of A is not unique. The reader may find it interesting to show that the columns of A are not strongly independent.

Corollary 3.2 *Strong independence of the columns of a matrix allows for a new equivalence between matrices, namely $A \sim \Gamma^A$, which relates a (possibly) rectangular matrix to a square matrix.*

Corollary 3.3 *If the columns of $A \in \mathrm{Hom}(S^m, S^n)$ are strongly independent, then:*

vii) M_A can be embedded in S^m.

Proof. The proof is straightforward, since $M_{\Gamma^A} \in S^m$. □

4 Examples

Example 4.1 *(4.3 of [29]) Let* $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix}$, *with $5 < a < 8$. It is not difficult to see that the columns of A are strongly independent, and* $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4 \\ 9^{-1} & \mathbb{1} & \mathbb{1} \\ 15^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}$.

The τ_{ij} of A and Γ^A are the same and equal to $(9, 11, 12)$ respectively. As stated in Theorem 1, M_A can be embedded in S^3, independently of the value of $a \in]5, 8[$.

Note also that, writing A_6 (resp. A_7) for the matrix with $a = 6$ (resp. 7), and M_6 (resp M_7) for the semimodule generated by the columns of A_6 (A_7), we have $M_6 \simeq M_7$. However, there is no isomorphism $S^6 \mapsto S^6$ whose restriction to M_6 yields M_7.

Example 4.2 *Let* $A = \begin{bmatrix} \mathbb{1} & 9 & 5 \\ \mathbb{1} & \mathbb{1} & 2 \\ \mathbb{1} & \underline{0} & \mathbb{1} \\ \underline{0} & 4 & \underline{0} \end{bmatrix}$. *We have* $\Gamma^A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \overline{0} \\ 9^{-1} & \mathbb{1} & \overline{0} & 4^{-1} \\ 5^{-1} & 2^{-1} & \mathbb{1} & \overline{0} \end{bmatrix} \star \begin{bmatrix} \mathbb{1} & 9 & 5 \\ \mathbb{1} & \mathbb{1} & 2 \\ \mathbb{1} & \underline{0} & \mathbb{1} \\ \underline{0} & 4 & \underline{0} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \underline{0} & \mathbb{1} \\ \underline{0} & \mathbb{1} & \underline{0} \\ 5^{-1} & \underline{0} & \mathbb{1} \end{bmatrix}$.

The torsion $\tau_{13} = 5$ in both M_A, and M_{Γ^A}. However, here we have $a_{\cdot,1} \vee a_{\cdot,2} = \begin{bmatrix} 9 \\ \mathbb{1} \\ \mathbb{1} \\ 4 \end{bmatrix} \leq a_{\cdot,2} \vee a_{\cdot,3} = \begin{bmatrix} 9 \\ 2 \\ \mathbb{1} \\ 4 \end{bmatrix} \leq 2 \begin{bmatrix} 9 \\ \mathbb{1} \\ \mathbb{1} \\ 4 \end{bmatrix}$. *More precisely the torsion coefficient $\tau_{c_1 \vee c_2, c_2 \vee c_3} = 2$ in M_A, while the corresponding coefficient in*

M_{Γ^A} is equal to 5. It follows that M_A, and M_{Γ^A} cannot be isomorphic. Therefore, by Theorem 1 the columns of A cannot be strongly independent.

Indeed, by inspection,we get $M_{c_1,c_2} \bigcap M_{c_2,c_3} \ni \begin{bmatrix} 13 \\ 4 \\ 1 \\ 8 \end{bmatrix} \notin M_{c_2}$. The interested reader may be interested to use the test of vi) in Theorem 1 mentioned in Remark 3.2.

Our next example shows that checking the strong independence of the columns of A directly on the matrix, should not be overlooked.

Example 4.3 *Let $A = \begin{bmatrix} \mathbb{1} & \underline{0} & \mathbb{1} \\ \mathbb{1} & \underline{0} & \underline{0} \\ \underline{0} & \mathbb{1} & \mathbb{1} \\ \underline{0} & \mathbb{1} & \underline{0} \end{bmatrix}$. Taking the columns of A two by two seems to show they are strongly independent, since neither $\xi_1 c_1 \vee \xi_2 c_2 = \lambda_1 c_2 \vee \lambda_3 c_3$, nor $\xi_1 c_1 \vee \xi_3 c_3 = \lambda_1 c_2 \vee \lambda_3 c_3$ has a non-trivial solution. However, $c_3 < c_1 \vee c_2$, hence $c_1 \vee c_2 = c_1 \vee c_2 \vee c_3$, and the columns of A are not strongly independent. We have $\Gamma^A = \begin{bmatrix} \mathbb{1} & \underline{0} & \underline{0} \\ \underline{0} & \mathbb{1} & \underline{0} \\ \underline{0} & \underline{0} & \mathbb{1} \end{bmatrix} = I_3$, and $A \cdot \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ \underline{0} \end{bmatrix} = A \cdot \begin{bmatrix} c\mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} c\mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \end{bmatrix}$, thus A is not injective on M_{Γ^A}.*

This example also shows that the strong independence of the columns of Γ^A does not imply that of the columns of A.

In our next example, we revisit Example 4.1 in which we set $a = \underline{\mathbf{0}}$ and reorder rows and columns for convenience.

Example 4.4 *Let $A = \begin{bmatrix} ccc\mathbb{1} & \mathbb{1} & 4 \\ \mathbb{1} & 3 & 13 \\ \mathbb{1} & 1 & 12 \\ \mathbb{1} & 5 & 5 \\ \mathbb{1} & 10 & 11 \\ \underline{0} & \underline{0} & \mathbb{1} \end{bmatrix}$, $\Gamma^A = \begin{bmatrix} cccccc\mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \overline{\mathbf{0}} \\ \mathbb{1} & 3^{-1} & 1^{-1} & 5^{-1} & 10^{-1} & \overline{\mathbf{0}} \\ 4^{-1} & 13^{-1} & 12^{-1} & 5^{-1} & 11^{-1} & \mathbb{1} \end{bmatrix} \star \begin{bmatrix} ccc\mathbb{1} & \mathbb{1} & 4 \\ \mathbb{1} & 3 & 13 \\ \mathbb{1} & 1 & 12 \\ \mathbb{1} & 5 & 5 \\ \mathbb{1} & 10 & 11 \\ \underline{0} & \underline{0} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} ccc\mathbb{1} & \mathbb{1} & 4 \\ 10^{-1} & \mathbb{1} & \mathbb{1} \\ \underline{0} & \underline{0} & \mathbb{1} \end{bmatrix}$.*

Here M_A is isomorphic to the semi-direct sum $M_B \tilde{\oplus} M_C$ introduced in [27], where M_B is generated by the columns of $B = \begin{bmatrix} cc\mathbb{1} & \mathbb{1} \\ \mathbb{1} & 3 \\ \mathbb{1} & 1 \\ \mathbb{1} & 5 \\ \mathbb{1} & 10 \end{bmatrix}$ and M_C by the column vector $C = (4\ 13\ 12\ 5\ 11\ \mathbb{1})^t$. Since column $\lambda_{12} = 10 (= \tau_{12})$, and $\lambda_{13} = 4$, $\lambda_{23} = \mathbb{1}$, the sum $M_B \tilde{\oplus} M_C$ cannot be a direct sum.

By Theorem 1 $M_B \simeq M_{\Gamma^B}$, with $\Gamma^B = \begin{bmatrix} cc\mathbb{1} & \mathbb{1} \\ 10^{-1} & \mathbb{1} \end{bmatrix}$. By Theorem 1 $M_A \simeq M_{\Gamma^A}$, which is isomorphic to the semi-direct sum $M_B \tilde{\oplus} M_D$, with M_D generated by the column vector $D = (4\ 1\ \mathbb{1})^t$.

5 Conclusion

We conclude below by exhibiting the short table of (some) algebraic invariants mentioned in the introduction.

ALGEBRAIC INVARIANTS

category	specify	char.
vector space	field F, n	F^n
module	PID	free $\oplus$ torsion.
idempotent semimodule	strongly indep. basis	Λ-matrix

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