

**An Efficient Numerical Method for Pricing
Long-maturity American Put Options**

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Abstract: We propose a new numerical method for evaluating long-maturity American put options. Most existing numerical approaches are based on the time discretization of the exercise strategy, and their convergence to continuous exercise opportunities is very slow. Instead of assuming a finite number of exercise dates, we allow the option holder to exercise continuously, using an optimal barrier approach combined with a cubic spline interpolation technique. Our method is shown to converge to the American option price much more quickly than the Bermudian approximation, making it especially appropriate for long-maturity options.

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1 Introduction

An American option gives its holder the right to exercise a contract (e.g. buy or sell the underlying asset) at a date prior to maturity. This early exercise right complicates the evaluation of such options, even in the most simple “vanilla” case (the so-called *American put*). The option holder’s exercise strategy is usually characterized by an *exercise frontier* such that the option is exercised if the underlying asset price falls below the frontier. In the absence of an analytical solution, the literature offers a multitude of numerical approaches and approximation formulas for determining the exercise frontier, and evaluating the price of the American put (see Barone-Adesi 2005 for a survey).

The first numerical approaches proposed in the literature include Brennan and Schwartz (1977), who use a finite-difference approximation of the solution of the partial differential equation characterizing the option value, and Cox, Ross and Rubinstein (1979), who introduce the binomial tree to represent the evolution of its value. Despite their relative inefficiency in terms of computation time, these two approaches are still the most widely used in practice because of their simplicity and flexibility.

Johnson (1983) and Broadie and Detemple (1996) exploit the fact that the value of the American put can be circumscribed in an interval bounded by the price of two European puts, making it obtainable through a weighted average of these two bounds. To improve the method’s accuracy, the authors propose an optimization procedure for the weights.

Geske and Johnson (1984) assume that exercise is possible at a finite number of dates, thus equating the American put to a compound option. The formula for evaluating the compound option involves the solution of an implicit equation, and a multivariate normal distribution whose complexity increases rapidly with the number of exercise dates. Bunch and Johnson (1992) propose extrapolation and optimization approaches to select exercise dates in order to improve the accuracy of the compound option method.

Omberg (1987) and Chesney and Lefoll (1996) approach the subject from the point of view of the exercise frontier. By restricting the exercise strategy to an exponential function of time, they obtain a closed-form expression for the value of the option, using the joint distribution of the underlying asset price and the first passage time through the exercise frontier. The approximate price for the American put is obtained by optimizing the expression of the exercise frontier.

Barone-Adesi and Whaley (1987) propose a closed-form analytical approximation using the partial differential equation characterizing the option. The simple expression they obtain for the price of the American put is useful for very short and very long maturities, but its accuracy deteriorates for average maturities.

Longstaff and Schwartz (2001) propose a recursive method for the evaluation of the American put at discrete times, where the value of holding the option is evaluated by Monte Carlo simulation on a finite grid and then interpolated by ordinary least squares. This method has the advantage of being easily applicable when the value of the option depends on many underlying assets or factors. In the same family of approaches, Ben Ameur et al. (2009) propose a dynamic program for the valuation of Bermudian options under a GARCH specification, using a piecewise polynomial interpolation of the option value.

These last two methods, based on a discrete-time dynamic programming (DP) model, liken the American put option to an equivalent Bermudian put with a large number of exercise opportunities. This equivalence is also used in the compound option, binomial tree and finite difference approaches. This assumption is justified by the view that the option holder does not observe the underlying asset price in continuous time. However, an increasing volume of transactions are now performed by high-frequency transaction algorithms. A study by the Aite Group, reported in *The Economist* (February 2012), shows that the percentage of algorithmic trading was 65% of equity trading and 30% of option trading in 2012, with a high forecasted positive trend for option trading. Algorithmic trading in financial markets is closer to the assumption of continuous-time than that of discrete-time exercise strategies. The aim of this paper is to propose a dynamic programming pricing algorithm that can efficiently price American options under the assumption that the price of the underlying asset is observed in continuous time.

The rest of the paper is organized as follows. Section 2 analyzes the convergence of Bermudian option prices under a general DP model as the time discretization becomes finer. We note that the convergence

of Bermudian option prices to their American counterparts is slow when the number of exercise opportunities becomes infinitely large, making the Bermudian approximation unsatisfactory for long-maturity options. Section 3 introduces a technique based on the successive optimization of a portfolio of barrier options, which yields better convergence in terms of time steps, although the computational burden of the successive optimization steps is significant. The third approach, presented in Section 4, significantly reduces the computation time while approaching the precision of the portfolio of barrier options. Our numerical experiments show the efficiency of this method, especially for long-maturity options. A brief conclusion is presented in Section 5.

2 The Bermudian approximation

Most existing numerical approaches for the evaluation of American options are based on an approximation using Bermudian options with a high exercise frequency. Indeed, the Bermudian option price converges to that of an American option with the same characteristics when the time interval between two exercise dates tends to 0. This section presents numerical experiments showing that the convergence of the Bermudian approximation is relatively slow, making this family of approaches computationally costly for long-maturity options.

2.1 Evaluation of a Bermudian put option

We first present a general algorithm for the computation of the price of a Bermudian put option on an underlying asset having a price process denoted by S_t and assumed to satisfy the assumptions of the Black and Scholes (1973) model. Under these assumptions, the evolution of the underlying asset price under the risk-neutral measure is described by

$$\frac{dS_t}{dt} = rdt + \sigma dW_t$$

where

r is the risk-less rate,

σ is the underlying asset price volatility, and

W_t is a standard Brownian motion.

The solution of this diffusion equation yields the price of the asset at a given date $t > 0$:

$$S_t = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right). \quad (1)$$

Define

$$\mu \equiv r - \frac{\sigma^2}{2}.$$

Using (1), the return of the price process in a time interval δ is a random variable Y_δ satisfying

$$\ln Y_\delta = \delta\mu + \sigma\sqrt{\delta}\epsilon$$

where $\epsilon \sim N(0, 1)$.

Consider a Bermudian put option of maturity T where the holder has the right to exercise at a finite number M of equally spaced *exercise dates* denoted by $t_m = m\frac{T}{M}$, $m = 1, \dots, M$, with $t_0 = 0$. Denote by $\delta = \frac{T}{M}$ the interval between two exercise dates. At a given exercise date, when the underlying asset price is s , the *exercise value* of the option is given by $(K - s)^+$, where $(x)^+$ is the function $\max\{x, 0\}$ and K is the contractual *strike price*. At each exercise date, the holder chooses between exercising his option, or holding it until at least the next exercise date. If the holder is using an optimal strategy, the Bermudian option value at any date t_m , $m = 0, \dots, M$, when the underlying asset price is s is given by the dynamic program

$$V_M(s) = (K - s)^+ \quad (2)$$

$$V_{m-1}(s) = \max \left\{ (K - s)^+; \exp(-r\delta) \mathbb{E}[V_m(sY_\delta)] \right\}, \quad m = 2, \dots, M \quad (3)$$

$$V_0(s) = \exp(-r\delta) \mathbb{E}[V_1(sY_\delta)] \quad (4)$$

where $\mathbb{E}[\cdot]$ denotes the expectation under the risk-neutral measure. The term

$$\exp(-r\delta) \mathbb{E}[V_m(sY_\delta)] \quad (5)$$

is the *holding value* of the option at (t_{m-1}, s) ; it represents the value of the potential future cash flows of the option, and is defined recursively by the expected value of the option at the next exercise date, discounted at rate r , given the current observed price of the underlying asset.

The numerical solution of the dynamic program (2)–(4) consists in recursively computing V_m at discrete points of the state space $(0, \infty)$, starting from the known function V_M . Clearly, the issue lies in the computation of the holding value (5), which involves evaluating the expected value of a function that is only known on a discrete subset of the state space.

We propose the use of cubic spline interpolation functions, which are continuous, twice differentiable piecewise polynomials that can be integrated analytically under the log-normal return assumption. Consider a grid $\mathcal{G}_n \equiv \{s_i\}_{i=1, \dots, n} \subset (0, \infty)$, where $s_0 = 0 < s_1 < s_2 < \dots < s_n < s_{n+1} = \infty$ and a function $V : (0, \infty) \rightarrow \mathbb{R}$, the value of which is known on \mathcal{G}_n . The cubic spline interpolation of V is given by

$$\hat{V}(s) = \sum_{i=0}^n \sum_{d=0}^3 c_{id} s^d \mathbb{I}_s[s_i, s_{i+1}) \quad (6)$$

where $\mathbb{I}_x \mathcal{I}$ is the indicator function

$$\mathbb{I}_x \mathcal{I} = \begin{cases} 1 & \text{if } x \in \mathcal{I} \\ 0 & \text{otherwise,} \end{cases}$$

and where the coefficients c_{id} are chosen so that \hat{V} is twice differentiable and coincides with V on \mathcal{G}_n .¹

Assume that the value of V_m is known on \mathcal{G}_n . The holding value at a given (t_{m-1}, s) is obtained by approximating $\mathbb{E}[V_m(sY_\delta)]$ by $\mathbb{E}[\hat{V}_m(sY_\delta)]$. We obtain, using (6):

$$\begin{aligned} & \mathbb{E}[\hat{V}_m(sY_\delta)] \\ &= \mathbb{E} \left[\sum_{i=0}^n \sum_{d=0}^3 \left(c_{id}^m (sY_\delta)^d \right) \mathbb{I}_{sY_\delta}[s_i, s_{i+1}) \right] \\ &= \sum_{i=0}^{n-1} \sum_{d=0}^3 c_{id}^m s^d \mathbb{E} \left[(Y_\delta)^d \mathbb{I}_{Y_\delta} \left[\frac{s_i}{s}, \frac{s_{i+1}}{s} \right) \right] \\ &= \sum_{i=0}^{n-1} \sum_{d=0}^3 c_{id}^m s^d \mathbb{E} \left[\exp(d\delta\mu + d\sigma\sqrt{\delta}\epsilon) \mathbb{I}_\epsilon[a_i, a_{i+1}) \right] \\ &= \sum_{i=0}^{n-1} \sum_{d=0}^3 c_{id}^m s^d \exp(d\delta\mu) \mathbb{E} \left[\exp(d\sigma\sqrt{\delta}\epsilon) \mathbb{I}_\epsilon[a_i, a_{i+1}) \right] \end{aligned}$$

where

$$a_i = \frac{\log\left(\frac{s_i}{s}\right) - \mu\delta}{\sigma\sqrt{\delta}} \quad \text{and} \quad \epsilon \sim N(0, 1).$$

Define

$$\Omega(s, i, d) \equiv \exp(d\delta\mu) \mathbb{E} \left[\exp(d\sigma\sqrt{\delta}\epsilon) \mathbb{I}_\epsilon[a_i, a_{i+1}) \right], \quad \epsilon \sim N(0, 1);$$

¹This set of condition yields a linear system of equations; additional conditions are needed at the boundaries to completely specify the interpolation coefficients. The MATLAB function *spline* can be used to obtain these coefficients.

this function can be obtained analytically from the normal cumulative function (details are given in Appendix 6.1). We finally get

$$\mathbb{E} \left[\widehat{V}_m(sY_\delta) \right] = \sum_{i=0}^n \sum_{d=0}^3 c_{id}^m s^d \Omega(s, i, d). \quad (7)$$

The numerical procedure for the evaluation of a Bermudian put option is summarized in the following algorithm:

Algorithm 1

1. Define parameters $T, M, n, \delta = \frac{T}{M}, s_n, K$.
2. Discretize time: $t_m = m \frac{T}{M}, m = 0, \dots, M$ and asset price space: $s_i = i \frac{s_n}{n}, i = 0, \dots, n$.
3. Compute $\Omega(s_i, j, d)$ for $i = 0, \dots, n, j = 0, \dots, n-1$ and $d = 0, \dots, 3$.
4. Compute coefficients c_{id}^M for $i = 0, \dots, n$ and $d = 0, \dots, 3$ interpolating V_M using (2).
5. For $m = M-1$ until $m = 1$
 - 5.1 Compute $V_m(s_i), i = 0, \dots, n$ using (3) and (7)
 - 5.2 Compute coefficients c_{id}^m for $i = 0, \dots, n$ and $d = 0, \dots, 3$ interpolating V_m .
6. Compute $V_0(s_i), i = 0, \dots, n$ using (4) and (7) and stop.

The choice of the grid \mathcal{G}_n depends on the option's volatility and maturity; in fact, the boundary s_n should be chosen so that the error due to extrapolation outside the evaluation grid is negligible. This can be achieved by ensuring that the probability of the underlying asset price attaining s_n is small, or that the option value for $s > s_n$ is negligible (the value of a put option vanishes when the underlying asset price is very high). On the other hand, the interval $[s_0, s_n]$ cannot be arbitrarily large since increasing the size of the interpolation interval means that a larger number n of grid points is required to ensure that the interpolation error is small. In all our numerical experiments, we use $s_n = K \exp(\mu T + 4\sigma\sqrt{T})$, and equally spaced grid points.

Figure 1 plots the relative interpolation error as a function of the number of grid points, for a European put option maturing in one year evaluated at $(0, s)$ using Algorithm 1 with two time steps (without exercise opportunities). As can be seen, when the number of interpolation points reaches 200, the error is around 10^{-8} . In our numerical experiments, the number of interpolation points is chosen in such a way that the interpolation error is smaller than the precision of the reported results, given the number of time steps over which this error can accumulate. In that case, the difference in value between an American and a Bermudian option can be assigned to the number of exercise opportunities.

2.2 Approximation of an American option by a Bermudian option

Because the American put gives its holder an infinite number of exercise opportunities, one expects the Bermudian put to become a good approximation of the American put when the number of exercise dates is sufficiently high. Numerical methods based on time discretization usually assume a day-long time interval between two exercise opportunities. Assuming that the holder can only observe the underlying asset price at discrete times, one observation per day seems a reasonable approximation of the holder's exercise strategy. We now evaluate the quality of this approximation by examining the convergence of the Bermudian option with increasing frequency of exercise opportunities. Representative results are presented in Tables 1 and 2.

In reviewing these results, we observe that convergence to a stable value with an increasing frequency of exercise opportunities is slow, particularly for long-maturity² or high-volatility options. In both cases, the number of exercise dates needs to be very high to approximate the continuous case. Recall that the numerical evaluation of the Bermudian option requires a spline interpolation at each exercise date, and therefore, the interpolation error may accumulate when the number of time steps becomes very large; moreover, the

²Long-maturity options include LEAPS (Long-term Equity Anticipation Securities, up to 3 years maturity on the CBOE) and put options embedded in protected capital notes (5 to 10 years maturity).

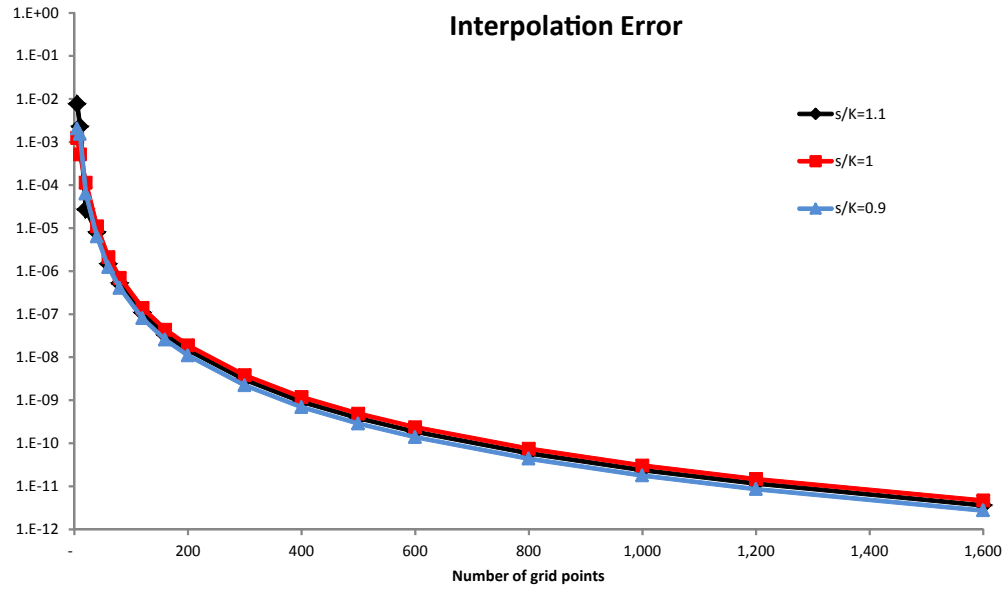


Figure 1: Convergence of spline interpolation to the exact value of the European put. Parameter values are $\sigma = 20\%$, $r = 4\%$, $K = 100$, $T = 1$ and $M = 2$.

Table 1: Value of a Bermudian option as a function of the number of exercise opportunities. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 20\%$, $n = 200$ for $T = 1$ and $n = 300$ for $T = 5$.

M	$T = 1$ year			$T = 5$ years		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	10.8414	6.0040	3.0476	11.4751	8.5766	6.3926
2	11.3951	6.2020	3.1018	13.4674	9.8514	7.1863
4	11.6227	6.2937	3.1457	14.4277	10.4229	7.5420
8	11.7173	6.3464	3.1739	14.8472	10.6843	7.7294
16	11.7614	6.3746	3.1894	15.0285	10.8168	7.8308
32	11.7839	6.3891	3.1977	15.1186	10.8873	7.8843
64	11.7953	6.3966	3.2021	15.1654	10.9221	7.9109
128	11.8009	6.4004	3.2043	15.1901	10.9406	7.9247
512	11.8052	6.4032	3.2060	15.2059	10.9541	7.9353
1024	11.8061	6.4038	3.2063	15.2084	10.9561	7.9370
1500	11.8065	6.4040	3.2064	15.2093	10.9567	7.9375
2000	11.8067	6.4041	3.2065	15.2098	10.9571	7.9378

interpolation grid domain $[0, s_n]$ also expands with increasing maturity and volatility. For such options, the use of a fine time discretization may be computationally expensive.

3 Approach using a portfolio of barrier options

The numerical method proposed in this section is based on Ingersoll (1998), where the American put is equated with a portfolio containing one European “down-and-out” (DO) put option, and one digital “first-touch” (FT) option, both options having the same barrier. In doing so, the author assumes that the holder of the American put uses a constant-barrier exercise strategy. He then shows numerically that the value of the American put can be approximated by the value of the equivalent portfolio when the (constant) barrier is optimal.

Table 2: Value of a Bermudian option as a function of the number of exercise opportunities. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 40\%$. Value of a Bermudian option as a function of the number of exercise opportunities. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 20\%$, $n = 300$ for $T = 1$ and $n = 500$ for $T = 5$.

M	$T = 1 \text{ year}$			$T = 5 \text{ years}$		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	17.9818	13.6572	10.2849	25.7637	23.0630	20.7195
2	18.2946	13.8303	10.3772	27.7787	24.6866	22.0320
4	18.4325	13.9313	10.4523	28.5908	25.3507	22.5907
8	18.5082	13.9915	10.4985	28.9455	25.6678	22.8788
16	18.5479	14.0234	10.5236	29.1390	25.8429	23.0369
32	18.5683	14.0401	10.5369	29.2338	25.9312	23.1187
64	18.5786	14.0486	10.5438	29.2793	25.9731	23.1575
128	18.5837	14.0528	10.5472	29.3033	25.9943	23.1766
512	18.5876	14.0561	10.5499	29.3243	26.0126	23.1929
1024	18.5883	14.0567	10.5504	29.3284	26.0161	23.1960
1500	18.5886	14.0569	10.5505	29.3296	26.0171	23.1968
2000	18.5888	14.0571	10.5507	29.3301	26.0175	23.1971

3.1 The options

A DO put option is knocked out worthless if the asset falls to the barrier B before maturity, and, if it does not, the holder receives the payoff of a vanilla put. Denote by τ_B the first passage time of S_t under the barrier B :

$$\tau_B = \inf \{ \tau \geq 0 : S_\tau \leq B \}.$$

The price of a DO put option of maturity δ at s is given by the following risk-neutral expectation, where $\mathbb{E}_s[\cdot]$ indicates that the expectation is conditional to the current price observation s :

$$D(s; B, \delta) = e^{-r\delta} \mathbb{E}_s \left[(K - sY_\delta)^+ \mathbb{I}_\tau(\delta, \infty) \right].$$

This price can be obtained in closed form under the Black-Scholes assumptions:

$$\begin{aligned} D(s; B, \delta) &= Ke^{-r\delta} \left(N_2 \left(\frac{s}{K} \right) - N_2 \left(\frac{s}{B} \right) \right) - s \left(N_1 \left(\frac{s}{K} \right) - N_1 \left(\frac{s}{B} \right) \right) \\ &\quad - \left(\frac{B}{s} \right)^{\frac{2\mu}{\sigma^2}} \left(Ke^{-r\delta} \left(N_2 \left(\frac{B^2}{sK} \right) - N_2 \left(\frac{B}{s} \right) \right) + B^2 \left(N_1 \left(\frac{B^2}{sK} \right) - N_1 \left(\frac{B}{s} \right) \right) \right) \end{aligned}$$

where the functions N_1 and N_2 are defined by:

$$\begin{aligned} N_1(x) &= \Phi \left(-\frac{\ln(x) + \mu\delta}{\sigma\sqrt{\delta}} - \sigma\sqrt{\delta} \right) \\ N_2(x) &= \Phi \left(-\frac{\ln(x) + \mu\delta}{\sigma\sqrt{\delta}} \right) \end{aligned}$$

and where Φ is the cumulative standard normal distribution function.

The holder of a digital FT option receives the strike price at the underlying asset price first passage time under the barrier. The value of a FT option of maturity δ at s is then given by

$$F(s; B, \delta) = (K - B) \mathbb{E}_s \left[e^{-r\tau} \mathbb{I}_\tau[0, \delta] \right],$$

which also has an analytical expression under the Black-Scholes assumptions:

$$\begin{aligned} F(s; B, \delta) &= (K - B) \left(\left(\frac{s}{B} \right)^{\frac{\beta - \mu}{\sigma^2}} \Phi(-N^+) - \left(\frac{s}{B} \right)^{\frac{\beta + \mu}{\sigma^2}} \Phi(-N^-) \right) \\ \beta &= \sqrt{\mu^2 + 2r\sigma^2} \\ N^+ &= \frac{\ln\left(\frac{s}{B}\right) + \beta\delta}{\sigma\sqrt{\delta}}; N^- = \frac{\ln\left(\frac{s}{B}\right) - \beta\delta}{\sigma\sqrt{\delta}}. \end{aligned} \tag{8}$$

3.2 Constant exercise strategy

For a given barrier B , the value at (t, s) of a portfolio containing the two options and maturing at date T is therefore equal to $D(s; B, T - t) + F(s; B, T - t)$. The approximation of the American put proposed by Ingersoll (1998) is obtained by finding the barrier maximizing the value of this portfolio, that is,

$$P_t^1(s) = \max_{0 \leq B \leq s} \{D(s; B, T - t) + F(s; B, T - t)\}. \quad (9)$$

In fact, the exact value of the American put at (t, s) is the solution of the following optimization problem:

$$\begin{aligned} \overline{P}(t, s) &= \max_{U \in \mathcal{U}([t, T])} \mathbb{E}_s \left[e^{-r\tau} (K - S_\tau)^+ \right] \\ \text{s.t.} \quad \tau &= \inf \{ \tau \geq t : S_\tau \leq U(\tau) \}, \end{aligned} \quad (10)$$

where $\mathcal{U}(X)$ represents the set of functions $U : X \rightarrow \mathbb{R}_+$. Thus, to evaluate an American put at (t, s) , one needs to identify the optimal exercise frontier among the functions $U : [t, T] \rightarrow \mathbb{R}_+$, i.e., the optimal exercise frontier is a function of time. Program (9), which restricts the solution space to barriers that are constant over time, thus provides a lower bound for the value of the American put. To get a better approximation, one can expand the solution space of the exercise strategy optimization problem.

A simple alternative is to consider the set of piecewise constant exercise barriers. Thus, we assume that the option holder can change the composition of his portfolio of barrier options several times before maturity, which should give a better approximation of the price of the American put.

3.3 An exercise strategy involving two barriers

First assume that the option holder can modify his portfolio at date $T/2$; The optimization problem with two barriers is then written at $(0, s)$:

$$\begin{aligned} P^2(s) &= \max_{B_1, B_2 \in \mathcal{V}} \left\{ e^{-rT} \mathbb{E}_s \left[(K - Y_\delta s) \mathbb{I}_{\tau_2}(T, \infty) \mathbb{I}_{\tau_1} \left(\frac{T}{2}, \infty \right) \right] \right. \\ &\quad + (K - B_2) \mathbb{E}_s \left[e^{-r\tau_2} \mathbb{I}_{\tau_2} \left[\frac{T}{2}, T \right] \right] \\ &\quad \left. + (K - B_1) \mathbb{E}_s \left[e^{-r\tau_1} \mathbb{I}_{\tau_1} \left[0, \frac{T}{2} \right] \right] \right\} \\ \text{s.t.} \quad \tau_1 &= \inf \{ \tau \geq 0 : S_\tau \leq B_1 \} \\ \tau_2 &= \inf \left\{ \tau \geq \frac{T}{2} : S_\tau \leq B_2 \right\} \end{aligned} \quad (11)$$

where \mathcal{V} represents the set of functions $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, that is, the exercise barriers $B_1(\cdot)$ and $B_2(\cdot)$ are not constants, but feedback strategies. Thus, at date $\frac{T}{2}$, the option holder chooses the best barrier B_2 according to the observed asset price $S_{\frac{T}{2}}$, maximizing the value of his portfolio given the information available to him at that date. In the same way, at date 0, the optimal value for B_1 depends on s .

Notice that the second and third terms of (11) are equal to the value of FT options of maturity $\frac{T}{2}$. The following recursive representation of problem (11) is more helpful for the characterization of feedback strategies, and will be generalized in the sequel:

$$\begin{aligned}
P^2(s) &= \max_{B_1} \left\{ e^{-r\frac{T}{2}} \mathbb{E}_s \left[P_{T/2}^1(sY_{T/2}) \mathbb{I}_{\tau_1} \left(\frac{T}{2}, \infty \right) \right] + F \left(s; B_1, \frac{T}{2} \right) \right\} \\
\text{s.t.} \\
\tau_1 &= \inf \{ \tau \geq 0 : S_\tau \leq B_1 \},
\end{aligned} \tag{12}$$

where $P_{T/2}^1$ is given by (9).

The solution of (11) reveals that the optimal barriers B_1 and B_2 actually depend on the level of the underlying asset price. This is a direct consequence of the choice of a piecewise constant exercise barrier. In fact, determining the optimal constant barrier at a given s involves the probability that the price of the underlying asset will eventually reach the barrier, which obviously depends on the current price.

It should be noted that the optimization problem (11) is different from that of an a priori approximation of the exercise frontier by a piecewise constant function. Indeed, deciding on constant values B_1 and B_2 at date 0 would commit the option holder to successive levels that would be independent of the evolution of the underlying asset, while the feedback strategy obtained by the solution of (11) will necessarily yield a higher value for the option.

3.3.1 An M -piecewise constant exercise strategy

We now generalize the approach presented in the previous section by using the following recursive principle: the set of exercise strategies involving k constant barriers is a proper subset of the set of strategies involving $k+1$ constant barriers. Thus, by increasing the number of times that the option holder can adjust his portfolio by modifying the barriers of the two options, the value of this portfolio necessarily increases, attaining at the limit, the price of the American put when the exercise frontier becomes a continuous function of time:

$$P_0^1(s) \leq P^2(s) \leq P^k(s) \leq P^\infty(s) = \bar{P}(0, s).$$

We also note that, when their decision dates t_m , $m = 1, \dots, M$ coincide, an exercise strategy involving M constant barriers gives the holder the same exercise opportunities as a Bermudian put option at t_m , and also allows him to exercise during the time interval (t_m, t_{m+1}) for $m = 1, \dots, M-1$, which implies

$$V_0(s) \leq P^M(s) \leq \bar{P}(0, s).$$

These two ordered relations can be used to design an approximation algorithm for the American put. The introduction of FT options makes it possible to consider continuous-time exercise opportunities, which improves the time-step convergence with respect to the Bermudian option approximation.

Consider again a finite number M of equally spaced dates denoted by $t_m = m\delta$, $m = 1, \dots, M$, $\delta = T/M$. Denote by $P_m(s)$ the value of a portfolio of barrier options at (t_m, s) when the option holder uses an exercise strategy involving $M-m$ constant barriers. The recursive formulation of the option holder's optimization problem defines the dynamic program:

$$P_M(s) = (K - s)^+ \tag{13}$$

$$P_{M-1}(s) = \max_{0 \leq B_M \leq s} \{ D(s; B_M, \delta) + F(s; B_M, \delta) \} \tag{14}$$

$$\begin{aligned}
P_{m-1}(s) &= \max_{B_m; \tau_m = \inf \{ \tau \geq 0 : S_\tau \leq B_m \}} \{ e^{-r\delta} \mathbb{E}_s [P_m(sY_\delta) \mathbb{I}_{\tau_m}(t_m, \infty)] \\
&\quad + F(s; B_m, \delta) \}, \quad m = 1, \dots, M-1.
\end{aligned} \tag{15}$$

The numerical solution of the dynamic program (13)–(15) consists in evaluating the function P_m on a grid \mathcal{G}_n , and interpolating it using a cubic spline \hat{P}_m . At date t_{M-1} , since the option holder uses a constant barrier strategy, the value of $P_{M-1}(s)$ is given by Ingersoll (1998), as in (9).

For $m = 1, \dots, M-1$, the first term of (15) involves the computation of the expected value of a function that is only known on \mathcal{G}_n . Using (6):

$$\begin{aligned}
\mathbb{E}_s [P_m(sY_\delta)\mathbb{I}_{\tau_m}(t_m, \infty)] &\simeq \mathbb{E}_s [\hat{P}_m(sY_\delta)\mathbb{I}_{\tau_m}(t_m, \infty)] \\
&= \sum_{i=0}^n \sum_{d=0}^3 c_{id}^m \mathbb{E}_s [(sY_\delta)^d \mathbb{I}_{sY_\delta}(s_i, s_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty)]
\end{aligned}$$

where

$$Y_\delta = \exp(\mu\delta + \sigma W_\delta).$$

Applying the following change of measure:

$$\widetilde{W}_t = \frac{\mu}{\sigma}t + W_t,$$

\widetilde{W}_t is also a Brownian motion and

$$Y_\delta = \exp(\sigma \widetilde{W}_\delta).$$

By Girsanov theorem:

$$\begin{aligned}
&\mathbb{E}_s [(sY_\delta)^d \mathbb{I}_{sY_\delta}(s_i, s_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty)] \\
&= \mathbb{E}_s^{\widetilde{Q}} \left[(sY_\delta)^d \mathbb{I}_{sY_\delta}(s_i, s_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty) \exp \left(-\frac{(\frac{\mu}{\sigma})^2}{2} \delta + \frac{\mu}{\sigma} \widetilde{W}_\delta \right) \right] \\
&= s^d \mathbb{E}_s^{\widetilde{Q}} \left[\exp(d\sigma \widetilde{W}_\delta) \mathbb{I}_{sY_\delta}(s_i, s_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty) \exp \left(-\frac{\mu^2 \delta}{2\sigma^2} + \frac{\mu}{\sigma} \widetilde{W}_\delta \right) \right] \\
&= s^d \exp \left(-\frac{\mu^2 \delta}{2\sigma^2} \right) \mathbb{E}_s^{\widetilde{Q}} \left[\exp \left(\left(d\sigma + \frac{\mu}{\sigma} \right) \widetilde{W}_\delta \right) \mathbb{I}_{\widetilde{W}_\delta}(a_i, a_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty) \right]
\end{aligned}$$

where

$$a_i = \frac{1}{\sigma} \ln \left(\frac{s_i}{s} \right).$$

We finally have

$$\mathbb{E}_s [\hat{P}_m(sY_\delta)\mathbb{I}_{\tau_m}(t_m, \infty)] = \sum_{i=0}^{n-1} \sum_{d=0}^3 c_{id}^m s^d \Psi(s, i, d) \quad (16)$$

where function Ψ can be written in closed-form using the normal cumulative distribution (details are provided in Appendix 6.2).

The approximation of the American put can be obtained using the following algorithm:

Algorithm 2

1. Define parameters $T, M, n, \delta = \frac{T}{M}, s_n, K$.
2. Discretize time: $t_m = m \frac{T}{M}$, $m = 0, \dots, M$ and asset price space: $s_i = i \frac{s_n}{n}$, $i = 0, \dots, n$.
3. Compute $\Psi(s_i, j, d)$ for $i = 0, \dots, n$, $j = 0, \dots, n-1$ and $d = 0, \dots, 3$.
4. Compute $P_{M-1}(s_i)$, $i = 0, \dots, n$ using (14).
5. For $m = M-1$ until $m = 0$
 - 5.1 Compute coefficients c_{id}^m for $i = 0, \dots, n$ and $d = 0, \dots, 3$ interpolating P_m
 - 5.2 Compute $P_m(s_i)$, $i = 0, \dots, n$ using (15) and (16).
6. Stop.

In our implementation, the optimization in (14) and (15) is performed by a derivative-free line search routine (e.g. the *fminsearch* Matlab function). Our numerical experiments show that Algorithm 2 converges very rapidly with increasing values of M . Tables 3 and 4 present a representative illustration of these numerical experiments, reporting the approximation of the option value as a function of M . We see that, even with only two time steps, the approximation is very close to the American put value, and that penny accuracy

is reached in 16 time steps (corresponding to a 3-month time interval) for the long-maturity, high-volatility options. However, each time step requires the solution of n optimization problems, which is computationally intensive—for instance, 1200 cpu seconds (20 mins) were required to solve on a desktop computer the instance involving 128 time steps with 300 interpolation points.

Table 3: American put evaluation using a portfolio of barrier options. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 20\%$, $n = 200$ for $T = 1$ and $n = 300$ for $T = 5$.

M	$T = 1$ year			$T = 5$ years		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	11.7439	6.3542	3.1780	15.1126	10.8789	7.8703
2	11.7848	6.3826	3.1912	15.1841	10.9274	7.9085
4	11.7988	6.3950	3.1994	15.2016	10.9457	7.9256
8	11.8036	6.4003	3.2033	15.2080	10.9530	7.9330
16	11.8054	6.4025	3.2051	15.2107	10.9562	7.9363
32	11.8062	6.4035	3.2059	15.2118	10.9576	7.9377
64	11.8065	6.4039	3.2063	15.2123	10.95826	7.9383
128	11.8066	6.4040	3.2064	15.2125	10.9584	7.9386
256	11.8066	6.4041	3.2065	15.2127	10.9586	7.9388

Table 4: American put evaluation using a portfolio of barrier options. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 40\%$, $n = 300$ for $T = 1$ and $n = 500$ for $T = 5$.

M	$T = 1$ year			$T = 5$ years		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	18.4921	13.9781	10.4891	28.8674	25.6526	22.9015
2	18.5490	14.0202	10.5186	29.2470	25.9336	23.1138
4	18.5722	14.0408	10.5358	29.3012	25.9874	23.1661
8	18.5822	14.0503	10.5442	29.3200	26.0070	23.1860
16	18.5862	14.0543	10.5480	29.3270	26.0145	23.1938
32	18.5878	14.0560	10.5496	29.3301	26.0179	23.1973
64	18.5885	14.0567	10.5503	29.3313	26.0193	23.1989
128	18.5887	14.0570	10.5506	29.3320	26.0200	23.1997
256	18.5889	14.0571	10.5507	29.3330	26.0210	23.2007

4 An adjusted Bermudian option approach

Convergence with respect to the time step is usually the major problem when numerically evaluating long-maturity American options. Indeed, the continuous exercise feature of American options requires a fine time discretization, and the corresponding computational burden may become impracticable for long-maturity options. In addition, the evaluation of a long-maturity option usually requires large evaluation grids, to account for a broader range of possible values for the state variables over long periods. The barrier-option portfolio approach proposed in the preceding section does not require a fine time discretization, but we saw that this approach is still computationally costly due to the optimization problems that must be solved at each point of the asset price discretization grid.

This section proposes an alternative method for efficiently pricing long-maturity American puts. This method is an adaptation of the Bermudian approximation; however we allow the option holder to exercise his option in continuous time. Thus, we again discretize time into decision dates. At each decision date, the option holder makes two separate decisions: he chooses whether or not to immediately exercise the option, and if not, he also chooses a barrier that will trigger the exercise of his option if it is reached by the underlying asset price before the next decision date.

It should be noted that the exercise frontier $U^*(t)$ of the American put defined by (10) is actually independent of the underlying asset price level, and that the exercise frontier s_m^b of the corresponding Bermudian put at t_m satisfies $s_m^b > U^*(t_m)$. On the other hand, the barrier portfolio approach consists in approximating

$U^*(t)$ by a constant function on $[t_m, t_{m+1})$. As was observed in the previous section, the best approximation of $U^*(t)$ at t_m depends on the current price of the underlying asset.

In the implementation presented here, the optimum barrier chosen by the option holder at each decision date is independent of the price of the underlying asset. At each decision date t_m , we find the optimal barrier at a single point, chosen near $U^*(t_m)$. This iteratively provides an approximation of $U^*(t)$ by a piecewise constant function, and this exercise frontier is used to evaluate the option on all grid points. Thus, compared to the approach implemented in Algorithm 2, the barrier is optimized only once, which significantly reduces the computation time. We denote by $A_m(s)$ the evaluation of the American put obtained by this third approach, which is defined by the following dynamic program:

$$A_M(s) = (K - s)^+ \quad (17)$$

$$B_M = \arg \max_{0 \leq B \leq s_M^*} \{D(s_M^*; B, \delta) + F(s_M^*; B, \delta)\} \quad (18)$$

$$A_{M-1}(s) = D(s; B_M, \delta) + F(s; B_M, \delta) \quad (19)$$

$$B_m = \arg \max_{0 \leq B \leq s_m^*} \{e^{-r\delta} \mathbb{E}_{s_m^*} [A_m(s_m^* Y_\delta) \mathbb{I}_{\tau_m}(t_m, \infty)] + F(s_m^*; B, \delta) : \tau_m = \inf \{\tau \geq 0 : S_\tau \leq B\}\}, m = 1, \dots, M-1 \quad (20)$$

$$A_{m-1}(s) = e^{-r\delta} \mathbb{E}_s [A_m(s Y_\delta) \mathbb{I}_{\tau_m}(t_m, \infty)] + F(s; B_m, \delta), \quad (21)$$

$$\tau_m = \inf \{\tau \geq 0 : S_\tau \leq B_m\}, m = 1, \dots, M-1,$$

where s_m^* satisfies

$$e^{-r\delta} \mathbb{E} [A_m(s_m^* Y_\delta)] = K - s_m^*. \quad (22)$$

It should be noted that s_m^* is not exactly equal to $U^*(t_{m-1})$ since this point is obtained from the approximation A_m of the American put. For the same reason, s_m^* is not exactly on the exercise frontier of the Bermudian option. It is easy to show that

$$\bar{P}(t_m, s) \geq A_m \geq V_m$$

and therefore

$$U^*(t_m) < s_{m+1}^* < s_m^b.$$

The cubic spline interpolation \hat{A}_m of A_m is given by (6), and the computation of expectations $\mathbb{E}_s [A_m(s Y_\delta) \mathbb{I}_{\tau_m}(t_m, \infty)]$ and $\mathbb{E} [A_m(s_m^* Y_\delta)]$ is obtained using (16) and (7) respectively. Algorithm 3 describes the implementation of the dynamic program (17)–(22).

Algorithm 3

1. Define parameters $T, M, n, \delta = \frac{T}{M}, s_n, K$
2. Discretize time: $t_m = m \frac{T}{M}, m = 0, \dots, M$ and asset price space: $s_i = i \frac{s_n}{n}, i = 0, \dots, n$.
3. Compute $\Omega(s_i, j, d)$ and $\Psi(s_i, j, d)$ for $i = 0, \dots, n, j = 0, \dots, n-1$ and $d = 0, \dots, 3$.
4. At t_{M-1} :
 - 4.1 Find s_M^* solving $\{(K - s)^+ - e^{-r\delta} \mathbb{E} [(K - s Y_\delta)^+] = 0, s \in \mathbb{R}_+\}$ using the Black-Scholes formula
 - 4.2 Find B_M using (18)
 - 4.3 Compute $A_{M-1}(s_i), i = 0, \dots, n$ using (19)
5. For $m = M-1$ until $m = 1$
 - 5.1 Compute coefficients c_{id}^m for $i = 0, \dots, n$ and $d = 0, \dots, 3$ interpolating A_m
 - 5.2 Find s_m^* solving $\{(K - s)^+ - e^{-r\delta} \mathbb{E} [A_m(s Y_\delta)] = 0, s \in \mathbb{R}_+\}$ using (7)
 - 5.3 Find B_m using (20) and (16)
 - 5.4 Compute $A_{m-1}(s_i), i = 0, \dots, n$ using (21) and (16)
6. Stop.

Tables 5 and 6 provide a representative illustration of our numerical experiments using Algorithm 3. Figures 2 and 3 assess the performance of this third approach by comparing it to the other two approaches discussed in this paper in terms of accuracy and computation time.³ The reference price p_{ref} used to compute the relative approximation error $|p_{ref} - p|/p_{ref}$ is obtained from an extrapolation by the epsilon algorithm (Wynn 1966) of the sequence of prices provided by Algorithm 3.

Table 5: American put evaluation using an adjusted Bermudan approach. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 20\%$, $n = 200$ for $T = 1$ and $n = 300$ for $T = 5$.

M	$T = 1 \text{ year}$			$T = 5 \text{ years}$		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	11.6570	6.3495	3.1775	13.3609	9.7986	7.2047
2	11.7789	6.3763	3.1860	14.6321	10.5809	7.6815
4	11.7946	6.3902	3.1956	15.0649	10.8534	7.8611
8	11.8019	6.3983	3.2017	15.1907	10.9403	7.9236
16	11.8048	6.4017	3.2044	15.2096	10.9549	7.9349
32	11.8059	6.4031	3.2056	15.2114	10.9570	7.9371
64	11.8064	6.4037	3.2062	15.2121	10.9579	7.9381
128	11.8066	6.4040	3.2064	15.2125	10.9583	7.9385
512	11.8068	6.4042	3.2066	15.2128	10.9587	7.9388
1024	11.8070	6.4044	3.2066	15.2130	10.9589	7.9390
1500	11.8072	6.4045	3.2067	15.2131	10.9591	7.9391

Table 6: American put evaluation using an adjusted Bermudan approach. Parameter values are $K = 100$, $r = 4\%$, $\sigma = 40\%$, $n = 300$ for $T = 1$ and $n = 500$ for $T = 5$.

M	$T = 1 \text{ year}$			$T = 5 \text{ years}$		
	$s = 90$	$s = 100$	$s = 110$	$s = 90$	$s = 100$	$s = 110$
1	18.4542	13.9709	10.4888	27.1345	24.2476	21.7532
2	18.5266	14.0027	10.5055	28.5846	25.3840	22.6491
4	18.5653	14.0346	10.5306	29.0746	25.7889	22.9912
8	18.5788	14.0470	10.5412	29.2618	25.9549	23.1392
16	18.5848	14.0529	10.5467	29.3204	26.0081	23.1877
32	18.5873	14.0554	10.5491	29.3291	26.0168	23.1962
64	18.5882	14.0565	10.5501	29.3309	26.0188	23.1983
128	18.5886	14.0569	10.5505	29.3318	26.0198	23.1995
512	18.5889	14.0572	10.5508	29.3351	26.0227	23.2021
1024	18.5891	14.0573	10.5509	29.3370	26.0243	23.2035
1500	18.5892	14.0574	10.5509	29.3374	26.0245	23.2037

5 Conclusion

This paper proposes two new numerical approaches for the evaluation of American put options. These approaches are particularly suited for the evaluation of long-maturity contracts where the number of exercise possibilities is large, and where an accurate approximation by a Bermudian option would require a large number of time steps. Our numerical experiments show that allowing for exercise between two time steps greatly improves the convergence of the approximation to the value of the American option.

In both cases, the exercise strategy and the value of the contract are characterized by a dynamic program assuming that the holder chooses an exercise barrier at each decision date, and exercises his option if the price of the underlying asset reaches the barrier at any time between two decision dates. The first approach assumes that the optimal exercise barrier depends on the level of the underlying asset, and the option value is likened to that of a portfolio of barrier options. This method yields a very good approximation of the American put, even for a small number of time steps. However, the computation of the exercise barrier at each point of the asset price discretization grid is computationally expensive. The second approach assumes

³Computation time is given in CPU seconds, on a Pentium® Dual-Core 2.8GHz computer.

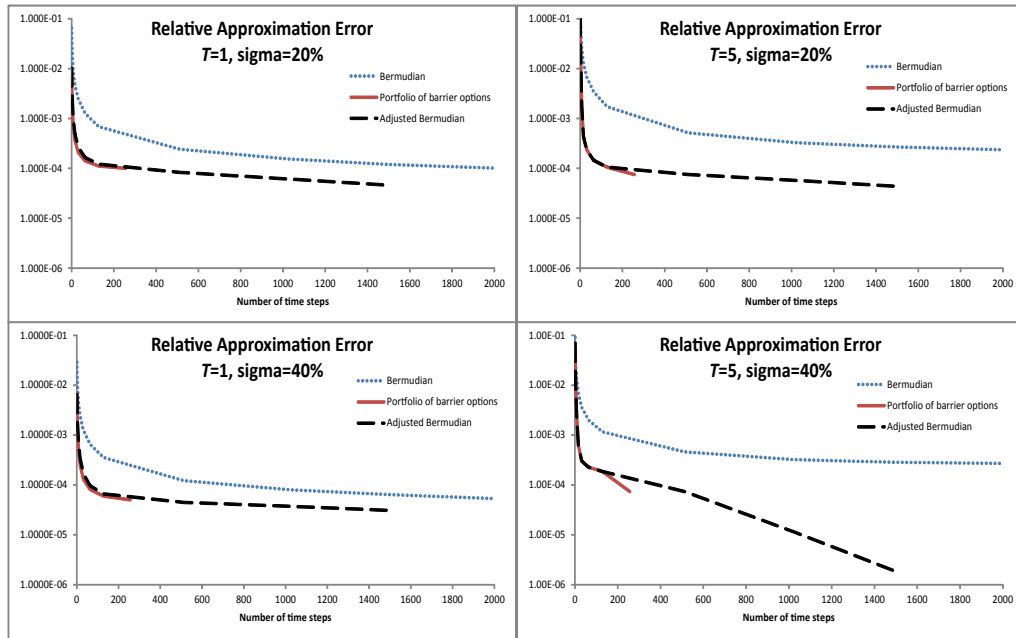


Figure 2: Convergence of the approximation error with the number of time steps, $r = 4\%$ and $K = 100$. Plotted values are averages for $s = 90, 100$ and 110 .

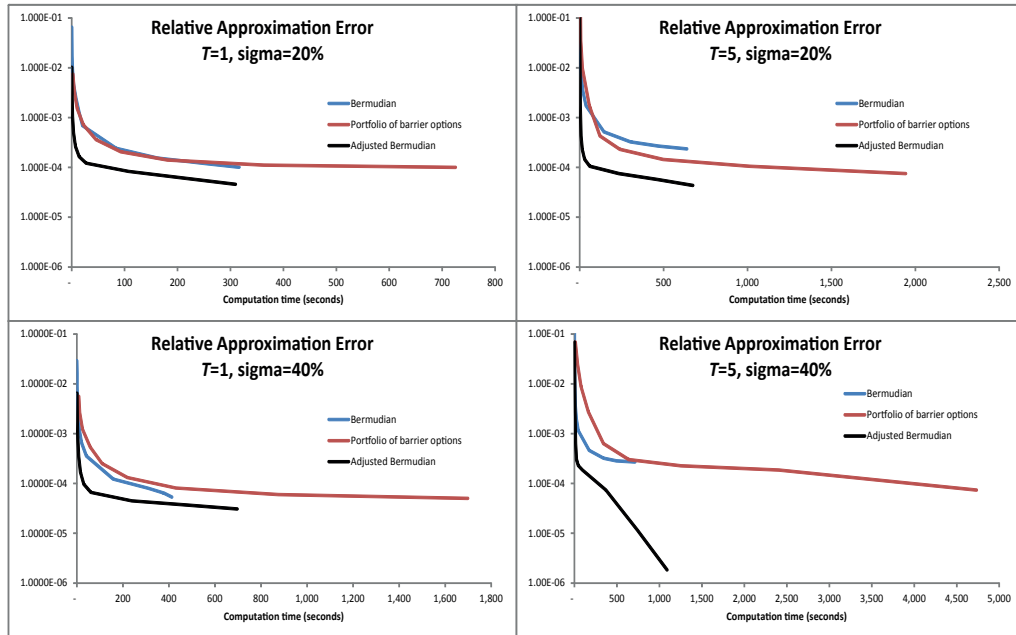


Figure 3: Approximation error as a function of the computation time, $r = 4\%$, $K = 100$. Plotted values are averages for $s = 90, 100$ and 110 .

that the optimal exercise barrier is independent of the level of the underlying asset price and corresponds to the approximation of the American put exercise frontier by a piecewise constant function. This hybrid approach is interpreted by assuming that the holder can choose whether or not to exercise at each decision date, as in the case of a Bermudian approximation, but that he also chooses, at each decision date, a constant barrier that will trigger exercise if the underlying asset price reaches it before the next decision date. This second approach is very efficient, yielding an accurate approximation of long-maturity American puts in a low computation time. Being solutions of dynamic programs, both methods provide in a single run the

price of options with various maturities, asset levels and strike prices, along with sensitivities and hedging parameters.

Our numerical experiments show that the use of a barrier between two discrete decision dates helps to converge rapidly to the value of an American option, and that the choice of a piecewise constant frontier independent of the underlying asset price level is the most efficient approach for long-maturity options. Developments are presented in the context of a log-normal market model. In this case, closed-form expressions are available for barrier options, and the interpolation of the value function by a cubic spline provides an analytical expression for its expectation. Both numerical algorithms could easily be applied to other market models; however, the numerical implementation could then require numerical integration for the computation of expected values.

6 Appendix

6.1 Computation of function Ω

Define

$$\Omega(s, i, d) \equiv \exp(d\delta\mu) \mathbb{E} \left[\exp(d\sigma\sqrt{\delta}\epsilon) \mathbb{I}_\epsilon(a_i, a_{i+1}) \right], \quad \epsilon \sim N(0, 1).$$

We have

$$\begin{aligned} \mathbb{E} \left[\exp(d\sigma\sqrt{\delta}\epsilon) \mathbb{I}_\epsilon(a_i, a_{i+1}) \right] &= \frac{1}{\sqrt{2\pi}} \int_{a_i}^{a_{i+1}} \exp\left(-\frac{w^2}{2}\right) \exp(d\sigma\sqrt{\delta}w) dw \\ &= \frac{1}{\sqrt{2\pi}} \int_{a_i}^{a_{i+1}} \exp\left(-\frac{w^2 - 2d\sigma\sqrt{\delta}w}{2}\right) dw \\ &= \frac{\exp\left(\frac{d^2\sigma^2\delta}{2}\right)}{\sqrt{2\pi}} \int_{a_i}^{a_{i+1}} \exp\left(-\frac{(w - d\sigma\sqrt{\delta})^2}{2}\right) dw \\ &= \frac{\exp\left(\frac{d^2\sigma^2\delta}{2}\right)}{\sqrt{2\pi}} \int_{a_i - d\sigma\sqrt{\delta}}^{a_{i+1} - d\sigma\sqrt{\delta}} \exp\left(-\frac{w^2}{2}\right) dw \\ &= \exp\left(\frac{d^2\sigma^2\delta}{2}\right) \left(\Phi(a_{i+1} - d\sigma\sqrt{\delta}) - \Phi(a_i - d\sigma\sqrt{\delta}) \right) \end{aligned}$$

where Φ is the standard normal cumulative distribution. We finally obtain

$$\begin{aligned} \Omega(s, i, d) &\equiv \exp\left(\frac{1}{2}d\delta(2\mu + d\sigma^2)\right) \left(\Phi(a_{i+1} - d\sigma\sqrt{\delta}) - \Phi(a_i - d\sigma\sqrt{\delta}) \right) \\ a_i &= \frac{\ln\left(\frac{s_i}{s}\right) - \mu\delta}{\sigma\sqrt{\delta}}, \quad i = 1, \dots, n. \end{aligned}$$

6.2 Computation of function Ψ

Define

$$\begin{aligned} \Psi(s, i, d) &= \exp\left(-\frac{\mu^2\delta}{2\sigma^2}\right) \mathbb{E}_{\tilde{Q}} \left[\exp\left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta\right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty) \right] \\ a_i &= \frac{1}{\sigma} \ln\left(\frac{s_i}{s}\right). \end{aligned}$$

We have

$$\begin{aligned} & \mathbb{E}_s^{\tilde{Q}} \left[\exp \left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta \right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \mathbb{I}_{\tau_m}(t_m, \infty) \right] \\ = & \mathbb{E}_s^{\tilde{Q}} \left[\exp \left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta \right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \right] \\ & - \mathbb{E}_s^{\tilde{Q}} \left[\exp \left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta \right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \mathbb{I}_{\tau_m}(t_m, t_{m+1}) \right]. \end{aligned}$$

Now

$$\begin{aligned} & \mathbb{E}_s^{\tilde{Q}} \left[\exp \left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta \right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \right] \\ = & \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp \left(\frac{\mu + d\sigma^2}{\sigma} \sqrt{\delta} w \right) \exp \left(-\frac{w^2}{2} \right) dw \\ = & \exp \left(\frac{\delta (\mu + d\sigma^2)^2}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp \left(-\frac{\left(w - \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \right)^2}{2} \right) dw \\ = & \exp \left(\frac{\delta (\mu + d\sigma^2)^2}{2\sigma^2} \right) \left(\Phi \left(\frac{a_{i+1}}{\sqrt{\delta}} - \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \right) - \Phi \left(\frac{a_i}{\sqrt{\delta}} - \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}_s^{\tilde{Q}} \left[\exp \left(\frac{\mu + d\sigma^2}{\sigma} \tilde{W}_\delta \right) \mathbb{I}_{\tilde{W}_\delta}(a_i, a_{i+1}) \mathbb{I}_{\tau_m}(t_m, t_{m+1}) \right] \\ = & \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp \left(w \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \right) \exp \left(-\frac{\left(w \sqrt{\delta} - \frac{2}{\sigma} \ln \left(\frac{B_m}{s} \right) \right)^2}{2} \right) dw. \end{aligned}$$

Using $L \equiv \frac{1}{\sigma} \ln \left(\frac{B_m}{s} \right)$,

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp \left(w \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \right) \exp \left(-\frac{\left(w \sqrt{\delta} - 2L \right)^2}{2} \right) dw \\ = & \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp \left(w \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} - \frac{\left(w \sqrt{\delta} - 2L \right)^2}{2} \right) dw \\ = & \exp \left(\frac{(d\sigma^2 + \mu)(d\sigma^2 + 4L\sigma + \mu)}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi}} \int_{\frac{a_i}{\sqrt{\delta}}}^{\frac{a_{i+1}}{\sqrt{\delta}}} \exp -\frac{1}{2} \left(\left(w \sqrt{\delta} - \left(2L + d\sigma + \frac{1}{\sigma} \mu \right) \right)^2 \right) dw \\ = & \exp \left(\frac{(d\sigma^2 + \mu)(d\sigma^2 + 4L\sigma + \mu)}{2\sigma^2} \right) \left(\Phi \left(a_{i+1} - 2L - d\sigma - \frac{1}{\sigma} \mu \right) - \Phi \left(a_i - 2L - d\sigma - \frac{1}{\sigma} \mu \right) \right) \end{aligned}$$

Finally:

$$\begin{aligned} \Psi(s, i, d) = & \exp \left(-\frac{\mu^2 \delta}{2\sigma^2} \right) \left(\exp \left(\frac{\delta (\mu + d\sigma^2)^2}{2\sigma^2} \right) (\Phi(a'_{i+1}) - \Phi(a'_i)) \right. \\ & \left. - \exp \left(\frac{(d\sigma^2 + \mu)(d\sigma^2 + 4L\sigma + \mu)}{2\sigma^2} \right) (\Phi(a''_{i+1}) - \Phi(a''_i)) \right) \end{aligned}$$

with

$$\begin{aligned}
 a_i &= \frac{1}{\sigma} \ln\left(\frac{s_i}{s}\right) \\
 a'_i &= \frac{a_i}{\sqrt{\delta}} - \sqrt{\delta} \frac{\mu + d\sigma^2}{\sigma} \\
 a''_i &= a_i - 2L - d\sigma - \frac{1}{\sigma} \mu \\
 L &= \frac{1}{\sigma} \ln\left(\frac{B_m}{s}\right).
 \end{aligned}$$

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