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Abstract: Introduced during the late nineties of the last century, Variable Neighborhood Search (VNS) was first designed for solving specific problems in combinatorial and global optimization. Nowadays, VNS is widely used as a general framework for solving many different scientific problems and even in scientific discovery. Actually, it is used for discovery in graph theory. The AutoGraphiX (AGX) system exploits different techniques from Variable Neighborhood Search to find extremal graphs, with respect to the maximization or minimization of a graph invariant, and then uses them for generating conjectures. AGX uses three approaches for conjecture–making: *analytic, algebraic* and *geometric*. In this paper, we describe the AutoGraphiX system and the VNS used in its optimization component. We present a survey of the conjectures and results obtained with AGX. Different forms of results that can be studied by AGX, and future development in the system are also discussed. Moreover, more than 150 open conjectures are mentioned.

Key Words: Variable Neighborhood Search; AutoGraphiX; Extremal graphs; Conjecture; Refutation; Automation; Computer assisted; Open problems on graphs.

Résumé : Introduite durant les dernières années du siècle passé, la Recherche à Voisinage Variable (RVV), fut d'abord conçue pour résoudre approximativement des problèmes spécifiques d'optimisation combinatoire ou globale. Aujourd'hui la RVV est un cadre largement utilisée pour la résolution de problèmes scientifiques nombreux et divers et même la découverte scientifique, en particulier en théorie des graphes. Le système AutoGraphiX (AGX) exploite différentes techniques de la RVV pour trouver des graphes extrêmaux ou quasi–extrêmaux maximisant ou minimisant un invariant graphique, et les utilise pour générer des conjectures. AGX utilise trois approches pour ce faire : analytique, géométrique et algébrique. Dans le présent article, nous décrivons le système AutoGraphiX et l'utilisation de la RVV dans sa composante d'optimisation. Nous passons en revue les conjectures et résultats obtenus avec AGX. Nous discutons également de différentes formes de résultats qui pourraient être étudiées avec AGX et les futurs développements potentiels de ce système. Enfin, nous mentionnons plus de 150 conjectures ouvertes.

Mots clés : Recherche à Voisinage Variable; AutoGraphiX; Graphes extrêmaux; Conjectures; Réfutation; Automatisation; Interactivité; Problèmes ouverts sur les graphes.

1 Introduction

Metaheuristics are general frameworks to build heuristics for solving combinatorial and global optimization problems. They have been the subject of intensive research since Kirkpatrick, Gellatt and Vecchi [133] proposed Simulated Annealing as a general scheme for building heuristics which get out of local minima. Several other metaheuristics were soon proposed. For discussion of the best-known of them the reader is referred to the books of surveys edited by Reeves [171], Glover and Kochenberger [94] and Burke and Kendall [46]. Some of the many successful applications of metaheuristics are also mentioned there.

Variable Neighborhood Search (VNS) [113, 114, 115, 116, 159] is a metaheuristic which exploits systematically the idea of neighborhood change, both in descent to local minima and in escape from the valleys which contain them. VNS exploits systematically the following observations:

- A local minimum with respect to one neighborhood structure is not necessary so for another.
- A global minimum is a local minimum with respect to all possible neighborhood structures.
- For many problems local minima with respect to one or several neighborhoods are relatively close to each other.

Unlike many other metaheuristics, the basic schemes of VNS and its extensions are simple and require few, and sometimes no parameters. Therefore, in addition to providing very good solutions, often in simpler ways than other methods, VNS gives insight into the reasons for such a performance, which, in turn, can lead to more efficient and sophisticated implementations.

```
  Function VNS(x, k_max, t_max);
1 repeat
2    k ← 1;
3    repeat
4       x' ← Shake(x, k)                 /* Shaking */;
5       x'' ← FirstImprovement(x')      /* Local search */;
6       NeighbourhoodChange(x, x'', k)  /* Change neighbourhood */;
     until k = k_max;
7    t ← CpuTime()
  until t > t_max;
```

Figure 1: Steps of the basic VNS.

The Basic VNS (BVNS) method [159] combines deterministic and stochastic changes of neighbourhood. Its steps are given in Figure 1. Often successive neighbourhoods will be nested. Observe that point x' is generated at random in Step 4 in order to avoid cycling, which might occur if deterministic rules were applied. In Step 5, several neighborhoods may be used. In this case, we speak about *variable neighborhood descent* (VND), the scheme of which is given in Figure 2. For more details about VNS ans its applications in solving problems in different domains of sciences see the recent survey [117] as well as the references therein.

```
Function VND(x, k'_max);
repeat
   k ← 1;
   repeat
      x' ← argmin_{y ∈ N'_k(x)} f(x) /* Find the best neighbor in N_k(x) */;
      NeighbourhoodChange(x, x', k) /* Change neighbourhood */;
   until k = k'_max;
until no improvement is obtained;
```

Figure 2: Steps of the basic VND.

In all its applications, VNS is used as an optimization tool. These applications are mainly solving specific optimization problems. However, VNS can also be used in *discovery science*, *i.e.*, help in the development of theories. The first domain to be addressed in this way was graph theory. VNS is the fundamental tool exploited in the system AutoGraphiX (AGX, for short) [12, 54, 55], which is devoted to conjecture–making, and therefore to scientific discovery, in graph theory. A long series of papers with the common title "*Variable neighborhood search for extremal graphs*" was published. The title are listed in Table 1. Several of the papers which use VNS without being included within this series are listed in Table 2. This system addresses the following problems:

- Find a graph satisfying given constraints;
- Find optimal or near optimal graphs for an invariant subject to constraints;
- Refute a conjecture;
- Suggest a conjecture (or repair or sharpen one);
- Provide a proof (in simple cases) or suggest an idea of proof.

A basic idea is then to consider all of these problems as parametric combinatorial optimization problems on the infinite set of all graphs (or in practice some smaller subset) solved with a generic heuristic. This is done by applying VNS to find extremal graphs, with a given number n of vertices (and possibly also a given number of edges). Then a VND with many neighbourhoods is used. Those neighborhoods are defined by modifications of the graphs such as the removal or addition of an edge, rotation of an edge, and so forth. Once a set of extremal graphs, parametrized by their order, is found, their properties are explored with various data mining techniques, leading to conjectures, refutations and simple proofs or ideas of proof.

Table 1: List of papers in the series "VNS for extremal graphs".

	Ref.	Author(s)	Title
1	[55]	Caporossi, Hansen	*The AutoGraphiX System.*
2	[49]	Caporossi, Cvetković, Gutman, Hansen	*Finding graphs with extremal energy.*
3	[74]	Cvetković, Simić, Caporossi, Hansen	*On the Largest Eigenvalue of Color-Constrained Trees.*
4	[51]	Caporossi, Gutman, Hansen	*Chemical trees with extremal connectivity index.*
5	[54]	Caporossi, Hansen	*Three ways to automate finding conjectures.*
6	[110]	Hansen, Mélot	*Analysing Bounds for the Connectivity Index.*
7	[93]	Fowler, Hansen, Caporossi, Soncini	*Polyenes with maximum HOMO-LUMO gap.*
8	[17]	Aouchiche, Caporossi, Hansen	*Variations on Graffiti 105.*
9	[109]	Hansen, Mélot	*Bounding the irregularity of a graph.*
10	[98]	Gutman, Hansen, Mélot	*Comparison of irregularity indices for chemical trees.*
11	[39]	Belhaiza, Abreu, Hansen, Oliveira	*Bounds on algebraic connectivity.*
12	[112]	Hansen, Mélot, Gutman	*A note on the variance of bounded degrees in graphs.*
13	[30]	Aouchiche, Hansen	*À propos de la maille (French).*
14	[12]	Aouchiche, Bonnefoy, Fidahoussen, Caporossi, Hansen, Hiesse, Lacheré, Monhait	*The AutoGraphiX 2 system.*
15	[118]	Hansen, Stevanović	*On Bags and Bugs.*
16	[11]	Aouchiche, Bell, Cvetković, Hansen, Rowlinson, Simić, Stevanović	*Some conjectures related to the largest eigenvalue of a graph.*
17	[35]	Aouchiche, Hansen, Stevanović	*Further conjectures and results about the index.*
18	[37]	Aouchiche, Hansen, Zheng	*Conjectures and results about the Randić index.*
19	[36]	Aouchiche, Hansen, Zheng	*Further conjectures and results about the Randić index.*
20	[15]	Aouchiche, Caporossi, Hansen	*Automated comparison of graph invariants.*
21	[13]	Aouchiche, Brinkmann, Hansen	*Conjectures and results about the independence number.*
22	[20]	Aouchiche, Favaron, Hansen	*Extending bounds for independence to upper irredundance.*
23	[119]	Hansen, Vukičević	*On the Randić index and the chromatic number.*
24	[173]	Sedlar, Vukičević, Aouchiche, Hansen	*Conjectures and results about the clique number.*
25	[174]	Sedlar, Vukičević, Aouchiche, Hansen	*Products of connectivity and distance measures.*
26	[19]	Aouchiche, Favaron, Hansen	*Nouveaux résultats sur la maille (French).*
27	[16]	Aouchiche, Caporossi, Hansen	*Families of extremal graphs.*

Table 2: A further list of papers on AGX and its conjectures.

	Ref.	Author(s)	Title
1	[1]	Abdo. Dimitrov, Gutman	*On the Zagreb indices equality.*
2	[2]	Abreu	*Old and new results on algebraic connectivity of graphs.*
3	[5]	Andova, Bogoev, Dimitrov, Pilipczuk, Škrekovski	*On the Zagreb index inequality of graphs with prescribed vertex degrees.*
4	[6]	Andova, Cohen, Škrekovski	*A note on Zagreb indices inequality for trees and unicyclic graphs.*
5	[7]	Andova, Cohen, Škrekovski	*Graph classes (dis)satisfying the Zagreb indices inequality.*
6	[8]	Andriantiana	*Unicyclic bipartite graphs with maximum energy.*
7	[9]	Andriantiana, Wagner	*Unicyclic graphs with large energy.*
8	[10]	Aouchiche	*Comparaison automatisée d'invariants en théorie des graphes.*
9	[14]	Aouchiche, Caporossi, Hansen	*Open problems on graph eigenvalues studied with AutoGraphiX.*
10	[18]	Aouchiche, Caporossi, Hansen, Laffay	*AutoGraphiX: A Survey.*
11	[21]	Aouchiche, Hansen	*Two Laplacians for the distance matrix of a graph.*
12	[22]	Aouchiche, Hansen	*Proximity, remoteness and girth in graphs.*
13	[24]	Aouchiche, Hansen	*The normalized revised Szeged index.*
14	[25]	Aouchiche, Hansen	*Proximity and remoteness in graphs: results and conjectures.*
15	[26]	Aouchiche, Hansen	*On a conjecture about the Szeged index.*
16	[27]	Aouchiche, Hansen	*Nordhaus-Gaddum Relations for Proximity and Remoteness in Graphs.*
17	[28]	Aouchiche, Hansen	*A survey of automated conjectures in spectral graph theory.*
18	[29]	Aouchiche, Hansen	*Bounding Average Distance Using Minimum Degree.*
19	[31]	Aouchiche, Hansen	*Automated Results and Conjectures on Average Distance in Graphs.*
20	[32]	Aouchiche, Hansen	*On a Conjecture about the Randić Index.*
21	[33]	Aouchiche, Hansen, Lucas	*On the extremal values of the second largest Q–eigenvalue.*
22	[34]	Aouchiche, Hansen, Stevanović	*A sharp upper bound on algebraic connectivity using domination number.*
23	[38]	Bekkai, Kouider	*On mean distance and girth.*
24	[42]	Bykoğlu, Leydold	*Graphs of given order and size and minimum algebraic connectivity.*
25	[43]	Bogoev	*A proof of an inequality related to variable Zagreb indices for simple connected graphs.*
26	[47]	Caporossi	*Découverte par Ordinateur en Théorie des Graphes.*
27	[48]	Caporossi, Chasset, Furtula	*Some conjectures and properties of distance energy.*
28	[50]	Caporossi, Dobrynin, Gutman, Hansen	*Trees with Palindromic Hosoya Polynomials.*
29	[52]	Caporossi, Gutman, Hansen, Pavlović	*Graphs with maximum connectivity index.*
30	[53]	Caporossi, Hansen	*A learning optimization algorithm in Graph Theory. Versatile Search for extremal graphs using a learning algorithm.*
31	[56]	Caporossi, Hansen,	*Finding relations in polynomial time.*
32	[57]	Caporossi, Hansen, Vukičević	*Comparing Zagreb indices of cyclic graphs.*
33	[58]	Caporossi, Paiva, Vukičević, Segatto	*Centrality and betweenness: vertex and edge decomposition of the wiener index.*
34	[59]	Cardoso, Cvetković, Rowlinson, Simić	*A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph.*
35	[61]	Chang, Tam, Wu	*Theorems on partitioned matrices revisited and their applications to graph spectra.*
36	[62]	Chen, Li, Liu	*The (revised) Szeged index and the Wiener index of a nonbipartite graph.*
37	[63]	Chen, Li, Liu	*On a relation between the Szeged index and the Wiener index for bipartite graphs.*
38	[69]	Cvetković, Rowlinson, Simić	*Eigenvalue bounds for the signless Laplacian.*
39	[70]	Cvetković, Simić	*Towards a spectral theory of graphs based on the signless Laplacian, I.*
40	[71]	Cvetković, Simić	*Towards a spectral theory of graphs based on the signless Laplacian, II.*
41	[72]	Cvetković, Simić	*Towards a spectral theory of graphs based on the signless Laplacian, III.*
42	[75]	Cygan, Pilipczuk, Škrekovski	*On the inequality between radius and Randić index for graphs.*
43	[76]	Das	*Proof of conjectures on adjacency eigenvalues of graphs.*

44	[77]	Das	*Proof of conjectures involving the largest and the smallest signless Laplacian eigenvalues of graphs*.
45	[78]	Das	*Proof of conjecture involving the second largest signless Laplacian eigenvalue and the index of graphs.*
46	[79]	Das	*Conjectures on index and algebraic connectivity of graphs*.
47	[80]	Das	*On conjectures involving second largest signless Laplacian eigenvalue of graphs.*
48	[81]	Das	*On comparing Zagreb indices of graphs.*
49	[82]	Deng, Tang, Zhang	*On a conjecture of Randić index and graph radius.*
50	[84]	Divnić, Pavlović	*Proof of the first part of the conjecture of Aouchiche and Hansen about the Randić index.*
51	[85]	Dvořàk, Lidický, Škrekovski	*Randić index and the diameter of a graph.*
52	[90]	Feng, Yu	*On three conjectures involving the signless Laplacian spectral radius of graphs.*
53	[99]	Gutman, Miljković, Caporossi, Hansen	*Alkanes with small and large Randić connectivity indices.*
54	[101]	Hansen	*How far is, should and could be conjecture-making in graph theory an automated process?*
55	[102]	Hansen	*Computers in Graph Theory.*
56	[105]	Hansen, Caporossi	*AutoGraphiX: an automated system for finding conjectures in graph theory*.
57	[106]	Hansen, Hertz, Kilani, Marcotte, Schindl	*Average distance and maximum induced forest.*
58	[107]	Hansen, Lucas	*Bounds and conjectures for the signless Laplacian index of graphs*.
59	[108]	Hansen, Lucas	*An inequality for the signless Laplacian index of a graph using the chromatic number.*
60	[111]	Hansen, Mélot	*Computers and Discovery in Algebraic Graph Theory.*
61	[120]	Hansen, Vukičević	*Comparing the Zagreb indices.*
62	[122]	Horoldagva, Das	*On comparing Zagreb indices of graphs.*
63	[123]	Horoldagva, Lee	*Comparing Zagreb indices for connected graphs.*
64	[124]	Hou	*Unicyclic graphs with minimal energy*.
65	[125]	Hou, Gutman, Woo	*Unicyclic graphs with maximal energy.*
66	[126]	Hua	*Bipartite unicyclic graphs with large energy.*
67	[127]	Hua, Das, Zhang, Xu	*Proof of a conjecture involving remoteness and radius of graphs*.
68	[128]	Huo, Ji, Li	*Solutions to unsolved problems on the minimal energies of two classes of graphs*.
69	[129]	Huo, Li, Shi	*Complete solution to a problem on the maximal energy of unicyclic bipartite graphs.*
70	[130]	Huo, Li, Shi	*Complete solution to a conjecture on the maximal energy of unicyclic graphs.*
71	[131]	Ilić	*On the extremal properties of the average eccentricity*.
72	[132]	Ilić, Stevanović	*On comparing Zagreb indices.*
73	[136]	Larson	*A survey of research in automated mathematical conjecture–making*.
74	[138]	Li, Liang	*Notes on “A proof for a conjecture on the Randić index of graphs with diameter”*.
75	[139]	Li, Liu	*Bicyclic graphs with maximal revised Szeged index.*
76	[140]	Li, Liu	*A proof of a conjecture on the Randić index of graphs with given girth*.
77	[141]	Li, Liu, Liu	*Complete solution to a conjecture on Randić index.*
78	[142]	Li, Shi	*On a relation between the Randić index and the chromatic number*.
79	[143]	Li, Shi	*Randić index, diameter and the average distance.*
80	[144]	Li, Shi	*A survey on the Randić index.*
81	[145]	Li, Shi	*Corrections of proofs for Hansen and Mélot’s two theorems.*
82	[146]	Li, Shi, Wang	*On a relation between Randić index and algebraic connectivity*.
83	[147]	Li, Zhang, Wang	*On bipartite graphs with minimal energy*.
84	[148]	Liang, Liu	*A proof of two conjectures on the Randić index and the average eccentricity.*
85	[149]	Liang, Liu	*On the Randić index and girth of graphs.*
86	[150]	Lima, Oliveira, Abreu, Nikiforov	*The smallest eigenvalue of the signless Laplacian.*
87	[151]	Liu	*On a conjecture about comparing Zagreb indices*.
88	[152]	Liu, Gutman	*On a conjecture on Randić indices*.

89	[153]	Liu, Liang, Cheng, Liu	*A proof for a conjecture on the Randić index of graphs with diameter.*
90	[154]	Liu, Pavlović, Divnić, Liu, Stojanović	*On the conjecture of Aouchiche and Hansen about the Randić index.*
91	[155]	Liu, You	*A survey on comparing Zagreb indices.*
92	[156]	Ma, Wu, Zhang	*Proximity and average eccentricity of a graph.*
93	[157]	Majstorovic, Caporossi	*Bounds and relations involving adjusted centrality of the vertices of a tree.*
94	[158]	Mélot	*On Automated and Computer Aided Conjectures in Graph Theory.*
95	[163]	Oliveira, Lima, Abreu, Kirkland	*Bounds on the Q-spread of a graph.*
96	[165]	Pavlović	*Comment on "Complete solution to a conjecture on Randić index".*
97	[169]	Rada	*Lower bounds for the energy of digraphs.*
98	[172]	Sedlar	*Remoteness, proximity and few other distance invariants in graphs.*
99	[175]	Sedlar, Vukicević, Hansen	*Using size for bounding expressions of graph invariants.*
100	[177]	Stevanović	*Comparing the Zagreb indices of the NEPS of graphs.*
101	[178]	Stevanović	*Resolution of AutoGraphiX conjectures relating the index and matching number of graphs.*
102	[179]	Stevanović	*Research problems from the Aveiro Workshop on Graph Spectra.*
103	[181]	Stevanović	*On a relation between the Zagreb indices.*
104	[182]	Stevanović, Aouchiche, Hansen	*On the spectral radius of graphs with a given domination number.*
105	[183]	Stevanović, Hansen	*The minimum spectral radius of graphs with a given clique number.*
106	[184]	Stevanović, Ilić	*Spectral properties of distance matrix of graphs.*
107	[185]	Stevanović, Milanić	*Improved inequality between Zagreb indices of trees.*
108	[186]	Sun, Chen	*Comparing the Zagreb indices for graphs with small difference between the maximum and minimum degrees.*
109	[187]	Sun, Wei	*Comparing the Zagreb indices for connected bicyclic graphs.*
110	[189]	Vukičević, Caporossi	*Network descriptors based on betweenness centrality and transmission and their extremal values.*
111	[190]	Vukičević, Graovac	*Comparing Zagreb M1 and M2 indices for acyclic molecules.*
112	[191]	Vukičević, Gutman, Furtula, Andova, Dimitrov	*Some observations on comparing Zagreb indices.*
113	[193]	Wu, Liu, An, Yan, Liu	*A conjecture on average distance and diameter of a graph.*
114	[194]	Yang, Lu	*The Randić index and the diameter of graphs.*
115	[195]	Yang, Wu, Yan	*On the sum of independence number and average degree of a graph.*
116	[196]	Ye, Fan, Wang	*Maximizing signless Laplacian or adjacency spectral radius of graphs subject to fixed connectivity.*
117	[197]	You, Liu	*On a conjecture of the Randić index.*
118	[198]	Yu, Lu, Tian	*New upper bounds for the energy of graphs.*
119	[199]	Zhang, Liu	*On a conjecture about the Randić index and diameter.*
120	[200]	Zuo	*About a conjecture on the Randić index of graphs.*

The rest of the paper is organized as follows. In the next section we describe the AutoGraphiX system, and how it uses VNS. We also briefly report on the earliest results of AGX. Section 3 summarizes AutoGraphiX conjectures that are bounds on single graph invariants. Section 4 is a survey of results of the form called *AGX Form 1*. These results were obtained as the outcome of systematic comparison of more than twenty graph invariants. Other forms of AutoGraphiX results such as bounds on a combination of more than two invariants or Nordhaus–Gaddum type relations, are over viewed in Section 5. In Section 6, we discuss different forms of results that can be studied using AutoGraphiX. To finish some conclusions are drawn, in terms of desirable properties of conjectures and how much they are shared by those found with AGX.

2 The AutoGraphiX system

Among the first application of VNS, a computer program, called the *AutoGraphiX system* (AGX, for short) [12, 54, 55], was built for conjecture–making in graph theory. This system has been developed at GERAD, Montreal, since 1997. Conjectures obtained with AGX were proved by the present authors or by graph theorists from several countries, mainly Serbia and China.

A graph invariant is a function of a graph G which does not depend on labeling of G's vertices or edges. Examples of graph invariants are the diameter, the radius, the average distance, the independence number and the index (definitions will be given below). Graph theory is replete with theorems involving graph invariants. They are either *algebraic*, *i.e.*, equalities or inequalities involving one or several invariants, or *structural*, *i.e.*, characterizations of the families of graphs for which an invariant takes an extremal value. Both types of results can be conjectured by AGX, in a fully automated way, or in some cases, to be carefully distinguished, in an assisted way.

Let $\mathscr{G}_n$ and $\mathscr{G}_{n,m}$ denote respectively the sets of all graphs with n vertices, and with n vertices and m edges. Two basic ideas underlie the systems AGX:

- *Most problems of extremal graph theory can be viewed as problems of parametric combinatorial optimization of the form*
$$\min/\max_{G\in\mathscr{G}_n} i(G) \quad \text{or} \quad \min/\max_{G\in\mathscr{G}_{n,m}} i(G) \tag{1}$$
for some invariant $i(G)$ with parameters n and m, or the exploitation of their solutions (in practice only moderate values of n and m will be considered);
- *All problems of the form (1) can be solved approximately by a generic heuristic.*

To obtain such a heuristic, the Variable Neighborhood Search metaheuristic (VNS) is specialized. VNS exploits systematically changes in neighborhoods used in the search, both in a descent phase to obtain a locally extremal graph, and in a ”shaking” phase, to get out of the corresponding valley (or away from the corresponding mountain) in order to find a better graph.

Rules of VNS applied in AGX are the following:

1. Select the set of neighborhood structures N_k, $k = 1, \dots k_{max}$ that will be used in the search for a better locally optimal graph, and a stopping condition. Choose an initial graph G.
   Repeat until the stopping condition is met:
2. Set $k = 1$;
3. Until $k = k_{max}$, repeat the following steps:
   (a) (*shaking*) generate a graph G' from the k^{th} neighborhood of G ($G' \in N_k(G)$);
   (b) (*descent*) apply *Variable Neighborhood Descent (VND)* with G' as initial graph; denote with G'' the locally optimal graph obtained;
   (c) (*improvement or continuation*) if $i(G'')$ is better than $i(G)$, the best value of i for a previously visited graph, move there, *i.e.*, replace G by G'', and continue search within $N_1(G)$; otherwise, set $k \leftarrow k+1$.

The stopping condition is usually a maximum computing time. The optimization routine of VNS is called *variable neighborhood descent*. It exploits systematically larger and larger neighborhoods of the current graph, and performs a move whenever it is profitable (fast improvement) or is also best within its neighborhood (best improvement). The neighborhoods used initially in AGX are the following: remove, add, move, detour, short cut, 2–opt, insert pending vertex, add pending vertex, and remove vertex. They are illustrated in Figure 3.

In the most recent version of AGX, the VND routine is replaced by *Learning Descent* (LD), in order to keep track of which transformations are the most fruitful and to reinforce their use. The learning descent used in AGX was described in [53]; it is an improvement of the optimization algorithm that was described in [12]. The Learning Descent (LD) is based upon a meta-transformations that could eventually be used within the VND frame. However, by itself, the learning descent replaces most of the classical transformations as those available in the early version of AGX for example. Each transformation is described as the replacement of an induced subgraph g' of G by another subgraph g''. In the current implementation, the order of g' (and g'') is 4, which implies at most 6 edges. There are $2^6 = 64$ possible labelled subgraphs to be considered. Each induced subgraph g' of G is identified and the substitution of g' by any other subgraph g'' is considered. As enumerating and evaluating all the alternative subgraphs g'' to replace g' would be very time consuming, replacing g' by g'' will only be evaluated if there are good reasons to believe it is worthwhile.

The implementation of this method encodes each subgraph g' or g'' as a label (number) based upon the 64 patterns as follows. After relabeling its vertices from 1 to 4 by preserving their order, each subgraph g' is characterized by a unique label from 0 to 63 as follows:

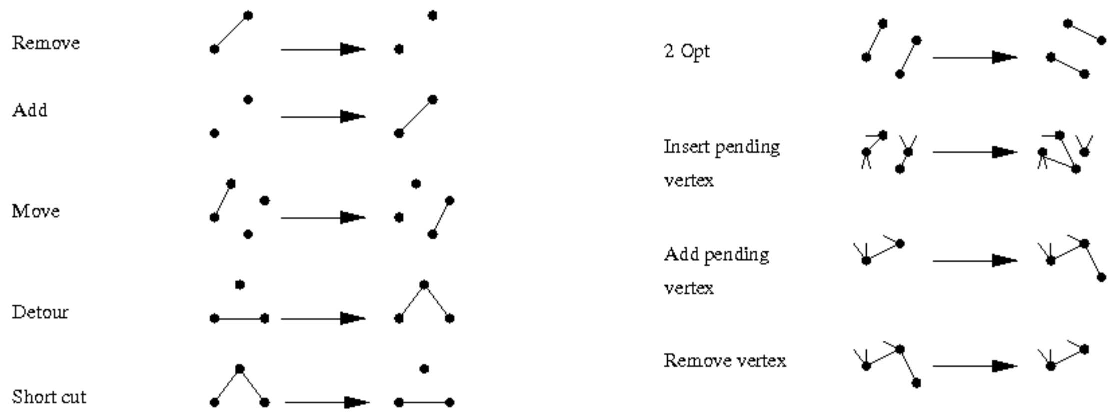

Figure 3: Neighborhoods initially implemented in AGX.

pattern 0 (vector = 000000): empty subgraph
pattern 1 (vector = 000001): $E = \{(1,2)\}$
pattern 2 (vector = 000010): $E = \{(1,3)\}$
:
pattern 13 (vector = 001101): $E = \{(1,2),(1,4),(2,3)\}$
:
pattern 63 (vector = 111111): complete subgraph on 4 vertices.

A 64×64 transformation matrix $T = \{t_{ij}\}$ is used to store information on the performance of each possible transformation from pattern i to pattern j.

The LD algorithm on Figure 4 could be described by the following observations:

1. The pertinence of changing g' into g'' (replacing pattern p' by pattern p'') is memorized in a 64×64 matrix T which is initially set to $T = \{t_{ij} = 0\}$.
2. During the optimization, each induced subgraph g' is considered for replacement by any possible alternative subgraph g" but this replacement will not necessarily be evaluated.
3. The probability to test the replacement of pattern i (g') by j (g") is $p = sig(t_{ij}) = \frac{1}{1+e^{-t_{ij}}}$. The initial probability to test a replacement is 50% according to point 1.
4. For any tested transformation, if changing g' (with pattern p') to g'' (with pattern p'') improves the solution, the entry $t_{p',p''}$ of T is increased by δ^+ (and reduced by δ^- otherwise), with $\delta^+ > \delta^-$ because it is more important to use an improving transformation than to avoid a bad one. Also, a good transformation may fail, specially if the graph already has a good performance (here, we use $\delta^+ = 1$ and $\delta^- = 0.1$). The probability to test a transformation increases when it succeeds, but decreases if it does not.

As it is often the case in neural networks, the sigmoid function $sig(x)$ is used to define the probability to test a transformation. Figure 5 represents the replacement of *pattern 60* by *pattern 27* on a given graph G for the induced subgraph g' defined by vertices 1, 3, 5 and 6.

Note that if the algorithm were restricted to Step 2, it would tend to reduce the probability to use any transformation when good solutions are encountered since few transformations would improve such solutions. To avoid this problem, the matrix T is centered after each local search to an average value $\overline{t} = 0$.

Once a set of (presumably) extremal graphs has been found, conjectures can be stated by one of the following 3 approaches [54]:

(i) a *numerical method* which applies the mathematics of Principle Component Analysis [56] to determine, in polynomial time, a basis of affine relations between invariants, satisfied by the extremal graphs found.

```
Step 1: Initialization
Load the last version of the matrix T for the problem under study if it exists
and initialize T = {t_ij = 0} otherwise.
Step 2: Apply Local Search
set improved ← true
while improved = true do:
    set improved ← false
    For each subgraph g' of G on n' vertices do:
        let p_i be the corresponding pattern
        For each alternative pattern p_j
        (corresponding to g''):
            let x be an uniform 0-1random number.
            if x ≤ sig(t_ij) do:
                if replacing g' by g'' in G improves the
                solution:
                    apply the change
                    set improved ← true
                    set t_ij = t_ij + δ+.
                otherwise:
                    set t_ij = t_ij − δ−.
            done
    done
done
Step 3: Scale the matrix T
    Let t̄ be the average value of the terms t_ij ≠ 0.
    For each t_ij ≠ 0:
        set t_ij = t_ij − t̄.
Step 4: Save the matrix T for future usage
```

Figure 4: Rules of the Learning Descent.

(ii) a *geometric method* which views extremal graphs as points in invariants space and applies a "gift-wrapping" algorithm to find their convex hull and linear inequality relations associated with its facets. Note that a similar approach is used in the recent system GraPHedron [60];

(iii) an *algebraic method* [10, 15, 12] which recognizes to which family (or families) of graphs the extremal graphs belong, then uses a database of formulae for invariants in function of the order of G to obtain conjectures.

3 Bounding invariants

The AutoGraphiX system was built for finding extremal graphs with respect to a given invariant or an algebraic combination of invariants, *i.e.* finding graphs that minimize or maximize a given invariant function. Once the extremal graphs obtained, research is done for finding a lower bound, in the case of minimization, or an upper bound in case of maximization, on the invariant function under study. Thus, naturally, the first AGX task is bounding one invariant at time, *i.e.*, without considering combinations of invariants.

The *degree* of a vertex v in G, denoted by $d(v) = d_G(v)$ is the number of vertices adjacent to v in G. The minimum, average and *maximum degrees* in G are denoted by δ, $\overline{d}$ and Δ respectively. The distance $d(u,v) = d_G(u,v)$ between two vertices u and v in a graph G is the *length* (number of edges) of a shortest path between u and v. The *average distance* is denoted by $\overline{l}$.

The problem of upper bounding the average distance in terms of order and minimum degree was studied using AutoGraphiX in [29]. Six conjectures were obtained, one of which was proved. First, we state the proved result.

Theorem 3.1 ([29]) *Let $G = (V,E)$ be a connected graph on $n \geq 7$ vertices with average distance $\overline{l}$ and minimum degree $\delta \geq 2$. Then*

$$\overline{l} \leq \frac{n+1}{3} - \frac{8}{n} + \frac{4}{n-1}$$

with equality if and only if G is composed of two triangles linked by a path.

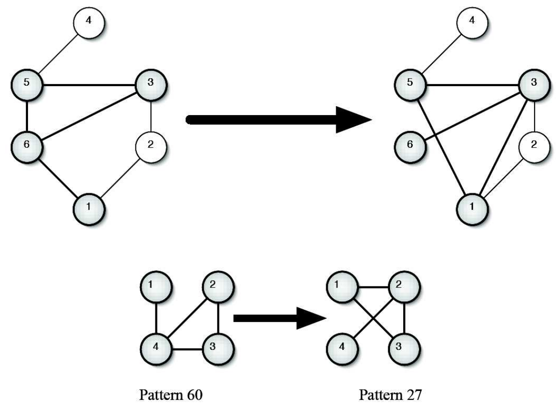

Figure 5: Illustration of the transformation of G (left) to G' (right).

After the above result, we progressively generalized our experiments according to the value of the minimum degree: $\delta = 3$, $\delta = 4$ and $\delta = 5$. Then, the general case, with a given lower bound on δ was considered. Among the obtained conjectures, we recall only the next two. Some graph definitions are needed.

(a) Let n and δ be integers such that $n = q(\delta+1)$ with $q \geq 2$ and $\delta \geq 3$. Consider the graph G obtained from the graph composed of q copies of $K_{\delta+1}$, say $K^i_{\delta+1}$ for $i = 1, 2, \ldots q$, by removing an edge $u^i v^i$ from each $K^i_{\delta+1}$ for $i = 2, \ldots q-1$, then adding the edges $v^i u^{i+1}$, uu^2 and $v^{q-1}v$ where u is any vertex from $K^1_{\delta+1}$ and v any vertex from $K^q_{\delta+1}$. If $q = 2$, there are two copies of $K_{\delta+1}$, then we add only the edge uv. See Figure 6 for $(n, \delta) = (25, 4)$.

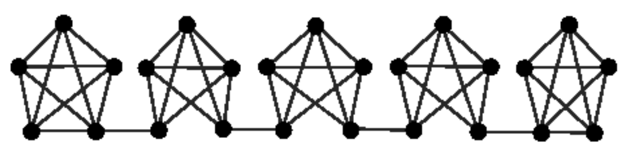

Figure 6: Presumably extremal graph for $(n, \delta) = (25, 4)$.

(b) Let n and δ be integers such that $n = q(\delta+1)+2$ with $q \geq 2$ and $\delta \geq 3$. Consider the graph G obtained from the graph described in (a) by replacing each of $K^1_{\delta+1}$ and $K^q_{\delta+1}$ by the graph H obtained from $K_{\delta+2}$ on the set of vertices $\{w_1, w_2, \ldots w_{\delta+2}\}$, by deleting the edges w_1w_2, w_1w_3 and w_iw_{i+1} for $i = 4, 6, \cdots p+1$, where $p = \delta$ if δ is even and $p = \delta+1$ if δ is odd. The vertices u and v from the graph described in (a) correspond to w_1 from each copy of H respectively. Again, if $q = 2$, there are two copies of H, then we add only the edge uv. See Figure 7 for $(n, \delta) = (22, 4)$.

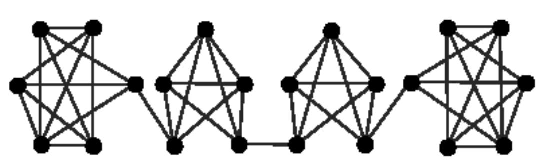

Figure 7: Presumably extremal graph for $(n, \delta) = (22, 4)$.

Conjecture 3.2 ([29]) *Let $G = (V,E)$ be a connected graph on n vertices with minimum degree $\delta \geq 3$ where $n = (\delta+1)\cdot k$ for some integer $k \geq 2$. Then the average distance $\bar{l}$ of G satisfies*

$$\bar{l} \leq \frac{n+1}{\delta+1} - \frac{4\delta}{n} + \frac{4\delta^2-\delta-2}{(\delta+1)(n-1)}$$

with equality if and only if G is obtained as described in (a).

Note that Kouider and Winkler [134] gave the extremal graphs of Conjecture 3.2 as extremal cases, without a proof, for the case $n = (\delta+1)k$. However, the corresponding bound does not appear to be generalizable for all integers n and δ. If true, the next conjecture provides a global and sharp upper bound on $\bar{l}$ in terms of δ.

Conjecture 3.3 ([29]) *Let $G = (V,E)$ be a connected graph on n vertices with minimum degree $\delta \geq 3$. Then the average distance $\bar{l}$ of G satisfies*

$$\bar{l} \leq \frac{n+1}{\delta+1} - \frac{2\delta^2-14\delta+36}{n} + \frac{12\delta^2-75\delta+150}{(\delta+1)(n-1)}.$$

The bound is reachable only if $n = (\delta+1)\cdot k+2$ for some integer $k \geq 2$, in which case the extremal graph G is the graph obtained as described in (b).

The *adjacency matrix A* of G is a 0–1 $n \times n$–matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the A–spectrum of G, *i.e.*, the spectrum of the adjacency matrix of G, and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The *spectral spread* of G is defined by $s(G) = \lambda_1(G) - \lambda_n(G)$. The problem of finding the maximum value of $s(G)$ among the class of connected graphs of given order n is an open problem. Experiments were done with the AutoGraphiX system to study the problem, and the extremal graphs were found. Little can be found in the literature concerning the spectral spread of a graph. All graphs whose spectral spread does not exceed 4 are determined in [166]. The spectral spread of unicyclic graphs has been studied in [176]. The problem of maximizing $s(G)$ over the class of connected graphs was studied using AutoGraphiX in [11] (see also [14, 28]). A conjecture was obtained, but before its statement, recall the following definition. A *complete split graph* with parameters n,q $(q \leq n)$, denoted by $CS(n,q)$, is a graph on n vertices consisting of a clique on q vertices and an independent set on the remaining $n-q$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. An example of a complete split graph is given in Figure 8.

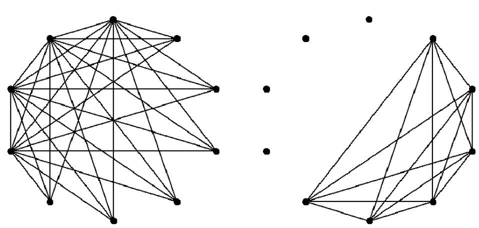

Figure 8: The complete split graph $CS(10,4)$ and its complement.

The spectrum of a complete split graph $CS(n,q)$ is

$$\begin{pmatrix} \frac{q-1}{2} + \frac{\sqrt{4qn-3q^2-2q+1}}{2} & 0 & -1 & \frac{q-1}{2} - \frac{\sqrt{4qn-3q^2-2q+1}}{2} \\ 1 & n-q-1 & q-1 & 1 \end{pmatrix}.$$

Now, we can state the conjecture which seems to be very hard to prove.

Conjecture 3.4 ([14]) *Let G be a connected graph on $n \geq 3$ vertices. Then*

$$s(G) \leq \sqrt{4qn-3q^2-2q+1}$$

with equality if and only if G is the complete split graph $CS(n,q)$ with an independent set of size $n-q = \left\lceil \frac{n}{3} \right\rceil$ and a clique of size $q = \left\lfloor \frac{2n}{3} \right\rfloor$.

Note that the above conjecture did appear in [95], in terms of extremal graphs only, where it has been verified by computer for graphs up to 9 vertices, but remained unsolved.

The *energy* $E(G)$ of a graph G, introduced by Gutman [97] in 1978 (see [96] for a survey), is defined as the sum of the absolute values of its eigenvalues, *i.e.*

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)| = 2\sum_{\lambda_i>0} \lambda_i(G) = 2\sum_{\lambda_i<0} |\lambda_i(G)|.$$

A lollipop $Lol_{n,g}$, with $n \geq g \geq 3$, is a graph obtained from a cycle C_g and a path P_{n-g} by adding an edge between a vertex from the cycle and an endpoint from the path (see Figure 9 for $Lol_{10,6}$). $Lol_{n,n-1}$ is called the short lollipop while $Lol_{n,3}$ is the long lollipop and $Lol_{n,n}$ is the cycle C_n.

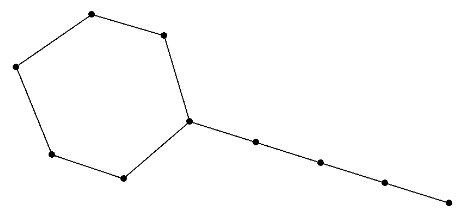

Figure 9: The lollipop $Lol_{10,6}$.

In order to find lower and upper bounds on the energy, Caporossi, Cvetković, Gutman and Hansen [49] used the AGX system. They found the following conjectures afterwards proved by hand.

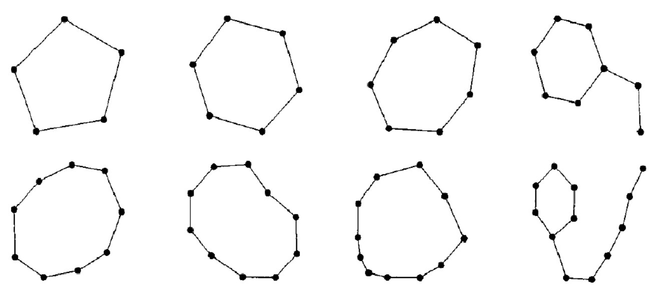

Figure 10: Unicyclic graphs with largest energy for $n = 5,\ldots,12$.

Theorem 3.5 *Let G be a simple graph on n vertices and m edges with energy E. Then*

1. $E \geq 4m/n$*;*
2. $E \geq 2\sqrt{m}$ *with equality if and only if G is a complete bipartite graph plus possibly some isolated vertices;*
3. *if G is connected,* $E \geq 2\sqrt{n-1}$ *with equality if and only if G is the star* S_n*;*
4. $E \leq 2m$ *with equality if and only if G is composed of disjoint edges and possibly isolated vertices.*

In this study, the particular case of unicyclic graphs was considered. Some unicyclic graphs that maximize the energy are given in Figure 10. The following conjecture was stated.

Conjecture 3.6 *Among unicyclic graphs on n vertices the cycle* C_n *has maximal energy if* $n \leq 7$ *and* $n = 9, 10, 11, 13$ *and* 15*. For all other values of n the unicyclic graph with maximum energy is the lollipop* $Lol_{n,6}$.

This conjecture was studied and partial results were found in Andriantiana [8], Andriantiana and Wagner [9], Hua [126], Hou, Gutman and Woo [125], Huo, Li and Shi [129, 130].

The problem of finding bicyclic graphs with maximum energy was also widely studied. It was posed by Gutman and Vidović [100]. In [14], the authors considered a more general form of the problem. First, we need the following definitions. Let p, q, r be integers such that $r = p + q$. For an integer n such that $n \geq 6r + 2$, let $P_n^{p\times 6; q\times 6}$ be the graph

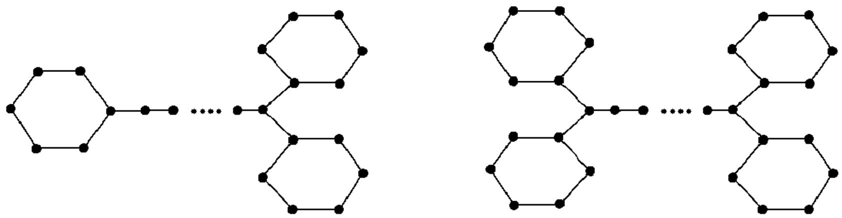

Figure 11: The tricyclic $P_n^{2\times 6;1\times 6}$ and the quadricyclic $P_n^{2\times 6;2\times 6}$ graphs.

obtained from r copies of C_6 and a path P_{n-6r} with endpoints u and v, by adding an edge between u and each of p copies of C_6 and an edge between v and each of the q other copies of C_6. See Figure 11 for $P_n^{2\times 6;1\times 6}$ and $P_n^{2\times 6;2\times 6}$. Now the general conjecture is the following.

Conjecture 3.7 *Let r and n be positive integers such that $n \geq 6r+4$. Then*

$$E(G) \leq E(P_n^{p\times 6;q\times 6}),$$

where $p = \lceil r/2 \rceil$ and $q = \lfloor r/2 \rfloor$, with equality if and only if $G \equiv P_n^{p\times 6;q\times 6}$.

The *Laplacian* of a graph G is the matrix defined by $Q = Deg - A$, where Deg is the diagonal matrix whose diagonal entries are the vertex degrees in G and A is the adjacency matrix of G. The *Laplacian spectrum* of G is the spectrum of Q and is denoted by $\mu_1, \mu_2, \ldots \mu_n$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. The second smallest Laplacian eigenvalue of a graph G is called *algebraic connectivity* of G [91] and denote $a = a(G)$. In [39], Belhaiza, Abreu, Hansen and Oliveira performed experiments using AGX and obtained lower and upper bounds on the algebraic connectivity. Among their results the following upper bound.

Theorem 3.8 ([39]) *Let G be a connected graph on n vertices and m edges with algebraic connectivity a. If $G \ncong K_n$, then*

$$a \leq \left\lfloor -1 + \sqrt{1+2m} \right\rfloor.$$

Moreover, the bound is sharp for all $m \geq 2$.

The *signless Laplacian* of a graph G is the matrix defined by $Q = Deg + A$, where Deg is the diagonal matrix whose diagonal entries are the vertex degrees in G and A is the adjacency matrix of G. The *signless Laplacian spectrum* of G is the spectrum of Q and is denoted by $q_1, q_2, \ldots q_n$, where $q_1 \geq q_2 \geq \cdots q_n$. For more details about Q and its spectrum, see [70, 71, 72, 69]. The paper [69], by Cvetković, Rowlinson and Simić, reports on AutoGraphiX conjectures obtained at GERAD and related to the signless Laplacian spectrum of a graph. Some examples of these results are given below and a complete list is given in Table 10 in the Appendix. Hansen and Lucas [108] used AutoGraphiX for studying the problem of upper bounding the largest signless Laplacian eigenvalue q_1 in terms of order n and chromatic number χ of G (the minimum number of colors that can be assigned to the vertices of a graph such that two adjacent vertices are not assigned the same color), and also in therms of order and clique number ω (the maximum number of pairwise adjacent vertices in a graph). The bounds obtained using AGX and then proved are gathered in the next theorem.

Theorem 3.9 ([108]) *Let G be a graph on n vertices with largest signless Laplacian q_1, chromatic number χ and clique number ω. Then*

$$q_1 \leq \frac{2n(\chi-1)}{\chi}$$

with equality if and only if G is is a complete regular χ-partite graph; and

$$q_1 \leq \frac{2n(\omega-1)}{\omega}$$

with equality if and only if G is is a complete regular ω-partite graph.

The *Albertson irregularity* $Al = Al(G)$ of a graph $G = (V,E)$, introduced by Albertson [3] in 1997, is defined as the of the absolute values of the differences between the degrees of the end-vertices of the edges of G, *i.e.*,

$$Al = Al(G) = \sum_{uv \in E} |d(u) - d(v)|.$$

Note that the difference $|d(u) - d(v)|$, for an edge uv is called by Albertson [3] the *imbalance* of uv.

Hansen and Mélot [109] used AutoGraphiX to find an upper bound on the Albertson irregularity in terms of order n and size m. Their experiments did not only conjecture a bound but also did suggest a clear idea for proving it. Some of the extremal graphs suggested by AutoGraphiX are presented in Figure 12. These graphs belong to the well–known family of *fanned complete split graphs*. A *fanned complete split graph* with parameters $n,q,t (n \geq q \geq t)$, denoted by $FCS(n,q,t)$, is a graph (on n vertices) obtained from a complete split graph $CS(n,q)$ by connecting a vertex from the stable set by edges to t other vertices of the stable set. The curves of the irregularity for $9 \leq n \leq 12$ and $n-1 \leq m \leq n(n-1)/2$ are given in Figure 13.

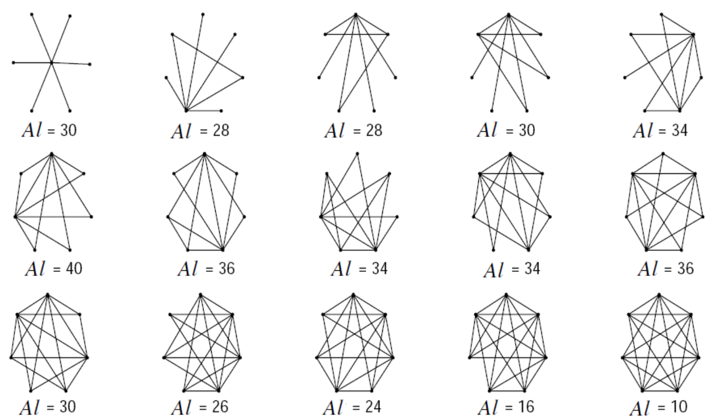

Figure 12: The extremal graphs for Al with $n = 7$ and $6 \leq m \leq 20$.

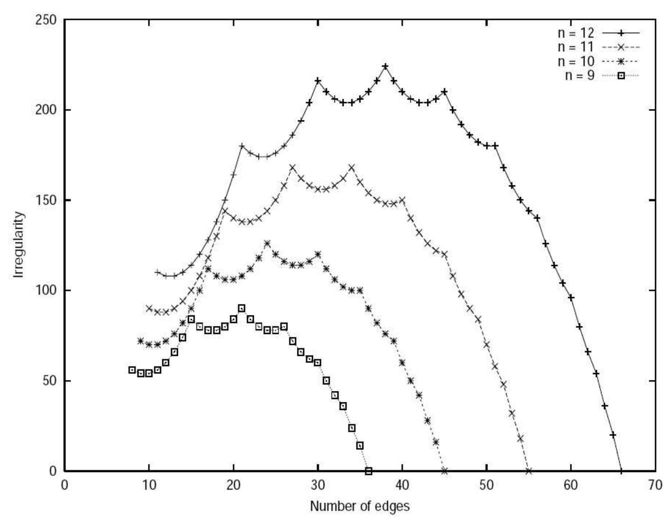

Figure 13: The curves of Al for $9 \leq n \leq 12$.

The extremal graphs were obtained by AutoGraphiX using a single move: *the rotation of an edge* (if $uv \in E$ and $uw \notin E$, the rotation of the uv to uw is the suppression of uv and the addition of uw). From where the proof idea: show that for any non-optimal graph, there exists an edge rotation that increases the irregularity. This proof works and the result is the following:

Theorem 3.10 ([109]) *If G is a graph on n vertices and m edges, then*

$$Al(G) \leq s(n-s)(n-s+1)+t(t-2s-1)$$

where

$$s = \left\lfloor n-\frac{1}{2}-\sqrt{\left(n-\frac{1}{2}\right)^2-2m}\right\rfloor \text{ and } t = m-s(n-s)-\frac{s(s-1)}{2}.$$

Moreover, the bound is attained if and only if G is fanned complete split graph.

4 AGX Form 1

In this section, we report on a particular form of results obtained using AGX. More precisely, in our experiments, we considered a set of invariants (20 at first and then few others were added) and sought expressions of the following form (called AGX Form 1):

$$\underline{b}(n) \leq i_1 \oplus i_2 \leq \overline{b}(n) \tag{2}$$

where i_1 and i_2 are invariants of a graph G from the chosen set, $\oplus$ denotes one of the 4 operations $+, -, /$ and $\times$, $\underline{b}(n)$ and $\overline{b}(n)$ are, respectively, lower and upper bounding functions depending on the *order* n, or number of vertices, of G which are *best possible*, *i.e.*, such that for each value of n (except possibly very small ones, due to border effects) there is a graph G for which the bound is tight. The order of invariants i_1 and i_2 in (2) is arbitrary. For $\oplus$ equal to $+$ or $\times$, changing this order has no effect; for $\oplus$ equal to $-$ or $/$, such a change permutes lower and upper bounds (bounds being multiplied by -1 in the former case and ratios in the bounds inverted also in the latter one). Note that the form (2) is reminiscent of the well-known Nordhaus-Gaddum relations [161]; however, it generalizes this last form in two ways:

- (i) the operations $-$ and $/$ are considered in addition to $+$ and $\times$;
- (ii) the invariants i_1 and i_2 are independent instead of having $i_2(G) = i_1(\overline{G})$, where $\overline{G}$ denotes the complementary graph of G, in which an edge joins vertices v_i and v_j if and only if there is no such edge in G.

In the thesis [10] expressions of AGX Form 1 were systematically studied for all pairs of invariants among a list of 20, given in Table 3. This amounts to 1520 cases. Results are summarized in Table 8 given in the Appendix. For each case, we give the formulae for the lower and upper bounds together with the status of the conjecture: known (K), trivial (T), open (O), assisted open (AO), structural open (SO), refuted (R). For a proved automated, assisted or structural conjecture, we refer to the paper where it is proved, and we indicate that no result is obtained (NR) whenever it is the case. Statistics on the numbers of cases which fall in these categories are given in Table 4. It appears that:

- (i) cases in which no result was obtained (because the graphs obtained by AGX do not present sufficient regularity) are rare (3.62%);
- (ii) known results rediscovered by AGX are also rare (2.43%);
- (iii) complete results, *i.e.*, algebraic formulae and extremal graphs, are frequent (82.89%). They comprise obvious results, usually proved automatically by AGX (55.59%), and non trivial results proved by hand either at GERAD or by graph theorists of various countries (23.75%), in such cases references to the proofs are given;
- (iv) in some other cases only structural conjectures, *i.e.*, only families of extremal graphs are obtained (11.06%), in some cases formulas are obtained by hand (5.67%);
- (v) cases where AGX conjectures were refuted are rare (3.62%);
- (vi) there remains a consequent number of open conjectures (8.42%). This is due to the fact that our systematic effort done to prove some families of conjectures was not enough or that some invariants appearing there are hard to handle or that some conjectures appear to be hard.

Results for a pair of invariants can be *complete*, *i.e.*, consist of both conjectured best possible functions $\underline{b}(n)$ and $\overline{b}(n)$ and the corresponding characterizations of the extremal graphs, or *structural*, *i.e.*, consist of the characterizations of extremal graphs only. This last case occurs when algebraic expressions for $\underline{b}(n)$ and $\overline{b}(n)$ are too difficult for AGX to

Table 3: The 20 invariants considered in [10] for the AGX Form 1.

Δ	The maximum degree.
δ	The minimum degree.
$\overline{d}$	The average degree.
$\bar{l}$	The average distance between all pairs of vertices.
D	The diameter.
r	The radius.
g	The girth, the length of the smallest cycle in a graph.
ecc	The average eccentricity.
π	The proximity or minimum normalized transmission.
ρ	The remoteness is maximum normalized transmission.
λ_1	The index or spectral radius.
Ra	The Randić index.
a	The algebraic connectivity or second smallest Laplacian eigenvalue.
ν	The vertex connectivity.
κ	The edge connectivity.
α	The independence number.
β	The domination number.
ω	The clique number.
χ	The chromatic number.
μ	The matching number.

Table 4: Summary of results.

Known results reproduced	37	(2.43 %)
Obvious results	845	(55.59 %)
Complete results proved by hand	361	(23.75 %)
Proved structural results and formulae by hand	46	(3.03 %)
Proved structural results only	21	(1.38 %)
Open complete results	33	(2.17 %)
Open structural results and formulae by hand	34	(2.24 %)
Open structural results only	61	(4.01 %)
Refuted complete results	21	(1.38 %)
Refuted structural results and formulae by hand	6	(0.40 %)
Refuted structural results only	0	(0.00 %)
No results	55	(3.62 %)
Total	1520	(100 %)

obtain, or when such expressions do not exist, *e.g.* because they correspond to solutions of an equation of degree 5 or more.

In some fairly frequent cases, complete results are simple and can be proved by AGX in a fully automated way; we then refer to them as *observations*. If results are structural, algebraic expressions for $\underline{b}(n)$ and $\overline{b}(n)$ can sometimes be deduced, in an assisted way, from the characterization of extremal graphs. In some fairly rare cases the graphs obtained by AGX and conjectured to be extremal present very little or no regularity and no results are obtained.

In each case, *i.e.*, each bound, graphs with 5 to 20 vertices were considered. Computing time on Intel Xeon with 2.66 GHz and 2 Gb RAM, at that moment, varied from less than 1 second in the frequent case in which a bound could be obtained automatically, without using VNS, up to 75 seconds per graph in the most complex cases, whether results were obtained or not. Trying longer computing times did not give better results.

Among all bounds conjectured in [10], 128 remain open, and among all possible cases, AGX failed to find a conjecture or a false conjecture in 82 cases. Under the assumption that these open conjectures are difficult to prove and that AGX failed when the cases are difficult to handle, we tried to figure out the reasons of these difficulties and we gathered the statistics summarized in Table 5 regarding the invariants, Table 6 regarding the operations and Table 7 regarding the bounds. In these tables, we use O, AO and SO for open, assisted open and structural open conjecture, respectively, and R and AR for refuted conjecture and refuted assisted conjecture. NR is used to say that no result is obtained in the corresponding case. T–O and T–R are used for the total over open conjectures and cases with no result or with refuted conjectures, respectively. Total indicates the sum of T–O and T–R.

In Table 5, the invariants are sorted in a decreasing way according to their total occurrences in the cases considered as difficult (open or refuted conjecture or no result is obtained in the corresponding case). According to these statistics, the most difficult invariant to handle is the domination number β with a total of 46 occurrences over 420 (10.95 %). The second most difficult invariant seems to be the Randić index Ra with 39 occurrences (9.51 %). Then comes a set of three invariants with 35 occurrences each (8.33 %). Two of these three invariants are eigenvalues, the index λ_1 and the algebraic connectivity a, and the third is a metric invariant, namely, the remoteness ρ. After that, we can find three sets each containing two invariants with almost the same occurrences: the average eccentricity ecc and the average distance $\bar{l}$ with 30 and 29 occurrences (7.14 % and 6.91 %), respectively, the proximity π and the independence number α with 25 occurrences each (5.95 %), and the radius r and the maximum degree Δ with 20 and 19 occurrences, respectively. The remaining nine invariants can be split into three sets each with three invariants with almost the same number of occurrences: the average degree $\overline{d}$, the diameter D (the maximum distance in a graph) and the chromatic number χ, with 14, 13 and 13 occurrences, respectively, form the first set, the minimum degree δ, the edge connectivity κ and the clique number ω, with 9, 9 and 8 occurrences, respectively, form another set, and finally, the set of the less frequent invariants is composed of the matching number μ, the girth g and the vertex connectivity ν with 6, 5 and 5 occurrences, respectively.

Table 5: Difficulties regarding the invariants.

Invariant	O	AO	SO	T–O	NR	R	AR	T–R	Total
β	9	11	12	32	12	0	2	14	46
Ra	9	12	6	27	8	4	0	12	39
λ_1	5	1	11	17	10	7	1	18	35
a	4	6	19	29	6	0	0	6	35
ρ	3	8	17	28	6	1	0	7	35
ecc	5	6	9	20	6	4	0	10	30
$\bar{l}$	3	5	9	17	9	2	1	12	29
π	6	3	8	17	8	0	0	8	25
α	5	4	7	16	7	2	0	9	25
r	5	4	2	11	8	1	0	9	20
Δ	0	2	3	5	12	1	1	14	19
$\overline{d}$	2	1	6	9	1	1	3	5	14
D	3	0	2	5	3	4	1	8	13
χ	0	2	6	8	2	2	1	5	13
δ	2	0	2	4	1	4	0	5	9
κ	2	1	0	3	2	4	0	6	9
ω	0	0	2	2	3	1	2	6	8
μ	1	1	0	2	3	1	0	4	6
g	1	0	1	2	1	2	0	3	5
ν	1	1	0	2	2	1	0	3	5

If we consider the difficulty with respect to the operations, it is easy to see that the product is the most difficult combination to handle. It occurs 79 times over 210 (37.62 %). The other three operations appear to present the same degree of difficulty: 41 occurrences (19.52 %) for the addition, 43 occurrences (20.48 %) for the subtraction and 47 occurrences (22.38 %) for the division.

Table 6: Difficulties regarding the operations.

Operation	O	AO	SO	T–O	NR	R	AR	T–R	Total
$-$	11	9	11	31	6	4	2	12	43
$+$	5	7	8	20	14	6	1	21	41
$/$	5	10	18	33	12	1	1	14	47
$\times$	12	8	24	44	23	10	2	35	79

If we distinguish between lower and upper bound, it is almost the same degree of difficulty in both cases even if the upper bounds seems to be slightly more difficult than the lower bound with 117 (55.71 %) cases among 210.

Table 7: Difficulties regarding the bounds.

Bound	O	AO	SO	T–O	NR	R	AR	T–R	Total
Lower	11	18	22	51	31	9	2	42	93
Upper	22	16	39	77	24	12	4	40	117

Among the bounds considered in the thesis [10], some were already known in graph theory literature. Among these results, we can cite

$$\delta \le \overline{d} \le \lambda_1 \le \Delta \quad [68];$$
$$\overline{l} \le \alpha \quad [64];$$
$$\chi \le \lambda_1 + 1 \quad [192];$$
$$a \le \tfrac{n\delta}{n-1} \quad [91];$$

where δ, $\overline{d}$ and Δ respectively denote the minimum, average and maximum degrees, $\overline{l}$ is the average distance, χ the chromatic number, α denotes the independence number (the maximum number of pairwise non adjacent vertices), λ_1 is the spectral radius of (the adjacency matrix of) a graph, and a denotes the algebraic connectivity (the second smallest eigenvalue of the Laplacian matrix of a graph).

Note that some of the above listed inequalities were obtained twice. For instance, the inequality $\lambda_1 \le \Delta$ was obtained as $\lambda_1 - \Delta \le 0$ and $\lambda_1/\Delta \le 1$.

Some of the bounds are naturally easy to obtain. When both invariants considered come from the same vector or matrix, say S, by taking its minimum ($m = \min S$), average $\overline{s} = \frac{1}{|S|}\sum_{s\in S} s$) or maximum value ($M = \max S$), it is obvious that

$$m \le \overline{s} \le M$$

with equality if and only if the entries of S are equal. Immediate consequences of this double inequality are

$$M - \overline{s} \ge 0, \quad \overline{s} - m \ge 0, \quad M - m \ge 0, \quad \frac{M}{\overline{s}} \ge 1, \quad \frac{\overline{s}}{m} \ge 1 \quad \text{and} \quad \frac{M}{m} \ge 1.$$

For example, for all connected graphs G with $n \ge 2$ vertices, diameter D (the maximum among all the distances in G) and average distance $\overline{l}$,

$$D - \overline{l} \ge 0 \quad \text{and} \quad \frac{D}{\overline{l}} \ge 1$$

with the equalities if and only if G is a complete graph.

There exists another kind of bounds easy to obtain. Actually, when the relevant families of extremal graphs for the invariants i_1 and i_2 are considered and if they have a non-empty intersection a proved and best possible bounding function is obtained. For our next example, we need the following definitions. The *eccentricity* $ecc(v)$ of a vertex v in G is the maximum among the distances from v to all other vertices in G. The *radius* $r = r(G)$ of a graph G is the maximum over the eccentricities of its vertices. The *Randić index* $Ra(G)$ of a graph $G = (V,E)$, introduced in [170], is defined by

$$Ra = Ra(G) = \sum_{uv\in E} \frac{1}{\sqrt{d_u d_v}},$$

where d_u and d_v denote the degree of the vertices u and v, respectively. It is well known that, on the one hand, the Randić index Ra is minimum for the star S_n, which is among the graphs that minimize the radius r, and on the other hand, Ra is maximum for any regular graph, and among the regular graph the cycle C_n maximizes r. Thus the following bounds are immediately obtained.

$$1+\sqrt{n-1} \leq Ra + r \leq \frac{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \sqrt{n-1} \leq Ra \cdot r \leq \frac{n}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor$$

with equality in both lower (resp. upper) bounds if and only if G is the star S_n (resp. cycle C_n).

Another example: the average distance $\bar{l}$ is minimum (resp. maximum) for the complete graph K_n (resp. path P_n) with $\bar{l}(K_n) = 1$ (resp. $\bar{l}(P_n) = (n+1)/3$, while the maximum degree Δ is maximum for K_n, with $\Delta(K_n) = n-1$, and minimum for P_n, with $\Delta(P_n) = 2$. Thus

$$2-n \leq \bar{l} - \Delta \leq \frac{n-5}{3} \quad \text{and} \quad \frac{1}{n-1} \leq \frac{\bar{l}}{\Delta} \leq \frac{n+1}{6}$$

with equality in both lower (resp. upper) bounds if and only if G is the complete graph K_n (resp. path P_n).

The other results were obtained as conjectures and can be divided into three types. A common step for all the three types is the VNS optimization. At that step, the optimization component of AutoGraphiX is executed and presumably extremal graphs are obtained. Then, a component, aimed for finding (linear) relations (see Section 2) between selected invariants, is executed. In case of success, we obtain a formulae: a lower bound for a minimizing problem or an upper bound for a maximizing problem. Thus, we get a conjecture containing a bound with corresponding extremal graphs and we speak about *complete conjectures*, that constitute the first type of results. Among such results, we cite the following theorems and conjectures.

Theorem 4.1 ([35]) *Let G be a connected graph on $n \geq 3$ vertices with index λ_1 and average distance $\bar{l}$. Then*

$$\lambda_1 + \bar{l} \leq n$$

with equality if and only if G is the complete graph K_n.

Conjecture 4.2 ([107]) *Let G be a connected graph on $n \geq 6$ vertices with signless Laplacian spectral radius q_1 and chromatic number χ. Then*

$$q_1 - \chi \leq \frac{3n-8}{2}$$

with equality if and only if G is the $\lfloor n/2 \rfloor$–partite graph $K_{p,2,2,\ldots,2}$, where $p = 2 + n \bmod (2)$.

A relation between the signless Laplacian index and the maximum degree, obtained by AGX, is proved by Cvetković, Rowlinson and Simić [69].

Theorem 4.3 ([69]) *Let G be a connected graph on $n \geq$ vertices with signless Laplacian index q_1 and maximum degree Δ. Then*

$$q_1 - \Delta \geq 1$$

with equality if and only if G is the star S_n.

The *girth* $g = g(G)$ of a connected graph G on $n \geq 3$ vertices with at least n edges, is the length (number of edges) of its smallest cycle. The following theorem, proved independently by Bekkai and Kouider [38] and in [19], was first conjectured by AGX.

Theorem 4.4 ([19, 38]) *Let G be a connected graph on $n \geq 3$ vertices and $m \leq n$ edges with girth g and average distance $\bar{l}$. Then*

$$\bar{l} \cdot g \leq \begin{cases} \frac{n^3}{4(n-1)} & \text{if } n \text{ is even,} \\ \frac{n^2+n}{4} & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \frac{\bar{l}}{g} \geq \begin{cases} \frac{n}{4(n-1)} & \text{if } n \text{ is even,} \\ \frac{n+1}{4n} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, both bounds are reached for cycles.

The *matching number* $\mu = \mu(G)$ of a graph is the maximum number of independent (pairwise non-incident) edges in G. The following result was conjectured using AGX and then proved by Stevanović [178].

Theorem 4.5 ([178]) *Let G be a connected graph, $G \ncong K_3$, on $n \geq 3$ vertices with adjacency index λ_1 and matching number μ. Then*

$$\lambda_1 - \mu \leq n - 1 - \left\lfloor \frac{n}{2} \right\rfloor$$

with equality if and only if G is the complete graph K_n. Also,

$$\frac{\lambda_1}{\mu} \leq \sqrt{n-1}$$

with equalities if and only if G is the star S_n.

Note that Stevanović [178] constructed an infinite family of counterexamples for the relation $\lambda_1 + \mu \geq \sqrt{n-1} + 1$ first conjectured by AGX.

When AGX could not provide a complete conjecture, an interactive procedure for recognizing the extremal graphs was launched. If the extremal graph are recognized and the corresponding formulas of the invariants under study are available in the database, substitutions are done and then bounds are obtained. The results so obtained are called *assisted conjectures*. First, recall that the *vertex* (resp. *edge*) *connectivity* $\nu = \nu(G)$ (resp. $\kappa = \kappa(G)$) of a connected graph G is the minimum number of vertices (resp. edges) whose removal disconnects G.

Theorem 4.6 ([79, 196]) *Let G be a connected graph on $n \geq 3$ vertices with index λ_1, vertex connectivity ν and edge connectivity κ. Then*

$$\lambda_1 - \nu \leq n - 3 + t; \quad \frac{\lambda_1}{\nu} \leq n - 2 + t; \quad \lambda_1 - \kappa \leq n - 3 + t; \quad \frac{\lambda_1}{\kappa} \leq n - 2 + t,$$

where t is such that $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2 - 3n + 1)t - 1 = 0$. Moreover, equalities hold if and only if G is the kite $Ki_{n,n-1}$.

Finding the bound in the above theorem in an automated way was not possible since it contains a factor that uses an implicit solution of a difficult to solve equation.

Another example with a complicated bound is the following theorem proved in [20].

Theorem 4.7 ([20]) *Let $G = (V, E)$ be a connected graph of order n with independence number α and maximum degree Δ. Then*

$$\alpha - \Delta \leq \max\left\{ \left\lfloor n - \frac{n-1}{\lceil \sqrt{n-1} \rceil} \right\rfloor - \left\lceil \sqrt{n-1} \right\rceil, \left\lfloor n - \frac{n-1}{\lfloor \sqrt{n-1} \rfloor} \right\rfloor - \left\lfloor \sqrt{n-1} \right\rfloor \right\}.$$

The bound is reached for every n.

For the above theorem, the difficulty is in the fact that the bound is an integer that implies a combination of fractions and square roots of integers. A similar difficulty is encountered in the next bound.

Theorem 4.8 ([35]) *Let G be a connected graph on $n \geq 2$ vertices with index λ_1 and independence number α. Then*

$$\alpha + \lambda_1 \leq \frac{n + \alpha' - 1 + \sqrt{(n - \alpha' - 1)^2 + 4\alpha'(n - \alpha')}}{2},$$

with equality if and only if G is the complete split graph $CS(n, n - \alpha')$, where α' is given by

$$\alpha' = \begin{cases} \left\lfloor \frac{n+1+\sqrt{n^2-n+1}}{3} \right\rfloor & \text{for } n = 3k \text{ or } n = 3k+2, \\ \left\lceil \frac{n+1+\sqrt{n^2-n+1}}{3} \right\rceil & \text{for } n = 3k+1. \end{cases}$$

Finally, when the recognition of the extremal graphs succeeded, but no formulae were found, we state a conjecture about the structure of the extremal graphs. In this case, we speak about *structural conjectures*.

The well–known result, in spectral graph theory, $\lambda_1(G) \geq \overline{d}(G)$ with equality if and only if G is a regular graph, was proved by Collatz and Sinogowitz [65] in 1957. Then, they proposed to consider the difference between the index and the average degree as a measure of the *irregularity* of a graph (other definitions of irregularity in graphs have been proposed, see [3, 40], and for a comparison between them see [98]). Thus the irregularity of a graph G is defined by $Irr(G) = \lambda_1(G) - \overline{d}(G)$. The problem of finding an upper bound on the irregularity and characterizing the most irregular graphs remains open. The following conjecture related to the irregularity of a graph have been formulated after some experiments with the system AGX. First, we need the following definition. A *pineapple* with parameters n, q $(q \leq n)$, denoted by $PA(n,q)$, is a graph on n vertices consisting of a clique (a set of pairwise adjacent vertices) on q vertices and an independent set (a set of pairwise non-adjacent vertices) on the remaining $n-q$ vertices in which each vertex of the independent set is adjacent to a unique and the same vertex of the clique. Some pineapples are illustrated in Figure 14.

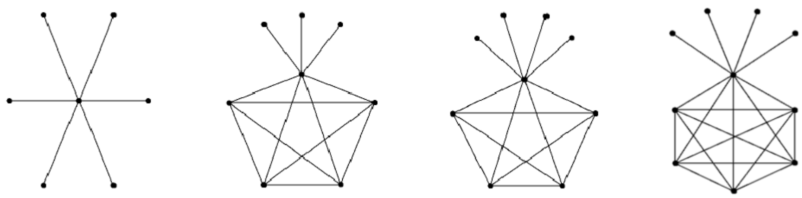

Figure 14: Presumably most irregular graphs for $n = 7, 8, 9, 10$.

Conjecture 4.9 ([11, 14]) *The most irregular connected graph on n $(n \geq 10)$ vertices is a pineapple $PA(n,q)$ in which the clique size q is equal to $\lceil \frac{n}{2} \rceil + 1$.*

The issue in the above theorem, as well as in the next, is the difficulty to get an explicit formulae of the index for some classes of graphs.

Theorem 4.10 ([19]) *Over all connected graphs on $n \geq 4$ vertices and $m \geq n$ edges with girth g and index λ_1, $g + \lambda_1$ is maximum for the kite $Ki_{n,3}$ (see Figure 15 for $Ki_{9,3}$). Moreover, for each $t > 0$, there exists an integer n_t such that for all $n \geq n_t$, $3 + \sqrt{5} - t < g(Ki_{n,3}) + \lambda_1(Ki_{n,3}) < 3 + \sqrt{5}$.*

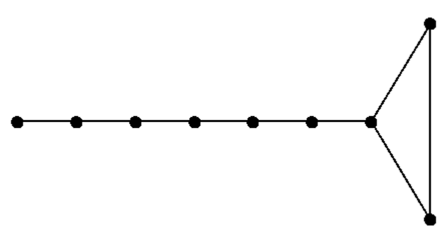

Figure 15: $Ki_{9,3}$: an extremal graph in Theorem 4.10.

A study similar to that of [10] was done by Hansen and Lucas [107] where the signless Laplacian spectral radius q_1 is compared to 19 other graph invariants. The results, to which belongs Conjecture 4.2, are summarized in Table 9.

In [175], Sedlar, Vukičević and Hansen introduced a first generalization of AGX Form 1 to AGX Form 2:

$$\underline{b}(m) \leq i_1 \oplus i_2 \leq \overline{b}(m) \tag{3}$$

in which the lower and upper bounding functions $\underline{b}(m)$ and $\overline{b}(m)$ depend on the size m (or number of edges) of the graph instead of its order. Otherwise the symbols have the same meaning and assumptions are the same. Among the AGX Form 2 results, we give the following theorem.

Theorem 4.11 ([175]) *Let G be a connected graph with size $m \geq 1$, radius r and minimum degree δ. Let k and l be integers such that $m = k(k-1)/2 - l$, where $0 \leq l < k-1$. Then*

$$\left.\begin{array}{ll} \text{if } l = 0, & 2-k \\ \text{if } 0 < l < k/2, & 3-k \\ \text{if } k/2 \leq l \leq k-1, & 4-k \end{array}\right\} \leq r - \delta \leq \left\lfloor \frac{m+1}{2} - 1 \right\rfloor; \tag{4}$$

$$\left.\begin{array}{ll} \text{if } l = 0, & 1/(1-k) \\ \text{if } 0 < l < k/2, & 1/(2-k) \\ \text{if } k/2 \leq l \leq k-1, & 1/(3-k) \end{array}\right\} \leq \frac{r}{\delta} \leq \left\lfloor \frac{m+1}{2} \right\rfloor; \tag{5}$$

$$2 \leq r + \delta \leq \left\lfloor \frac{m}{2} \right\rfloor + 2; \tag{6}$$

$$1 \leq r + \delta \leq \left\lfloor \frac{m}{2} \right\rfloor. \tag{7}$$

The lower bounds for (4) and (5) are attained by the complete graph K_k if $l = 0$, by $K_k \setminus M_l$, where M_l is a matching containing l edges, if $0 < l \leq k/2$ and by $K_k \setminus C_l$, where C_l is a cycle containing l edges, if $k/2 < l < k-1$. The lower bounds for (6) and (7) are attained by the star S_{m+1}.

The upper bounds for (4) and (5) are attained by the path P_{m+1}. The upper bounds for (6) and (7) are attained by the cycle C_m.

5 Other forms

Besides bounding invariants and bounds of AGX Form 1, several results of different forms were studied using AutoGraphiX. In this section, we report on relations that do not belong to those described in the two previous sections. As a first example, we give relations involving more than two graph invariants, in addition to the order n. Such relationships are rare in the graph theory literature. A second example is a result about one invariant, in which we consider the behavior of the invariant instead of its minimum or maximum values. Other examples are given below and more can be found in Aouchiche, Bell, Cvetković, Hansen, Rowlinson, Simić, Stevanović [11], Aouchiche, Caporossi and Hansen [14], Aouchiche and Hansen [27], Caporossi and Hansen [54], Cvetković, Rowlinson and Simić [69], Cvetković and Simić [70, 71, 72], Cvetković, Simić, Caporossi and Hansen [74], Hansen and Mélot [109],

Any tree is a bipartite graph and therefore its vertex set can be partitioned into two independent subsets. Let a be the number of vertices in one subset and b in the other. In this case, we speak about an (a,b)–partition. Assume, without loss of generality that $a \geq b$ and let $\mathscr{T}_{a,b}$ be the class of all trees that can be partitioned into an (a,b)–partition. In [74], the authors considered the problem of finding extremal trees $T \in \mathscr{T}_{a,b}$ with respect to the adjacency index $\lambda_1(T)$, *i.e.*, solving the problems

$$\min_{T \in \mathscr{T}_{a,b}} \lambda_1(T) \quad \text{and} \quad \max_{T \in \mathscr{T}_{a,b}} \lambda_1(T)$$

for given a and b. Among their results, we recall the following two theorems and conjecture.

Theorem 5.1 ([47, 74]) *For fixed order $n = a + b$ and for $T \in \mathscr{T}_{a,b}$, the minimal value of $\lambda_1(T)$ increases monotonously with $a - b$.*

Conjecture 5.2 ([47, 74]) *A vertex from the subset with a vertices in a minimal tree over the class $\mathscr{T}_{a,b}$, with respect to λ_1, has degree 1 or 2.*

For the statement of the next conjecture, we need the following definition. A *comet* $Co_{n,\Delta}$ is the tree obtained from a star S_Δ by inserting $n - \Delta$ vertices (of degree 2) into the same edge.

Theorem 5.3 ([47, 74]) *For $a = b+2$ and $n = a+b \geq 6$, trees $T^* \in \mathscr{T}_{a,b}$ with minimal λ_1 are comets $Co_{n,4}$. Moreover*

$$\lim_{n \to +\infty} \lambda_1(T^*) = 2.$$

In [54], after experiments using AutoGraphiX on trees in $\mathscr{T}_{a,b}$ with fixed a and b, the authors obtained the following unexpected conjecture involving five invariants.

Conjecture 5.4 ([47, 54]) *For fixed integers a and b, let* $T \in \mathscr{T}_{a,b}$ *with size m, independence number* α*, diameter D, radius r and* n_1 *pendent edges. Then*

$$2\alpha - m - n_1 + 2r - D = 0.$$

The above conjecture is not valid for the class of trees in general. Experiments done in [47, 54] with AutoGraphiX led to the following theorem, first obtained as a conjecture.

Theorem 5.5 ([47, 54]) *Let T be a tree on n vertices and m edges with independence number* α*, diameter D, radius r and* n_1 *pendent edges. Then*

$$m + n_1 + D - 2r - \left\lfloor \frac{n-2}{2} \right\rfloor \leq 2\alpha \leq m + n_1 + D - 2r$$

In 1956, Nordhaus and Gaddum [161] proved that

$$2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n+1 \qquad \text{and} \qquad n \leq \chi(G) \cdot \chi(\bar{G}) \leq \frac{(n+1)^2}{4},$$

where χ is the chromatic number of a graph. Finck [92] showed that these bounds were sharp (taking floors and ceilings if necessary) and characterized extremal graphs. Similar bounds were obtained for a large number of graph invariants by a variety of authors. Let $i(G)$ denote a graph invariant. Classical Nordhaus-Gaddum relations are of the following form:

$$l_1(n) \leq i(G) + i(\bar{G}) \leq u_1(n) \qquad \text{and} \qquad l_2(n) \leq i(G) \cdot i(\bar{G}) \leq u_2(n).$$

In more general form, the lower and upper bounding functions may depend on several variables. For an extensive survey of such relations see [23] and over 350 references therein. Here, we are interested in Nordhaus–Gaddum relations only for the index. Nosal [162] and Amin and Hakimi [4] independently proved that

$$n - 1 \leq \lambda_1(G) + \lambda_1(\bar{G}) \leq \sqrt{2}(n-1).$$

The lower bound (attained if and only if the graph is regular) has been proved independently by Nosal [162] in 1970 and Amin and Hakimi [4] in 1972, and has been improved in 2007 by Nikiforov [160] to

$$\lambda_1(G) + \lambda_1(\overline{G}) \geq n - 1 + \sqrt{2}\frac{div^2(G)}{n^3},$$

where $div(G) = \sum_{u \in V(G)} \left| d(u) - \frac{2m}{n} \right|$.

The best bound known up to now is proved by Csikvári [67] in 2009:

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \frac{1+\sqrt{3}}{2} n - 1.$$

The problem of finding an upper bound for the index of the Nordhaus–Gaddum type was studied using AGX [11, 14]. The AutoGraphiX conjecture about the upper bound is as follows.

Conjecture 5.6 ([11, 23]) *For any simple graph G, with complement* $\overline{G}$*, index* $\lambda_1(G)$ *and n vertices we have*

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \frac{4}{3}n - \frac{5}{3} - \begin{cases} f_1(n) & if \quad n \bmod (3) = 1 \\ 0 & if \quad n \bmod (3) = 2 \\ f_2(n) & if \quad n \bmod (3) = 0, \end{cases}$$

where $f_1(n) = \frac{3n-2-\sqrt{9n^2-12n+12}}{6}$ *and* $f_2(n) = \frac{3n-1-\sqrt{9n^2-6n+9}}{6}$.

This bound is sharp and attained if and only if G or $\overline{G}$ *is a complete split graph with an independent set on* $\lfloor \frac{n}{3} \rfloor$ *vertices (and also on* $\lceil \frac{n}{3} \rceil$ *vertices if* $n \bmod (3) = 2$*).*

We shall describe in some detail the use of AGX in formulating Conjecture 5.6.

Additional experiments have shown that maximal graphs for $\lambda_1 + \overline{\lambda}_1$ for given n and m are complete split graphs or fanned complete split graphs with a few exceptions.

When looking for extremal graphs with the system AGX, using Variable Neighborhood Search metaheuristic, we defined the objective function as $\lambda_1(G) + \lambda_1(\overline{G})$ to be maximized over the class of all graphs of order from 4 to 24. To be coherent in our investigations, we required the graph G, but not necessarily its complement $\overline{G}$, to be connected. This constraint is without loss of generality because of the fact that at least one of the complementary graphs G and $\overline{G}$ is connected.

For a fixed order n, the extremal graph G is composed of a clique on q vertices and an independent set with s vertices in which every vertex is connected to all vertices of the clique. When we observed the values of q and s for different graphs, we found the following:

$$q = \begin{cases} \lfloor \frac{n}{3} \rfloor & if \quad n\, mod(3) = 1 \\ \frac{n}{3} & if \quad n\, mod(3) = 0 \end{cases} \qquad \text{and} \qquad s = \begin{cases} \lceil \frac{2n}{3} \rceil & if \quad n\, mod(3) = 1 \\ \frac{2n}{3} & if \quad n\, mod(3) = 0. \end{cases}$$

While the experiments show regularity for the cases $n\, mod(3) = 0$ and $n\, mod(3) = 1$, it was not the case when $n\, mod(3) = 2$. Sometimes we have $q = \lfloor \frac{n}{3} \rfloor$ and $s = \lceil \frac{2n}{3} \rceil$ and at other times, we have $q = \lceil \frac{n}{3} \rceil$ and $s = \lfloor \frac{2n}{3} \rfloor$. We decided to examine the two cases interactively on *AGX* for every n up to 24, and we observed that the objective function has the same value in both cases ($q = \lfloor \frac{n}{3} \rfloor$ or $q = \lceil \frac{n}{3} \rceil$).

AGX did not find any conjecture on the relation between the objective function $\lambda_1(G) + \lambda_1(\overline{G})$ and the order when using all the presumably extremal graphs obtained by AGX. But when we separated the set of graphs into three subsets, with $n\ mod(3) = 0$ for the first subset, $n\ mod(3) = 1$ for the second one and $n\ mod(3) = 2$ for the third one, AGX did not find anything about the two first subsets but suggested the following linear relation for the third one ($n\ mod(3) = 2$)

$$\lambda_1(G) + \lambda_1(\overline{G}) = \frac{4}{3}n - \frac{5}{3}.$$

The difficulty in proving Conjecture 5.6 is that we have almost no lemmas on the behavior of the corresponding invariant under local graph transformations. Experiments with GRAPH [73], newGRAPH [180] and AGX could be useful in producing conjectures for such lemmas (e.g. adding an edge, rotating an edge etc.). Some partial results about that conjecture can be found in [11, 14].

The problem of finding Nordhaus–Gaddum inequalities was also considered with AutoGraphiX for the two other invariants.

The *transmission* $t(v)$ of a vertex v in a connected graph G, is the sum of the distances from v to all other vertices in G. It is said to be *normalized*, and then denoted $\tilde{t}(v)$, when divided by $n-1$. The *proximity* $\pi = \pi(G)$ and *remoteness* $\rho = \rho(G)$ [10, 15] of G are, respectively, the minimum and the maximum normalized transmission in G. That is

$$\pi = \min_{v \in V} \tilde{t}(v) \qquad \text{and} \qquad \rho = \max_{v \in V} \tilde{t}(v).$$

Some properties of proximity and remoteness are studied in [10, 15, 25, 22, 172]. In [27], the authors derived and proved Nordhaus–Gaddum type inequalities for π and for ρ. The results are stated below.

Theorem 5.7 ([27]) *For any connected graph G on $n \geq 5$ vertices for which $\overline{G}$ is connected*

$$\frac{2n}{n-1} \leq \pi + \overline{\pi} \leq \begin{cases} \frac{n+1}{4} + \frac{n+1}{n-1} & \text{if } n \text{ is odd,} \\ \frac{n}{4} + \frac{n}{4(n-1)} + \frac{n+1}{n-1} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound is attained if and only if $\Delta(G) = \Delta(\overline{G}) = n-2$. The upper bound is attained if and only if either G or $\overline{G}$ is the cycle C_n.

Theorem 5.8 ([27]) *For any connected graph G on $n \geq 5$ vertices for which $\overline{G}$ is connected*

$$\frac{n^2}{(n-1)^2} \leq \pi \cdot \overline{\pi} \leq \begin{cases} \frac{(n+1)^2}{4(n-1)} & \text{if } n \text{ is odd,} \\ \frac{n(n+1)}{4(n-1)} + \frac{n(n+1)}{4(n-1)^2} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound is attained if and only if $\Delta(G) = \Delta(\overline{G}) = n-2$. *The upper bound is attained if and only if either* G *or* $\overline{G}$ *is the cycle* C_n.

Theorem 5.9 ([27]) *For any connected graph* G *on* $n \geq 6$ *vertices for which* $\overline{G}$ *is connected*

$$3 \leq \rho + \overline{\rho} \leq \frac{n+2}{2} + \frac{2}{n-1}.$$

The lower bound is attained if and only if $n \geq 8$, G *is regular and* $D = \overline{D} = 2$. *The upper bound is attained if and only if* G *or* $\overline{G}$ *is the path* P_n, *the comet* $Co_{n,3}$ *or the path-complete graph* $PK_{n,n}$ *when* $n \geq 7$, *and if and only if* G *or* $\overline{G}$ *is the path* P_6, *the comet* $Co_{6,3}$, *the path-complete graph* $PK_{6,6}$ *or one of the graphs in Figure 16.*

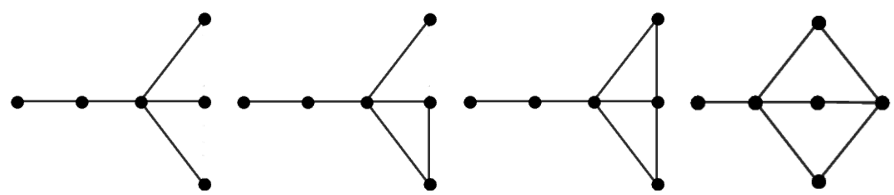

Figure 16: Graphs with $D = 3$ that maximize $\rho + \overline{\rho}$ for $n = 6$.

Theorem 5.10 ([27]) *For any connected graph* G *on* $n \geq 7$ *vertices for which* $\overline{G}$ *is connected*

$$\rho \cdot \overline{\rho} \leq \begin{cases} \frac{16n+20}{27} + \frac{8}{9(n-1)} + \frac{4}{27(n-1)^2} & \text{if} \quad n = 0 \ (mod\ 3), \\ \frac{16n+20}{27} + \frac{2}{3(n-1)} & \text{if} \quad n = 1 \ (mod\ 3), \\ \frac{16n+20}{27} + \frac{8}{9(n-1)} + \frac{5}{27(n-1)^2} & \text{if} \quad n = 2 \ (mod\ 3). \end{cases}$$

The upper bound is the best possible as shown by the comets $Co_{n,\lceil \frac{n}{3} \rceil + 1}$, *and* $Co_{n,\lceil \frac{n}{3} \rceil}$ *if* $n = 1 \ (mod\ 3)$.

Recall that the *Laplacian* of a graph is the matrix defined by $L = Deg - A$, where Deg is the diagonal matrix whose diagonal entries are the vertex degrees in G and A is the adjacency matrix of G. The *Laplacian eigenvalues* of a graph G are those of its Laplacian matrix. The are usually denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and indexed such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Among the relations obtained using AutoGraphiX and where more than two invariants were involved, we cite the following three conjectures.

Conjecture 5.11 ([69]) *Let* G *be a connected graph on* $n \geq 4$ *vertices with signless Laplacian index* q_1, *adjacency index* λ_1 *and average degree* $\overline{d}$. *Then*

$$q_1 - \lambda_1 - \overline{d} \leq n - 2 - \sqrt{n-1} + \frac{2}{n}$$

with equality if and only if G *is the star* S_n.

Conjecture 5.12 ([69]) *Let* G *be a connected graph on* $n \geq 4$ *vertices with signless Laplacian index* q_1, *Laplacian index* μ_1 *and adjacency index* λ_1. *Then*

$$\mu_1 + \lambda_1 - q_1 \leq \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil}$$

with equality if and only if G *is the complete bipartite graph* $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Conjecture 5.13 ([69]) *Let* G *be a connected graph on* $n \geq 4$ *vertices with signless Laplacian index* q_1, *smallest signless Laplacian eigenvalue* q_n *and independence number* α. *Then*

$$q_1 + q_n + 2\alpha \leq 3n - 2$$

with equality if and only if G *is the complete split graph* $CS(n, n-\alpha)$.

All the results discussed above deal, besides the bounding functions, also with the characterizations of the corresponding extremal graphs. AGX can also be used for finding characterizations of a given class of graphs. For instance, in [69], a characterization of the complete graph is obtained using AutoGraphiX.

Theorem 5.14 ([69]) *Let $e(Q)$ be the number of distinct eigenvalues of the signless Laplacian of a graph and $m(q_i)$ the multiplicity of the eigenvalue q_i of Q. Then*

$$e(Q) = 2 \iff m(q_2) = n-1 \iff G \cong K_n.$$

In this case $q_2 = n-2$.

6 Extensions

In the previous sections, we discussed several forms of conjectures obtained using the AutoGraphiX system, and therefore exploiting variable neighborhood search techniques. In this section we discuss forms of conjectures that can potentially be explored with the help of AGX. In [104], the authors addressed the following question: *What makes a mathematical result interesting?* Despite its obvious interest, this question of mathematical philosophy is scarcely discussed. Views of several famous scientists on this topic are interspersed with discussions of graph theoretical conjectures in the large *Written on the wall* file of Fajtlowicz [86]. Colton [66] and Larson [137], also address this question in detail. Some criteria of interestingness of a mathematical result were set in [104]:

- *Simplicity*: simple formulae are the most used ones, and thus the most likely to have many consequences. They also have the most potential falsifiers, as explained by Popper in his famous book *The Logic of scientific discovery* [167]. However, it may be hard to find many simple, new and true formulae. Moreover, some of them may be trivial, e.g., that the clique number of a graph is not larger than its chromatic number.
- *Centrality*: conjectures should preferably involve the most central concepts of graph theory as e.g. connectedness, stability, colorability, and so forth. To illustrate, some new concepts proved to be interesting and lead to numerous results, as e.g. pancyclicity or having elementary cycles of all possible lengths, introduced by Bondy [44], which is close to the basic concept of cycle. This is far from being always the case for the numerous new concepts which nowadays proliferate and, to some extent, threaten the unity of graph theory.
- *Problem solving*: instead of considering centrality in terms of concepts, one may examine it in terms of problems posed by scientists in a given field. This leads to another criterion, again stated by Popper in [167]: *Only if it is the answer to a problem a difficult, a fertile problem, a problem of some depth does a truth, or a conjecture about the truth, become relevant to science. This is so in pure mathematics, and it is so in the natural sciences.*
- *Surprisingness*: Conways answer to the question "*What makes a good conjecture?*", according to [86], was "*It should be outrageous*". This means a trained mathematician finds something contrary to what suggests his well-educated intuition, and so gets a new insight. Of course, it remains to be examined whether some explanation may be found, together with new results, or the conjecture will remain an isolated curiosity.
- *Distance between concepts*: a conjecture will be the more interesting the farther the concepts involved are one from another. This implies an operational notion of distance, either in the conjecture-making program or possibly in a lattice of graph-theoretical concepts.
- *Information-content* relative to databases of conjectures and graphs. A conjecture is interesting if it tells more, for at least one graph than the conjunction of all other conjectures. It also means the conjecture should not be redundant. This criterion is discussed in [101].
- *Sharpness*: the conjecture should be best possible in the weak sense, *i.e.*, sharp for some values of the parameters, or in the strong sense, *i.e.*, sharp for all values of the parameters compatible with the existence of a graph [101].

In addition to such abstract criteria one might take a pragmatic view and say that a conjecture is interesting if it has attracted the attention of mathematicians, whoever they may be. This is fairly tautological. Note, moreover, that popularity of a result depends not only on its intrinsic merits but also on its visibility (Journal where it was published, computer systems which mention it or give access to it, as well as relations and aptitude for marketing of its authors).

The following observations on AutoGraphiX behavior can be made in view of the results discussed above: simplicity, the AGX Form 1 and 2 are quite simple. Other simple forms could be explored also, as suggested in [104]; centrality, the invariants are chosen by the user of AGX who may focus on central ones; problem solving, AGX has been used to find conjectures in subfields of graph theory which are our topics, the main example being the study of the signless Laplacian of graphs [33, 69, 70, 71, 72, 107]; surprisingness, the generalization of Chung's theorem it i.e., $\bar{l} \leq \alpha$ to $\bar{l} \leq \alpha_2/2$ [106], where α_2 denotes the maximum cardinality of an induced bipartite subgraph. The generalization was done with the help of AGX and is surprising, particularly as no similar generalization holds for α_3, the cardinality of the maximum cardinality of an induced tripartite subgraph; distance between concepts has not been systematically studied yet. It could be a good guide for invariant selection; information-content, the concept of proximity and remoteness of a graph are easily derived from the concept of transmission. They have the advantage of being of order n as are several other distance based invariants, e.g., the radius and the diameter; sharpness, most of AGX conjectures are sharp. It is the case each time complete results are obtained.

Let us also mention that AGX results can suggest ideas of proof. A good example is the proof of the upper bound on the irregularity done in [60] (see the discussion on Theorem 3.10 in the present paper).

Another task for which AGX is proven to be useful is extending bounds about an invariants to another (but close) invariant. Actually, AutoGraphiX was successfully used by Aouchiche, Favaron and Hansen [20] to extend a series of bounds on the independence number, proved in [10, 13, 30], to bounds on the *upper irredundance*. For more details on upper irredundance see [87, 88, 121].

All conjectures obtained using AutoGraphiX are algebraic relations between graph invariants. The forms of the conjectures are of different types. Conjectures of AGX Form 1 (see Section 4) were systematically generated for more than 20 graph invariant. Conjectures of AGX Form 2 type were systematically generated for few invariants (see [175]). Thus AGX Form 2 type relations remain to be explored for many invariants.

A well–known relation on the *chromatic index* χ' of a graph (the minimum number of colors to assign to the edge of a graph such that any two incident edge are assigned different colors) is the double inequality $\Delta \leq \chi' \leq \Delta + 1$ proved by Vizing [188] and where Δ denote the maximum degree. The Vizing double inequality can be generalized in a natural way. Let i_1 and i_2 be two graph invariants and consider the problem of finding two function $L(i_2)$ and $U(i_2)$ such that

$$L(i_2) \leq i_1 \leq U(i_2)$$

for all graphs or at least for a given class of graphs. A basic relation of this type is that, well-known, between the order n and size m among the class of connected graphs:

$$n-1 \leq m \leq \frac{n(n-1)}{2}$$

with equality for the lower bound if and only if the graph is a tree, and for the upper bound if and only if the graph is complete. Contrary to the double inequality, the case of a single inequality (lower or upper bounding of a graph invariant) is widely studied in graph theory. This generalized form could be studied systematically using the AutoGraphiX system.

Another kind of results that can be studied with the help of AGX it the behavior of an invariant i_1 with respect to another invariant i_2, *i.e.*, *qualitative relations* between graph invariants. Results of this type are rare in graph theory, but quite frequent in other domains of sciences. Qualitative relations between i_1 and i_2 can be expressed by

invariant i_1 increases when invariant i_2 increases,

or

invariant i_1 increases when invariant i_2 decreases,

or using the usual differentiation notation:

$$\frac{\partial i_1}{\partial i_2} > 0 \qquad \text{or} \qquad \frac{\partial i_1}{\partial i_2} < 0.$$

Theorem 5.1 illustrates well such type of results.

The statement of sufficient conditions for a graph G to belong to a given class of graphs is a kind of results widely studied in graph theory. A few examples and well-known results of this type are gathered in the following theorem.

Theorem 6.1 *A graph G of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ is Hamiltonian if one of the following conditions holds:*

- $d_k \geq n/2$ *for all* $k = 1, 2, \ldots n$ *(Dirac [83]);*
- $d_u + d_v \geq n$ *for all pairs of non adjacent vertices* u *and* v *(Ore [164]);*
- $d_k > k$ *for all* k *with* $1 \leq k \leq n/2$ *(Pósa [168]);*
- $d_i + d_j \geq n$ *for all* i, j *with* $d_i \leq i$ *and* $d_j \leq j - 1$ *(Bondy [45]).*

The above conditions are expressed in terms of graph parameters, thus the use of AutoGraphiX in investigating then could be fruitful. Such investigations may be extended to the case of necessary conditions or both necessary and sufficient conditions in which case we speak about characterizations of classes of graphs.

Among the earliest results in graph theory the following: *a tree is a connected graph without cycles*. If one wants to express this result in terms of graph class, it could be *the class of trees is the intersection between the class of connected graphs and that of graphs without cycles*. The question that generalizes such a result could be *which class of graphs is an intersection of two or more classes*? Instead of speaking about *equals an intersection*, we may consider only inclusion *i.e., all graphs in class* $\mathscr{C}_1$ *belong to class* $\mathscr{C}_2$. A well–known result of this type: *all trees are bipartite graphs*. In the case of double inclusion, of course, we speak about equality such as in the Kuratowski theorem.

Theorem 6.2 ([135]) *A graph G is planar if and only if it does not contain an induced subgraph homeomorphic to* K_5 *or to* $K_{3,3}$.

In the graph theory literature, it is quite frequent to find that a relation R_1 can be deduced, immediately or by means of some algebraic manipulations, from another relation R_2. In such a case, we speak about *implication between relations*: $R_2 \Rightarrow R_1$. In some cases there is a double implication, and therefore, an *equivalence*: $R_2 \Leftrightarrow R_1$. For instance, The lower bound, proved by Berge [41], on the independence number α of a graph G on n vertices and m edges,

$$\alpha \geq \frac{n^2}{2m+n}$$

is implied by that proved by Favaron, Mahéo and Saclé [89],

$$\alpha \geq \left\lceil \frac{2n - \frac{2m}{\lceil \frac{2m}{n} \rceil}}{\lceil \frac{2m}{n} \rceil + 1} \right\rceil$$

which is equivalent to the lower bound proved by Hansen [103],

$$\alpha \geq \left\lceil n - \frac{2m}{1 + \lfloor \frac{2m}{n} \rfloor} \right\rceil + \left\lceil \frac{n - \left\lceil n - \frac{2m}{(1+\lfloor \frac{2m}{n} \rfloor)(1+\lfloor \frac{2m}{n} \rfloor)} \right\rceil}{2 + \lfloor \frac{2m}{n} \rfloor} \right\rceil,$$

and is best possible.

The use of the computer in the study of all these kinds of results could be fruitful and remains to be done.

Appendix

In the next table, we summarize the results of AGX Form 1 type obtained and studied in [10]. The invariants involved in that table are defined in Table 3. The first column of Table 8 contains the different combinations of pairs of invariants. Columns 1, 2 and 3 contain, when available, the lower bound, an extremal graph corresponding to the bound and its the status, respectively. The three last columns contain, when available, the lower bound, an extremal graph corresponding to the bound and its the status, respectively. We use the following notation for the status of the bound: K for a known result, T when the bound is trivial, O for an open conjecture with a formulae and extremal graphs, SO for an open structural conjecture and NR to indicate that no results were obtain in that case. Finally, when a result is proved, we refer to the paper containing the proof by its number in the bibliography.

For the extremal graphs, in addition to the notations defined above, we the following ones: ReG for an arbitrary (degree) regular graph; $TReG$ for an arbitrary transmission regular graph (all vertices having the same transmission); $Tu_{n,g}$ for a *turnip*, *i.e.*, the graph obtained from a cycle C_g by adding $n-g$ pending edges incident to the same vertex from the cycle; T_n^k for a balanced complete k–partite graph, *i.e.*, a complete k-partite graph in which the cardinalities of any two independent sets differ by at most 1; PTP for the graph obtained from a path P_{n-6} by attaching a triangle at each of its endpoints; Ctr for a caterpillar, *i.e.*, the tree obtained from a path by attaching pending edges at its internal vertices; Ur_n for an urchin graph, *i.e.*, the graph obtained from a clique $K_{\lceil n/2 \rceil}$ by attaching a pending edge at each of $\lfloor n/2 \rfloor$ of its vertices; $Tree$ for an arbitrary tree; $K_n - e$ for the graph obtained from K_n by the deletion of an edge; $K_n - M$ for the graph obtained from K_n be the deletion of $\lfloor n/2 \rfloor$ disjoint edges; $K_n - R$ for the graph obtained from K_n be the deletion of $n/2$ disjoint edges if n is even, or the deletion of $(n-3)/2$ disjoint edges and a path on the three vertices that are not incident to the deleted edges; $Clqs$ for a graph composed of a set of disjoint cliques of almost equal size connected with at most one edge between two cliques such that there is not cycle that is not entirely included in a clique; $Bag_{p,q}$ for a graph obtained from a complete graph K_p by replacing an edge with a path P_q; Bug_{p,q_1,q_2} for a graph obtained from a complete graph K_p by deleting an edge uv and attaching paths P_{q_1} and P_{q_2} at u and v, respectively; K_p^q the graph on $p+q-1$ vertices obtained from two cliques K_p and K_q by the coalescence of two vertices, one from each cliques; Ke_p^q the graph on $p+q$ vertices obtained from two cliques K_p and K_q by adding an edge between the cliques; $DC_{n,p,q}$ for a *double comet* on n vertices maximum and second maximum degrees p and q, *i.e.*, the tree obtained from two stars S_{p+1} and S_{q+1} and a path $P_{n-p-q-2}$ by adding an edge between an endpoint of the path to a pendent vertex from S_{p+1} another edge between the other endpoint of the path to a pendent vertex from S_{q+1}.

When a family of graphs is recognized but there is not enough regularity to derive its parameters, we use x to denote the missing parameter, such as in $Ki_{n,x}$ corresponding to the extremal graphs for the upper bound on $\overline{d}/a$, or x and y whenever there are two parameters such as in $DC_{n,x,y}$ corresponding to the extremal graphs for the upper bound on $\overline{l}-\beta$.

Table 8 bellow reads as follows. Consider the bloc corresponding the comparison of the average degree $\overline{d}$ with the algebraic connectivity a, *i.e.*,

$\overline{d}-a$	-1	K_n	[10]	$n-4+4/n$	$Ki_{n,n-1}$	O
$\overline{d}+a$	$4-\frac{2}{n}-2\cos\frac{\pi}{n}$	P_n	T	$2n-1$	K_n	T
$\overline{d}/a$	$\frac{n-1}{n}$	K_n	[10]		$Ki_{n,x}$	SO
$\overline{d}\cdot a$	$\left(4-\frac{4}{n}\right)\left(1-\cos\frac{\pi}{n}\right)$	P_n	T	$n(n-1)$	K_n	T

The lower and upper bounds on $\overline{d}+a$ and $\overline{d}\cdot a$ are trivial (T), so they can be stated as observations only:

Observation: *Let G be a graph on n vertices with average degree $\overline{d}$ and algebraic connectivity a. Then*

$$4-\frac{2}{n}-2\cos\frac{\pi}{n} \le \overline{d}+a \le 2n-1 \quad \textit{and} \quad \left(4-\frac{4}{n}\right)\left(1-\cos\frac{\pi}{n}\right) \le \overline{d}\cdot a \le n(n-1).$$

Moreover, the bounds are the best possible as shown by the P_n for both lower bounds and by the complete graph K_n for both upper bounds.

The lower bounds on $\overline{d}-a$ and $\overline{d}/a$ are not trivial and are proved in [10], so they can be stated as a theorem:

Theorem: *Let G be a graph on n vertices with average degree $\overline{d}$ and algebraic connectivity a. Then*

$$\overline{d}-a \ge -1 \quad \textit{and} \quad \overline{d}/a \ge \frac{n-1}{n}.$$

Moreover, the bounds are the best possible as shown by the complete graph K_n.

The upper bounds on $\overline{d}-a$ and $\overline{d}/a$ are stated as conjectures:

Conjecture: *Let G be a graph on n vertices with average degree $\overline{d}$ and algebraic connectivity a. Then*

$$\overline{d}-a \le n-4+\frac{4}{n},$$

and the bound is the best possible as shown by the short kite $Ki_{n,n-1}$; and $\overline{d}/a$ is maximum for some kite.

Table 8: List of AGX conjectures obtained in [10].

$i_1 \oplus i_2$	lower bound	G	st.	upper bound	G	st.
$\Delta - \delta$	0	ReG	T	$n-2$	S_n	T
$\Delta + \delta$	3	P_n	T	$2n-2$	K_n	T
Δ / δ	1	ReG	T	$n-1$	S_n	T
$\Delta \cdot \delta$	2	P_n	T	$(n-1)^2$	K_n	T
$\Delta - \overline{d}$	0	ReG	T	$(n-1)(n-2)/n$	S_n	T
$\Delta + \overline{d}$	$4-2/n$	P_n	T	$2n-2$	K_n	T
$\Delta / \overline{d}$	1	ReG	T	$n/2$	S_n	T
$\Delta \cdot \overline{d}$	$4-4/n$	P_n	T	$(n-1)^2$	K_n	T
$\Delta - \overline{l}$	$(5-n)/3$	P_n	T	$n-2$	K_n	T
$\Delta + \overline{l}$			NR	$n+1-2/n$	S_n	[31]
$\Delta / \overline{l}$	$6/(n+1)$	P_n	T	$n-1$	K_n	T
$\Delta \cdot \overline{l}$			NR		$PK_{n,m}$	[31]
$\Delta - D$	$3-n$	P_n	T	$n-2$	K_n	T
$\Delta + D$			NR	$n+1$	S_n	[10]
Δ / D	$2/(n-1)$	P_n	T	$n-1$	K_n	T
$\Delta \cdot D$			NR	$\lfloor (n+1)/2 \rfloor$	$Co_{n,\lfloor \frac{n+1}{2} \rfloor}$	[10]
$\Delta - r$	$2 - \lfloor \frac{n}{2} \rfloor$	P_n	T	$n-2$	K_n	T
$\Delta + r$			NR	n	K_n	[10]
Δ / r	$2 / \lfloor \frac{n}{2} \rfloor$	P_n	T	$n-1$	K_n	T
$\Delta \cdot r$			NR	$\begin{cases} \lfloor \frac{n+2}{2} \rfloor \cdot \lfloor \frac{n+2}{4} \rfloor & \text{if } n \equiv 2[4], \\ \lfloor \frac{n+4}{2} \rfloor \cdot \lfloor \frac{n+2}{4} \rfloor & \text{if } n \not\equiv 2[4]. \end{cases}$	$Co_{n,n-2\lfloor \frac{n+7}{4} \rfloor}$	AO
$\Delta - g$	$2-n$	C_n	T	$n-4$	K_n	T
$\Delta + g$	6	$Ki_{n,3}$	[30]	$n+2$	C_n	[30]
Δ / g	$2/n$	C_n	T	$(n-1)/3$	K_n	T
$\Delta \cdot g$	9	$Ki_{n,3}$	[30]	$\lfloor \frac{n+2}{2} \rfloor \lceil \frac{n+2}{2} \rceil$	$Tu_{n,\lfloor \frac{n+2}{2} \rfloor}$	[30]
$\Delta - ecc$	$\begin{cases} 2 - \frac{3n+1}{4}\frac{n-1}{n} & \text{if } n \text{ is odd} \\ 2 - \frac{3n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-2$	K_n	T
$\Delta + ecc$			NR	$n+1-\frac{1}{n}$	S_n	[10]
Δ / ecc	$\begin{cases} \frac{8}{3n+1}\frac{n}{n-1} & \text{if } n \text{ is odd} \\ \frac{8}{3n-2} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-1$	K_n	T
$\Delta \cdot ecc$			NR		$PK_{n,x}$	SO
$\Delta - \pi$	$\begin{cases} \frac{7-n}{4} & \text{if } n \text{ is odd} \\ \frac{8-n}{4} - \frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-2$	S_n	T
$\Delta + \pi$			NR	n	S_n	[10]
Δ / π	$\begin{cases} \frac{8}{n+1} & \text{if } n \text{ is odd,} \\ \frac{8(n-1)}{n^2} & \text{if } n \text{ is even.} \end{cases}$	P_n	T	$n-1$	S_n	T
$\Delta \cdot \pi$			NR		$PK_{n,x}$	SO
$\Delta - \rho$	$(4-n)/2$	P_n	T	$n-2$	K_n	T
$\Delta + \rho$	$\begin{cases} \frac{n+9}{4} & \text{if } n \text{ is odd,} \\ 2 + \frac{n^2}{4n-4} & \text{if } n \text{ is even} \end{cases}$	C_n	AO	$(n^2-2)/(n-1)$	S_n	[10]
Δ / ρ	$4/n$	P_n	T	$n-1$	K_n	T
$\Delta \cdot \rho$			NR		$PK_{n,x}$	[10]
$\Delta - \lambda_1$	0	ReG	K	$n-1-\sqrt{n-1}$	S_n	[35]
$\Delta + \lambda_1$	$2+2\cos\left(\frac{\pi}{n+1}\right)$	P_n	T	$2n-2$	K_n	T
Δ / λ_1	1	ReG	K	$\sqrt{n-1}$	S_n	[35]
$\Delta \cdot \lambda_1$	$4\cos\left(\frac{\pi}{n+1}\right)$	P_n	T	$(n-1)^2$	K_n	T
$\Delta - Ra$	$(4-n)/2$	C_n	T	$n-1-\sqrt{n-1}$	S_n	T
$\Delta + Ra$	$(n+1+2\sqrt{2})/2$	P_n	R	$(3n-2)/2$	K_n	T
Δ / Ra	$4/n$	C_n	T	$\sqrt{n-1}$	S_n	T
$\Delta \cdot Ra$	$n-3+2\sqrt{2}$	Pn	[36]	$n(n-1)/2$	K_n	T

$\Delta - a$	-1	K_n	[10]	$n-2$	S_n	[10]
$\Delta + a$	$4-2\cos(\frac{\pi}{n})$	P_n	T	$2n-1$	K_n	T
Δ / a	$(n-1)/n$	K_n	[10]		$Co_{n,\lfloor\frac{n}{2}\rfloor}$	SO
$\Delta \cdot a$	$4-4\cos(\frac{\pi}{n})$	P_n	T	$n(n-1)$	K_n	T
$\Delta - \nu$	0	C_n	K	$n-2$	S_n	T
$\Delta + \nu$	3	P_n	T	$2n-2$	K_n	T
Δ / ν	1	C_n	K	$n-1$	S_n	T
$\Delta \cdot \nu$	2	P_n	T	$(n-1)^2$	K_n	T
$\Delta - \kappa$	0	C_n	K	$n-2$	S_n	T
$\Delta + \kappa$	3	P_n	T	$2n-2$	K_n	T
Δ / κ	1	C_n	K	$n-1$	S_n	T
$\Delta \cdot \kappa$	2	P_n	T	$(n-1)^2$	K_n	T
$\Delta - \alpha$	$\lceil 2\sqrt{n-1}\rceil - n$		[20]	$n-2$	K_n	T
$\Delta + \alpha$	$\left\lceil \frac{n}{\lceil\sqrt{n}\rceil}\right\rceil + \lceil\sqrt{n}\rceil - 1$		[20]	$2n-2$	S_n	T
Δ / α	$\frac{2}{\lceil\frac{n}{2}\rceil}$	P_n or C_n	[13]	$n-1$	K_n	T
$\Delta \cdot \alpha$	$n-1$	K_n	[13]	$(n-1)^2$	S_n	T
$\Delta - \beta$	$3-\lfloor n/2\rfloor$	Ctr	[10]	$n-2$	S_n	T
$\Delta + \beta$	$\lceil n/3\rceil + 2$	P_n	AR	n	S_n	K
Δ / β	$2/\lceil n/3\rceil$	P_n	[10]	$n-1$	K_n	T
$\Delta \cdot \beta$	$2\lceil n/3\rceil$	P_n	[10]	$\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor$	Ur_n	[10]
$\Delta - \omega$	-1	K_n	K	$n-3$	S_n	T
$\Delta + \omega$	4	P_n	T	$2n-1$	K_n	T
Δ / ω	$(n-1)/n$	K_n	[173]	$(n-1)/2$	S_n	T
$\Delta \cdot \omega$	4	P_n	T	$n(n-1)$	K_n	T
$\Delta - \chi$	-1	K_n	K	$n-3$	S_n	T
$\Delta + \chi$	4	P_n	T	$2n-1$	K_n	T
Δ / χ	$\begin{cases} 3/2 & \text{if } n \text{ is odd,} \\ n/(n-1) & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} C_n \\ K_n \end{cases}$	[10]	$(n-1)/2$	S_n	T
$\Delta \cdot \chi$	4	P_n	T	$n(n-1)$	K_n	T
$\Delta - \mu$	$\lfloor (n-4)/2\rfloor$	P_n	T	$n-2$	S_n	T
$\Delta + \mu$			NR	$\lfloor n/2\rfloor + n - 1$	K_n	T
Δ / μ	$2/\lfloor n/2\rfloor$	P_n	T	$n-1$	S_n	T
$\Delta \cdot \mu$	$n-1$	S_n	[10]	$\lfloor n/2\rfloor(n-1)$	K_n	T
$\delta - \overline{d}$	$-(n-2)^2/n$	$Ki_{n,n-1}$	[10]	0	ReG	T
$\delta + \overline{d}$	$3-2/n$	$Tree$	T	$2n-2$	K_n	T
$\delta / \overline{d}$	$n/(n^2-3n+4)$	$Ki_{n,n-1}$	[10]	1	ReG	T
$\delta \cdot \overline{d}$	$2-2/n$	$Tree$	T	$(n-1)^2$	K_n	T
$\delta - \overline{l}$	$(2-n)/3$	P_n	T	$n-2$	K_n	T
$\delta + \overline{l}$	$(2n^2-4)/(n(n-1))$	$Ki_{n,n-1}$	[31]	n	K_n	[31]
$\delta / \overline{l}$	$3/(n+1)$	P_n	T	$n-1$	K_n	T
$\delta \cdot \overline{l}$	$(n^2+n-4)/(n(n-1))$	$Ki_{n,n-1}$	[31]	$n-1$	K_n	[31]
$\delta - D$	$2-n$	P_n	T	$n-2$	K_n	T
$\delta + D$	3	S_n	T	n	K_n	[10]
δ / D	$1/(n-1)$	P_n	T	$n-1$	K_n	T
$\delta \cdot D$	2	S_n	T	$2n-4$	$K_n - e$	R
$\delta - r$	$1-\lfloor n/2\rfloor$	P_n	T	$n-2$	K_n	T
$\delta + r$	2	S_n	T	n	K_n	[10]
δ / r	$1/\lfloor n/2\rfloor$	P_n	T	$n-1$	K_n	T
$\delta \cdot r$	1	S_n	T	$\begin{cases} 2n-6 & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$	$K_n - R$	O
$\delta - g$	$2-n$	C_n	T	$n-4$	K_n	T
$\delta + g$	4	$Ki_{n,3}$	T	$n+2$	K_n	[30]
δ / g	$1/(n-1)$	$Tu_{n,n-1}$	T	$(n-1)/3$	K_n	T

$\delta \cdot g$	3	$Ki_{n,3}$	T	$3n-3$	$K-n$	[30]
$\delta - ecc$	$\begin{cases} 1-\frac{(3n+1)(n-1)}{4n} & \text{if } n \text{ is odd,} \\ 1-\frac{3n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-2$	K_n	T
$\delta + ecc$	$3-1/n$	S_n	[10]	n	K_n	[10]
δ / ecc	$\begin{cases} \frac{4n}{(3n+1)(n-1)} & \text{if } n \text{ is odd,} \\ \frac{4}{3n-2} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-1$	K_n	T
$\delta \cdot ecc$	$2-1/n$	S_n	[10]	$\begin{cases} (n-2)(2-1/n) & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$	$K_n - M$	R
$\delta - \pi$	$\begin{cases} \frac{3-n}{4} & \text{if } n \text{ is odd,} \\ \frac{4-n}{4} - \frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-2$	K_n	T
$\delta + \pi$	2	S_n	T	n	K_n	[10]
δ / π	$\begin{cases} 4/(n+1) & \text{if } n \text{ is odd,} \\ (4n-4)/n^2 & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-1$	K_n	T
$\delta \cdot \pi$	1	S_n	T	$n-1$	K_n	[10]
$\delta - \rho$	$(2-n)/2$	P_n	T	$n-2$	K_n	T
$\delta + \rho$	$3-1/(n-1)$	S_n	[10]	n	K_n	[10]
δ / ρ	$2/n$	P_n	T	$n-1$	K_n	T
$\delta \cdot \rho$	$(2n-3)/(n-1)$	S_n	[10]			NR
$\delta - \lambda_1$	0	ReG	K	$3-n-t$, where $0<t<1$ and	$Ki_{n,n-1}$	[35]
				$t^3+(2n-3)t^2+(n^2-3n+1)t=1$		
$\delta + \lambda_1$	$1+2\cos\frac{\pi}{n+1}$	P_n	T	$2n-2$	K_n	T
δ / λ_1	1	ReG	K	$1/(n-2+t)$, where $0<t<1$ and	$Ki_{n,n-1}$	[35]
				$t^3+(2n-3)t^2+(n^2-3n+1)t=1$		
$\delta \cdot \lambda_1$	$2\cos\frac{\pi}{n+1}$	P_n	T	$(n-1)^2$	K_n	T
$\delta - Ra$	$-\frac{3n-13+\sqrt{6}+3\sqrt{2}}{6}$		[36]	$(n-2)/2$	K_n	[36]
$\delta + Ra$	$1+\sqrt{n-1}$	S_n	T	$(3n-2)/2$	K_n	T
δ / Ra	$6/(3n-7+\sqrt{6}+3\sqrt{2})$		[36]	$(2n-2)/n$	K_n	[36]
$\delta \cdot Ra$	$\sqrt{n-1}$	S_n	T	$n(n-1)/2$	K_n	T
$\delta - a$	-1	K_n	[10]		$K_{\lfloor\frac{n+1}{2}\rfloor,\lceil\frac{n+1}{2}\rceil}$	SO
$\delta + a$	$3-2\cos\frac{\pi}{n}$	P_n	T	$2n-1$	K_n	T
δ / a	$(n-1)/n$	K_n	K		TPT	SO
$\delta \cdot a$	$2-2\cos\frac{\pi}{n}$	P_n	T	$n(n-1)$	K_n	T
$\delta - \nu$	0	C_n	K	$\lfloor (n-3)/2 \rfloor$		[10]
$\delta + \nu$	2	P_n	T	$2n-2$	K_n	T
δ / ν	1	C_n	K	$\lfloor (n-1)/2 \rfloor$		[10]
$\delta \cdot \nu$	1	P_n	T	$(n-1)^2$	K_n	T
$\delta - \kappa$	0	C_n	K	$\lfloor (n-4)/2 \rfloor$		[10]
$\delta + \kappa$	2	P_n	T	$2n-2$	K_n	T
δ / κ	1	C_n	K	$\lfloor (n-2)/2 \rfloor$		[10]
$\delta \cdot \kappa$	1	P_n	T	$(n-1)^2$	K_n	T
$\delta - \alpha$	$2-n$	S_n	T	$n-2$	K_n	T
$\delta + \alpha$	3	$Ki_{n,n-1}$	[13]	n	S_n	[13]
δ / α	$1/(n-1)$	S_n	T	$n-1$	K_n	T
$\delta \cdot \alpha$	2	$Ki_{n,n-1}$	[13]	$\lceil n/2 \rceil \lfloor n/2 \rfloor$	$K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$	[13]
$\delta - \beta$	$1-\lfloor n/2 \rfloor$	Ur_n	T	$n-2$	K_n	T
$\delta + \beta$	2	S_n	T	n	K_n	[10]
δ / β	$1/\lfloor n/2 \rfloor$	Ur_n	T	$n-1$	K_n	T
$\delta \cdot \beta$	1	S_n	T	$\begin{cases} 2n-6 & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$		O
$\delta - \omega$	$2-n$	$K_{n,n-1}$	[173]	$\lfloor n/2 \rfloor - 2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	R
$\delta + \omega$	3	$Tree$	T	$2n-1$	K_n	T
δ / ω	$1/(n-1)$	$Ki_{n,n-1}$	[173]	$\lfloor n/2 \rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[173]
$\delta \cdot \omega$	2	$Tree$	T	$n(n-1)$	K_n	T
$\delta - \chi$	$2-n$	$K_{n,n-1}$	T	$\lfloor n/2 \rfloor - 2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	R
$\delta + \chi$	3	$Tree$	T	$2n-1$	K_n	T

δ/χ	$1/(n-1)$	$Ki_{n,n-1}$	T	$\lfloor n/2 \rfloor/2$	$K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$	[10]
$\delta \cdot \chi$	2	$Tree$	T	$n(n-1)$	K_n	T
$\delta - \mu$	$1 - \lfloor n/2 \rfloor$	P_n	T	$\lceil n/2 \rceil - 1$	K_n	[10]
$\delta + \mu$	2	S_n	T	$\lfloor n/2 \rfloor + n - 1$	K_n	T
δ/μ	$1/\lfloor n/2 \rfloor$	P_n	T	$(n-1)/\lfloor n/2 \rfloor$	K_n	[10]
$\delta \cdot \mu$	1	S_n	T	$(n-1)\lfloor n/2 \rfloor$	K_n	T
$\overline{d} - \overline{l}$	$(5-n)/3 - 2/n$	P_n	T	$n-2$	K_n	T
$\overline{d} + \overline{l}$	$4 - 4/n$	S_n	[31]	n	K_n	[31]
$\overline{d}/\overline{l}$	$(6n-6)/(n^2+n)$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot \overline{l}$	$4(n-1)^2/n^2$	S_n	[31]		$PK_{n,x}$	[31]
$\overline{d} - D$	$3 - n - 2/n$	P_n	T	$n-2$	K_n	T
$\overline{d} + D$	$4 - 2/n$	S_n	T	$n + 1 - 2/n$	P_n	T
$\overline{d}/D$	$2/n$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot D$	$4 - 4/n$	S_n	[10]		Har	[10]
$\overline{d} - r$	$2 - 2/n - \lfloor n/2 \rfloor$	P_n	T	$n-2$	K_n	T
$\overline{d} + r$	$3 - 2/n$	S_n	T	n	K_n	[10]
$\overline{d}/r$	$(2-2/n)/\lfloor n/2 \rfloor$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot r$	$2 - 2/n$	S_n	T			SO
$\overline{d} - g$	$2 - n$	C_n	T	$n-4$	K_n	T
$\overline{d} + g$	5	S_n^+	T	$n+2$	K_n	[30]
$\overline{d}/g$	$2/n$	C_n	T	$(n-1)/3$	K_n	T
$\overline{d} \cdot g$	6	S_n^+	T	$3n-3$	K_n	[30]
$\overline{d} - ecc$	$\begin{cases} 2 - \frac{2}{n} - \frac{3n+1}{4}\frac{n-1}{n} & \text{if } n \text{ odd,} \\ 2 - \frac{2}{n} - \frac{3n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-2$	K_n	T
$\overline{d} + ecc$	$4 - \frac{3}{n}$	S_n	[10]	n	K_n	[10]
$\overline{d}/ecc$	$\begin{cases} \frac{8}{3n+1} & \text{if } n \text{ odd,} \\ \frac{8}{3n-2}\frac{n-1}{n} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot ecc$	$4 - 3/n$	S_n	[10]		$PK_{n,x}$	SO
$\overline{d} - \pi$	$\begin{cases} \frac{7-n}{4} - \frac{2}{n} & \text{for } n \text{ odd} \\ 2 - \frac{2}{n} - \frac{n}{4} - \frac{n}{4n-4} & \text{for } n \text{ even} \end{cases}$	P_n	T	$n-2$	K_n	T
$\overline{d} + \pi$	$3 - 2/n$	S_n	T	n	K_n	[10]
$\overline{d}/\pi$	$\begin{cases} \frac{8n-8}{n(n+1)} & \text{if } n \text{ is odd,} \\ \frac{8(n-1)^2}{n^3} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot \pi$	$2 - 2/n$	S_n	T	$n-1$	K_n	O
$\overline{d} - \rho$	$2 - 2/n + 2/n$	P_n	T	$n-2$	K_n	T
$\overline{d} + \rho$	$4 - 1/(n-1) - 2/n$	S_n	[10]	n	K_n	[10]
$\overline{d}/\rho$	$4(n-1)/n^2$	P_n	T	$n-1$	K_n	T
$\overline{d} \cdot \rho$	$4 - 2/(n-1) - 4/n + 2/(n(n-1))$	S_n	[10]		$PK_{n,x}$	SO
$\overline{d} - \lambda_1$		$Pi_{n,x}$	SO	0	ReG	K
$\overline{d} + \lambda_1$	$2 - \frac{2}{n} + 2\cos\frac{\pi}{n+1}$	P_n	T	$2n-2$	K_n	T
$\overline{d}/\lambda_1$	$\frac{2\sqrt{n-1}}{n}$	S_n	[35]	1	ReG	K
$\overline{d} \cdot \lambda_1$	$\left(4 - \frac{4}{n}\right) \cdot cos\frac{\pi}{n+1}$	P_n	T	$(n-1)^2$	K_n	T
$\overline{d} - Ra$	$\frac{7-n-2\sqrt{2}}{2} - \frac{2}{n}$	P_n	[36]	$(n-2)/2$	K_n	[36]
$\overline{d} + Ra$	$2 - 2/n + \sqrt{n-1}$	S_n	T	$(3n-2)/2$	K_n	T
$\overline{d}/Ra$	$\frac{4}{n}\frac{n-1}{n-3+2\sqrt{2}}$	P_n	[36]	$\frac{2n-2}{n}$	K_n	[36]
$\overline{d} \cdot Ra$	$(2-2/n)\sqrt{n-1}$	S_n	T	$n(n-1)/2$	K_n	T
$\overline{d} - a$	-1	K_n	[10]	$n - 4 + 4/n$	$Ki_{n,n-1}$	O
$\overline{d} + a$	$4 - \frac{2}{n} - 2cos\frac{\pi}{n}$	P_n	T	$2n-1$	K_n	T
$\overline{d}/a$	$\frac{n-1}{n}$	K_n	[10]		$Ki_{n,x}$	SO
$\overline{d} \cdot a$	$\left(4 - \frac{4}{n}\right)\left(1 - cos\frac{\pi}{n}\right)$	P_n	T	$n(n-1)$	K_n	T
$\overline{d} - \nu$	0	C_n	K	$n - 4 + 4/n$	$Ki_{n,n-1}$	[10]
$\overline{d} + \nu$	$3 - 2/n$	$Tree$	T	$2n-2$	K_n	T
$\overline{d}/\nu$	1	C_n	K	$(n^2 - 3n + 4)/n$	$Ki_{n,n-1}$	[10]
$\overline{d} \cdot \nu$	$2 - 2/n$	$Tree$	T	$(n-1)^2$	K_n	T
$\overline{d} - \kappa$	0	C_n	K	$n - 4 + 4/n$	$Ki_{n,n-1}$	[10]

$\overline{d}+\kappa$	$3-2/n$	$Tree$	T	$2n-2$	K_n	T
$\overline{d}/\kappa$	1	C_n	K	$(n^2-3n+4)/n$	$Ki_{n,n-1}$	[10]
$\overline{d}\cdot\kappa$	$2-2/n$	$Tree$	T	$(n-1)^2$	K_n	T
$\overline{d}-\alpha$	$3-n-2/n$	S_n	T	$n-2$	K_n	T
$\overline{d}+\alpha$		$Clqs$	[195]	$\begin{cases} \frac{5n-2}{4}+\frac{1}{4n} & \text{if } n \text{ is odd,} \\ \frac{5n-2}{4} & \text{if } n \text{ is even} \end{cases}$	$CS_{n,\lceil \frac{n}{2} \rceil}$	[13]
$\overline{d}/\alpha$	$2/n$	S_n	T	$n-1$	K_n	T
$\overline{d}\cdot\alpha$		$Clqs$	SO	$\begin{cases} \frac{3(n+1)(n-1)^2}{8n} & \text{if } n \text{ is odd,} \\ \frac{3n^2-2n}{8} & \text{if } n \text{ is even} \end{cases}$	$CS_{n,\lceil \frac{n}{2} \rceil}$	R
$\overline{d}-\beta$	$2-2/n-\lfloor n/2 \rfloor$	Ctr	T	$n-2$	K_n	T
$\overline{d}+\beta$	$3-2/n$	S_n	T	n	K_n	[10]
$\overline{d}/\beta$	$(2-2/n)/\lfloor n/2 \rfloor$	Ctr	T	$n-1$	K_n	T
$\overline{d}\cdot\beta$	$2-2/n$	S_n	T			NR
$\overline{d}-\omega$	$\begin{cases} \frac{2-n}{4}-\frac{9}{4n} & \text{if } n \text{ is odd,} \\ \frac{2-n}{4}-\frac{2}{n} & \text{if } n \text{ is even} \end{cases}$		[173]	$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 2$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	AR
$\overline{d}+\omega$	$4-2/n$	$Tree$	T	$2n-1$	K_n	T
$\overline{d}/\omega$	$\frac{t^2-3t+2n}{nt}$ where $t=\lfloor \frac{n+1}{4} \rfloor + 2$		AR	$\frac{1}{2}\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	[173]
$\overline{d}\cdot\omega$	$4-4/n$	$Tree$	T	$n(n-1)$	K_n	T
$\overline{d}-\chi$	$\begin{cases} \frac{2-n}{4}-\frac{9}{4n} & \text{if } n \text{ is odd,} \\ \frac{2-n}{4}-\frac{2}{n} & \text{if } n \text{ is even} \end{cases}$		[10]	$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 2$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	AR
$\overline{d}+\chi$	$4-2/n$	$Tree$	T	$2n-1$	K_n	T
$\overline{d}/\chi$	$\frac{t^2-3t+2n}{nt}$ where $t=\lfloor \frac{n+1}{4} \rfloor + 2$		AO	$\frac{1}{2}\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	[10]
$\overline{d}\cdot\chi$	$4-4/n$	$Tree$	T	$n(n-1)$	K_n	T
$\overline{d}-\mu$	$2-2/n-\lfloor n/2 \rfloor$	P_n	T	$\lceil n/2 \rceil - 1$	K_n	[10]
$\overline{d}+\mu$	$3-2/n$	S_n	T	$n-1+\lfloor n/2 \rfloor$	K_n	T
$\overline{d}/\mu$	$(2-2/n)/\lfloor n/2 \rfloor$	P_n	T	$(n-1)/\lfloor n/2 \rfloor$	K_n	[10]
$\overline{d}\cdot\mu$	$2-2/n$	S_n	T	$(n-1)\lfloor n/2 \rfloor$	K_n	T
$\overline{l}-D$	$(4-2n)/3$	P_n	[31]	0	$TReG$	T
$\overline{l}+D$	2	K_n	T	$(4n-2)/3$	P_n	T
$\overline{l}/D$		Har	[193]	1	$TReG$	T
$\overline{l}\cdot D$	1	K_n	T	$(n^2-1)/3$	P_n	T
$\overline{l}-r$	$\begin{cases} \frac{8-(n-1)^3}{4n(n-1)} & \text{if } n \text{ is odd,} \\ \frac{-n(n-2)}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	C_n, Bag	AO			NR
$\overline{l}+r$	2	K_n	T	$\begin{cases} \frac{5n+2}{6} & \text{if } n \text{ is odd,} \\ \frac{5n-1}{6} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\overline{l}/r$		Bag	[118]	$2-2/n$	S_n	[31]
$\overline{l}\cdot r$	1	K_n	T	$\begin{cases} \frac{n^2+n}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2-1}{6} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\overline{l}-g$	$\begin{cases} \frac{n+1}{4}-n & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)}-n & \text{if } n \text{ is even} \end{cases}$	C_n	[31]	$\frac{(n+1)(n^2-10n+12)}{3n(n-1)}$	$Ki_{n,3}$	[31]
$\overline{l}+g$	4	K_n	T	$\begin{cases} \frac{5n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)}+n & \text{if } n \text{ is even} \end{cases}$	C_n	[31]
$\overline{l}/g$	$\begin{cases} \frac{n+1}{4n} & \text{if } n \text{ is odd,} \\ \frac{n}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	C_n	R	$\frac{n^3-7n+12}{9n(n-1)}$	$Ki_{n,3}$	[31]
$\overline{l}\cdot g$	3	K_n	T	$\begin{cases} \frac{n^2+n}{4} & \text{if } n \text{ is odd,} \\ \frac{n^3}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	C_n	[19]
$\overline{l}-ecc$	$\begin{cases} \frac{n+1}{3}-\frac{3n+1}{4}\frac{n-1}{n} & \text{if } n \text{ is odd,} \\ \frac{10-5n}{12} & \text{if } n \text{ is even} \end{cases}$	P_n	AO	0	K_n	T

$\bar{l}+ecc$	2	K_n	T	$\begin{cases} \frac{3n+1}{4}\,\frac{n-1}{n}+\frac{n+1}{3} & \text{if } n \text{ is odd,} \\ \frac{13n-2}{12} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\bar{l}/ecc$		$PK_{n,x}$	SO	1	K_n	T
$\bar{l}\cdot ecc$	1	K_n	T	$\begin{cases} \frac{3n+1}{4}\,\frac{n-1}{n}\,\frac{n+1}{3} & \text{if } n \text{ is odd,} \\ \frac{(3n-2)(n+1)}{12} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\bar{l}-\pi$	0	$TReG$	T		$3\times P$	[172]
$\bar{l}+\pi$	2	K_n	T	$\begin{cases} \frac{7n+7}{12} & \text{if } n \text{ is odd,} \\ \frac{7n+7}{12}+\frac{1}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\bar{l}/\pi$	1	$TReG$	T	$2-2/n$	S_n	O
$\bar{l}\cdot\pi$	1	K_n	T	$\begin{cases} \frac{(n+1)^2}{12} & \text{if } n \text{ is odd,} \\ \frac{(n+1)^2}{12}+\frac{n+1}{12n-12} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\bar{l}-\rho$	0	$TReG$	T		$PK_{n,x}$	SO
$\bar{l}+\rho$	2	K_n	T	$(5n+3)/6$	P_n	T
$\bar{l}/\rho$	1	$TReG$	T		$PK_{n,x}$	SO
$\bar{l}\cdot\rho$	1	K_n	T	$n(n+1)/6$	P_n	T
$\bar{l}-\lambda_1$	$2-n$	K_n	T	$\frac{n+1}{3}-2\cos\frac{\pi}{n+1}$	P_n	T
$\bar{l}+\lambda_1$			NR	n	K_n	[35]
$\bar{l}/\lambda_1$	$1/(n-1)$	K_n	T	$(n+1)/(6\cos\frac{\pi}{n+1})$	P_n	T
$\bar{l}\cdot\lambda_1$			NR	$\frac{1}{2}\left(n-3+\sqrt{n^2+2n-7}\right)\cdot\left(1+\frac{2}{n(n-1)}\right)$	K_n-e	AR
$\bar{l}-Ra$	$(2-n)/2$	K_n	T	$2-2/n-\sqrt{n-1}$	S_n	O
$\bar{l}+Ra$	$\begin{cases} \frac{n+2}{2} & \text{if } n\le 5 \\ 2-\frac{2}{n}+\sqrt{n-1} & \text{if } n\ge 6 \end{cases}$	$\begin{cases} K_n \\ S_n \end{cases}$	AO	$\frac{5n-7+6\sqrt{2}}{6}$	P_n	R
$\bar{l}/Ra$	$2/n$	K_n	T			NR
$\bar{l}\cdot Ra$	$\begin{cases} \frac{n}{2} & \text{if } n\le 12 \\ (2-2/n)\sqrt{n-1} & \text{if } n\ge 13 \end{cases}$	$\begin{cases} K_n \\ S_n \end{cases}$	AO	$\frac{n+1}{3}\cdot\frac{n-3+2\sqrt{2}}{2}$	P_n	[15]
$\bar{l}-a$	$1-n$	K_n	T	$(n+1)/3-2(1-cos\frac{\pi}{n})$	P_n	T
$\bar{l}+a$		$Ki_{n,n-3}$	SO	$n+1$	K_n	[35]
$\bar{l}/a$	$1/n$	K_n	T	$6(1-cos\frac{\pi}{n})/(n+1)$	P_n	T
$\bar{l}\cdot a$		TPT	SO	n	K_n	[35]
$\bar{l}-\nu$	$2-n$	K_n	T	$(n-2)/3$	P_n	T
$\bar{l}+\nu$	$(2n^2-4)/(n(n-1))$	$Ki_{n,n-1}$	[35]	n	K_n	[35]
$\bar{l}/\nu$	$1/(n-1)$	K_n	T	$(n+1)/3$	P_n	T
$\bar{l}\cdot\nu$	$(n^2+n-4)/(n(n-1))$	$Ki_{n,n-1}$	[35]	$n-1$	K_n	[35]
$\bar{l}-\kappa$	$2-n$	K_n	T	$(n-2)/3$	P_n	T
$\bar{l}+\kappa$	$(2n^2-4)/(n(n-1))$	$Ki_{n,n-1}$	[35]	n	K_n	[35]
$\bar{l}/\kappa$	$1/(n-1)$	K_n	T	$(n+1)/3$	P_n	T
$\bar{l}\cdot\kappa$	$(n^2+n-4)/(n(n-1))$	$Ki_{n,n-1}$	[35]	$n-1$	K_n	[35]
$\bar{l}-\alpha$	$3-n-2/n$	S_n	[13]	0	K_n	K
$\bar{l}+\alpha$	2	K_n	T	$n+1-2/n$	S_n	[13]
$\bar{l}/\alpha$	$2/n$	S_n	[13]	1	K_n	K
$\bar{l}\cdot\alpha$	1	K_n	T			NR
$\bar{l}-\beta$		Ur_n	SO		$DC_{n,x,y}$	SO
$\bar{l}+\beta$	2	K_n	T			NR
$\bar{l}/\beta$	$2/(1+2\lceil n/3\rceil)$	Ur_n	AO	$2-2/n$	S_n	O
$\bar{l}\cdot\beta$	1	K_n	T			NR
$\bar{l}-\omega$	$1-n$	K_n	T	$(n-5)/3$	P_n	T
$\bar{l}+\omega$	$\begin{cases} \frac{7}{2}-\frac{1}{2n} & \text{if } n \text{ is odd,} \\ \frac{7}{2}-\frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[173]	$n+1$	K_n	[173]
$\bar{l}/\omega$	$1/n$	K_n	T	$(n+1)/6$	P_n	T
$\bar{l}\cdot\omega$	$\begin{cases} 3-1/n & \text{if } n \text{ is odd,} \\ 3-1/(n-1) & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[173]		$Ki_{n,x}$	SO
$\bar{l}-\chi$	$1-n$	K_n	T	$(n-5)/3$	P_n	T
$\bar{l}+\chi$	$\begin{cases} \frac{7}{2}-\frac{1}{2n} & \text{if } n \text{ is odd,} \\ \frac{7}{2}-\frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]	$n+1$	K_n	[10]

$\bar{l}/\chi$	$1/n$	K_n	T	$(n+1)/6$	P_n	T
$\bar{l}\cdot\chi$	$\begin{cases} 3-1/n & \text{if } n \text{ is odd,} \\ 3-1/(n-1) & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]		$Ki_{n,x}$	SO
$\bar{l}-\mu$	$1-\lfloor n/2\rfloor$	K_n	T	$1-2/n$	S_n	[10]
$\bar{l}+\mu$	$3-2/n$	S_n	[10]	$\lfloor n/2\rfloor+(n+1)/3$	P_n	T
$\bar{l}/\mu$	$1/\lfloor n/2\rfloor$	K_n	T	$2-2/n$	S_n	[10]
$\bar{l}\cdot\mu$	$2-2/n$	S_n	[10]	$\lfloor n/2\rfloor(n+1)/3$	P_n	T
$D-r$	0	C_n	T	$\lfloor (n-1)/2\rfloor$	P_n	[10]
$D+r$	2	K_n	T	$n-1+\lfloor n/2\rfloor$	P_n	T
D/r	1	C_n	T	2	S_n	K
$D\cdot r$	1	K_n	T	$(n-1)\lfloor n/2\rfloor$	P_n	T
$D-g$	$-\lceil n/2\rceil$	C_n	[30]	$n-5$	$Ki_{n,3}$	[30]
$D+g$	4	K_n	T	$\begin{cases} \frac{3n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{3n}{2} & \text{if } n \text{ is even} \end{cases}$	C_n	[30]
D/g	$1/3$	K_n	[30]	$(n-2)/3$	$Ki_{n,3}$	[30]
$D\cdot g$	3	K_n	T	$\begin{cases} \frac{n^2-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n^2}{2} & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ll_{n,n-1} \\ C_n \end{cases}$	[30]
$D-ecc$	0	C_n	T	$\begin{cases} n-1-\frac{(3n+1)(n-1)}{4n} & \text{if } n \text{ is odd,} \\ \frac{n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	[10]
$D+ecc$	2	K_n	T	$\begin{cases} n-1+\frac{(3n+1)(n-1)}{4n} & \text{if } n \text{ is odd,} \\ \frac{7n-6}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
D/ecc	1	C_n	T	$2n/(n-2)$	K_n-e	[10]
$D\cdot ecc$	1	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)^2}{4n} & \text{if } n \text{ is odd,} \\ \frac{(3n-2)(n-1)}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$D-\pi$	0	K_n	T	$\begin{cases} \frac{3n-5}{4} & \text{if } n \text{ is odd,} \\ \frac{3n-5}{4}-\frac{1}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	[25]
$D+\pi$	2	K_n	T	$\begin{cases} \frac{5n-3}{4} & \text{if } n \text{ is odd,} \\ \frac{5n-3}{4}-\frac{1}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
D/π	1	K_n	T			NR
$D\cdot\pi$	1	K_n	T	$\begin{cases} (n^2-1)/4 & \text{if } n \text{ is odd,} \\ n^2/4 & \text{if } n \text{ is even} \end{cases}$	P_n	T
$D-\rho$	0	K_n	T	$(n-2)/2$	P_n	[25]
$D+\rho$	2	K_n	T	$(3n-2)/2$	P_n	T
D/ρ	1	K_n	T	$2-2/n$	P_n	[10]
$D\cdot\rho$	1	K_n	T	$n(n-1)/2$	P_n	T
$D-\lambda_1$	$2-n$	K_n	T	$n-1-2\cos\frac{\pi}{n+1}$	P_n	T
$D+\lambda_1$	$2+\sqrt{n-1}$	S_n	R	$n-1+2\cos\frac{\pi}{n+1}$	P_n	O
D/λ_1	$1/(n-1)$	K_n	T	$(n-1)/(2\cos\frac{\pi}{n+1})$	P_n	T
$D\cdot\lambda_1$	$2\sqrt{n-1}$	S_n	R		Bug	SO
$D-Ra$	$(2-n)/2$	K_n	T	$(n+1)/2-\sqrt{2}$	P_n	[194]
$D+Ra$	$2+\sqrt{n-1}$	S_n	T	$(3n-5+2\sqrt{2})/2$	P_n	[36]
D/Ra	$2/n$	K_n	T	$(2n-2)/(n-3+2\sqrt{2})$	P_n	[194]
$D\cdot Ra$	$2\sqrt{n-1}$	S_n	T	$(n-1)(n-3+2\sqrt{2})/2$	P_n	[36]
$D-a$	$1-n$	K_n	T	$n-3+2\cos\frac{\pi}{n}$	P_n	T
$D+a$	3	S_n	O	$n+1$	K_n	[10]
D/a	$1/n$	K_n	T	$(n-1)/(2+2\cos\frac{\pi}{n})$	P_n	T
$D\cdot a$			SO	$2n-4$	K_n-M	[174]
$D-\nu$	$2-n$	K_n	T	$n-2$	P_n	T
$D+\nu$	3	S_n	T	n	P_n	[10]
D/ν	$1/(n-1)$	K_n	T	$n-1$	P_n	T
$D\cdot\nu$	2	S_n	T	$2n-4$	K_n-M	[174]
$D-\kappa$	$2-n$	K_n	T	$n-2$	P_n	T
$D+\kappa$	3	S_n	T	n	$P'n,K_n$	[10]

D/κ	$1/(n-1)$	K_n	T	$n-1$	P_n	T
$D\cdot\kappa$	2	S_n	T	$2n-4$	K_n-M	R
$D-\alpha$	$3-n$	S_n	T	$\lfloor n/2\rfloor-1$	P_n	[13]
$D+\alpha$	2	K_n	T	$n-1+\lfloor n/2\rfloor$	P_n	[13]
D/α	$2/(n-1)$	S_n	T	$\begin{cases} 2-2/(n-1) & \text{if } n \text{ is odd,} \\ 2-2/n & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ki_{n,3} \\ P_n \end{cases}$	[13]
$D\cdot\alpha$	1	K_n	T	$\begin{cases} (n^2-1)/2 & \text{if } n \text{ is odd,} \\ (n^2-4)/2 & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \end{cases}$	[13]
$D-\beta$	$3-\lfloor n/2\rfloor$	Ur_n	O	$n-1-\lceil n/3\rceil$	P_n	[10]
$D+\beta$	2	K_n	T	$n-1+\lceil n/3\rceil$	P_n	[10]
D/β	$3/\lfloor n/2\rfloor$	Ur_n	[10]	$(n-1-n\,mod(3))/\lfloor n/3\rfloor$		[10]
$D\cdot\beta$	1	K_n	T	$\begin{cases} \frac{(n+3)(n-2)}{3} & \text{if } n\equiv 0[3] \\ \lceil \frac{n}{3}\rceil (n-1) & \text{if } n\not\equiv 0[3] \end{cases}$		AR
$D-\omega$	$1-n$	K_n	T	$n-3$	P_n	T
$D+\omega$	4	S_n	T	$n+1$	P_n	[173]
D/ω	$1/n$	K_n	T	$(n-1)/2$	P_n	T
$D\cdot\omega$	4	S_n	T	$\lfloor (n+1)/2\rfloor\lceil (n+1)/2\rceil$	$Ki_{n,\lfloor \frac{n+1}{2}\rfloor}$	[173]
$D-\chi$	$1-n$	K_n	T	$n-3$	P_n	T
$D+\chi$	4	S_n	T	$n+1$	P_n	[10]
D/χ	$1/n$	K_n	T	$(n-1)/2$	P_n	T
$D\cdot\chi$	4	S_n	T	$\lfloor (n+1)/2\rfloor\lceil (n+1)/2\rceil$	$Ki_{n,\lfloor \frac{n+1}{2}\rfloor}$	[10]
$D-\mu$	$1-\lfloor n/2\rfloor$	K_n	T	$\lceil (n+1)/2\rceil-1$	P_n	[10]
$D+\mu$	3	S_n	T	$n-1+\lfloor n/2\rfloor$	P_n	T
D/μ	$1/\lfloor n/2\rfloor$	K_n	T	2	S_n	[10]
$D\cdot\mu$	2	S_n	T	$(n-1)\lfloor n/2\rfloor$	P_n	T
$r-g$	$-\lceil (n+1)/2\rceil$	C_n	[30]	$\lfloor (n-1)/2\rfloor-3$		[30]
$r+g$	4	K_n	T	$\lfloor n/2\rfloor+n$	C_n	T
r/g	$1/3$	K_n	[30]	$\lfloor (n-1)/2\rfloor/3$		[30]
$r\cdot g$	3	K_n	T	$n\lfloor n/2\rfloor$	C_n	T
$r-ecc$	$\begin{cases} \frac{1}{4n}-\frac{n}{4} & \text{if } n \text{ is odd} \\ \frac{1}{n}-\frac{n}{4} & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \end{cases}$	[10]	0	C_n	T
$r+ecc$	2	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)}{4n}+\frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{5n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
r/ecc	$2-1/n$	S_n	[10]	1	C_n	T
$r\cdot ecc$	1	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)^2}{8n} & \text{if } n \text{ is odd,} \\ \frac{3n^2-2n}{8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$r-\pi$	0	S_n	T	$\begin{cases} \frac{n-1}{4}-\frac{1}{n-1} & \text{if } n \text{ is odd,} \\ \frac{n}{2}-\frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$		[25]
$r+\pi$	2	S_n	T	$\begin{cases} \frac{3n-1}{4} & \text{if } n \text{ is odd,} \\ \frac{3n+1}{4}+\frac{1}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
r/π	1	S_n	T			NR
$r\cdot\pi$	1	K_n	T	$\begin{cases} \frac{n^2-1}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2+n}{8}+\frac{1}{8(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$r-\rho$	$\frac{2t^2-nt}{n-1}$ where $t=\lfloor \frac{n+1}{4}\rfloor$	$Co_{n,n-2t+1}$	[25]	$\begin{cases} \frac{n-3}{4} & \text{if } n \text{ is odd,} \\ \frac{n}{2}-\frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$		[127]
$r+\rho$	2	K_n	T	$n/2+\lfloor n/2\rfloor$	P_n	T
r/ρ	$(n-1)/(2n-3)$	S_n	[10]	$\begin{cases} 2-\frac{4}{n+1} & \text{if } n \text{ is odd,} \\ 2-\frac{n}{2}-\frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$		O
$r\cdot\rho$	1	K_n	T	$(n/2)\lfloor n/2\rfloor$	P_n	T
$r-\lambda_1$	$2-n$	K_n	T	$\lfloor n/2\rfloor-2\cos\frac{\pi}{n+1}$	P_n	T
$r+\lambda_1$			NR	n	K_n	[35]

r/λ_1	$1/(n-1)$	K_n	T	$\lfloor n/2\rfloor/(2\cos\frac{\pi}{n+1})$	P_n	T
$r\cdot\lambda_1$	$\sqrt{n-1}$	S_n	R		Bag	SO
$r-Ra$	$(2-n)/2$	K_n	T	$\lfloor n/2\rfloor-(n-3+2\sqrt{2})/2$	P_n	O
$r+Ra$	$1+\sqrt{n-1}$	S_n	T	$\lfloor n/2\rfloor+n/2$	C_n	T
r/Ra	$2/n$	K_n	T	$2\lfloor n/2\rfloor/(n-3+2\sqrt{2})$	P_n	O
$r\cdot Ra$	$\sqrt{n-1}$	S_n	T	$(n/2)\lfloor n/2\rfloor$	C_n	T
$r-a$	$1-n$	K_n	T	$\lfloor n/2\rfloor-2(1-\cos\frac{\pi}{n})$	P_n	T
$r+a$	2	S_n	[10]	$n+1$	K_n	[10]
r/a	$1/n$	K_n	T	$\lfloor n/2\rfloor/(2-2\cos\frac{\pi}{n})$	P_n	T
$r\cdot a$			NR	$4\lfloor n/2\rfloor-4$	K_n-M	[174]
$r-\nu$	$2-n$	K_n	T	$\lfloor n/2\rfloor-1$	P_n	T
$r+\nu$	2	S_n	T	n	K_n	[10]
r/ν	$1/(n-1)$	K_n	T	$\lfloor n/2\rfloor$	P_n	T
$r\cdot\nu$	1	S_n	T	$4\lfloor n/2\rfloor-4$	K_n-M	[174]
$r-\kappa$	$2-n$	K_n	T	$\lfloor n/2\rfloor-1$	P_n	T
$r+\kappa$	2	S_n	T	n	K_n	[10]
r/κ	$1/(n-1)$	K_n	T	$\lfloor n/2\rfloor$	P_n	T
$r\cdot\kappa$	1	S_n	T	$4\lfloor n/2\rfloor-4$	K_n-M	O
$r-\alpha$	$2-n$	S_n	T	0	K_n	K
$r+\alpha$	2	K_n	T	n	P_n	K
r/α	$1/(n-1)$	S_n	T	1	K_n	K
$r\cdot\alpha$	1	K_n	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	P_n	[13]
$r-\beta$	$2-\lfloor n/2\rfloor$	Ur_n	[10]	$\lfloor n/6\rfloor$		AO
$r+\beta$	2	K_n	T	$\lfloor(5n+4)/6\rfloor$		AO
r/β	$2/\lfloor n/2\rfloor$	Ur_n	[10]			NR
$r\cdot\beta$	1	K_n	T			NR
$r-\omega$	$1-n$	K_n	T	$\lfloor n/2\rfloor-2$	P_n	T
$r+\omega$	3	S_n	T	$n+1$	K_n	[173]
r/ω	$1/n$	K_n	T	$\lfloor n/2\rfloor/2$	P_n	T
$r\cdot\omega$	2	S_n	T	$(n-2\lfloor n/4\rfloor)(\lfloor n/4\rfloor+1)$	$Ki_{n,\lfloor n/2\rfloor}$	[173]
$r-\chi$	$1-n$	K_n	T	$\lfloor n/2\rfloor-2$	P_n	T
$r+\chi$	3	S_n	T	$n+1$	K_n	[10]
r/χ	$1/n$	K_n	T	$\lfloor n/2\rfloor/2$	P_n	T
$r\cdot\chi$	2	S_n	T	$(n-2\lfloor n/4\rfloor)(\lfloor n/4\rfloor+1)$	$Ki_{n,\lfloor n/2\rfloor}$	[10]
$r-\mu$	$1-\lfloor n/2\rfloor$	K_n	T	0	P_n	[10]
$r+\mu$	2	S_n	T	$2\lfloor n/2\rfloor$	P_n	T
r/μ	$1/\lfloor n/2\rfloor$	K_n	T	1	P_n	[10]
$r\cdot\mu$	1	S_n	T	$\lfloor n/2\rfloor^2$	P_n	T

$g-ecc$	$\begin{cases}\frac{16-3n}{4}+\frac{3}{4n} & \text{if } n \text{ is odd,}\\ \frac{16-3n}{4}+\frac{1}{n} & \text{if } n \text{ is even}\end{cases}$	$Ki_{n,3}$	[10]	$\lceil n/2\rceil$	C_n	[10]
$g+ecc$	4	K_n	T	$n+\lfloor n/2\rfloor$	C_n	T
g/ecc	$\begin{cases}\frac{12n}{3n^2-4n-3} & \text{if } n \text{ is odd,}\\ \frac{12n}{3n^2-4n-4} & \text{if } n \text{ is even}\end{cases}$	C_n	[10]	3	K_n	[10]
$g\cdot ecc$	3	K_n	T	$n\lfloor n/2\rfloor$	C_n	T
$g-\pi$	$\begin{cases}\frac{11-n}{4}+\frac{1}{n-1} & \text{if } n \text{ is odd,}\\ \frac{4-n^2}{4n-4}+3 & \text{if } n \text{ is even}\end{cases}$	$Ki_{n,3}$	[22]	$\begin{cases}\frac{3n-1}{4} & \text{if } n \text{ is odd,}\\ \frac{3n^2-4n}{4n-4} & \text{if } n \text{ is even}\end{cases}$	C_n	[22]
$g+\pi$	4	K_n	T	$\begin{cases}\frac{5n+1}{4} & \text{if } n \text{ is odd,}\\ \frac{5n^2-4n}{4n-4} & \text{if } n \text{ is even}\end{cases}$	C_n	[22]
g/π	$\begin{cases}\frac{12n-12}{3n^2-7} & \text{if } n \text{ is odd,}\\ \frac{12n-12}{n^2-4} & \text{if } n \text{ is even}\end{cases}$	$Ki_{n,3}$	[22]	$\frac{(2\lfloor\sqrt{n}\rfloor+1)(n-1)}{n-1+\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor-1)}$	$Tr_{n,t+1}$; $t=2\lfloor\sqrt{n}\rfloor$	[22]
$g\cdot\pi$	3	K_n	T	$\begin{cases}\frac{n^2+n}{4} & \text{if } n \text{ is odd,}\\ \frac{n^3}{4n-4} & \text{if } n \text{ is even}\end{cases}$	C_n	[22]

$g-\rho$	$3-\frac{(n+1)(n-2)}{2n-2}$	$Ki_{n,3}$	[22]	$\begin{cases} \frac{3n-1}{4} & \text{if } n \text{ is odd,} \\ \frac{3n^2-4n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	C_n	[22]
$g+\rho$	4	K_n	T	$\begin{cases} \frac{5n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{5n^2-4n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	C_n	[22]
g/ρ	$\frac{6n-6}{(n+1)(n-2)}$	$Ki_{n,3}$	[22]	$\begin{cases} \frac{4n}{n+1} & \text{if } n \text{ is odd,} \\ \frac{4n-4}{n} & \text{if } n \text{ is even} \end{cases}$	C_n	O
$g\cdot\rho$	3	K_n	T			NR
$g-\lambda_1$	$4-n$	K_n	T	$2-n$	C_n	[30]
$g+\lambda_1$		$Ki_{n,3}$	[19]	$n+2$	Kn,C_n	[30]
g/λ_1	$\frac{3}{n-1}$	K_n	T	$n/2$	C_n	[30]
$g\cdot\lambda_1$		$Ki_{n,3}$	[19]	$3(n-1)$	Kn	R
$g-Ra$	$3-n/2$	K_n	T	$n/2$	C_n	[36]
$g+Ra$	$\frac{n-3+\sqrt{2}}{\sqrt{n-1}}+\frac{7}{2}$	S_n^+	[19]	$3n/2$	C_n	T
g/Ra	$6/n$	K_n	T	2	C_n	[36]
$g\cdot Ra$	$\frac{3n-9+3\sqrt{2}}{\sqrt{n-1}}+\frac{3}{2}$	S_n^+	[19]	$n^2/2$	C_n	T
$g-a$	$3-n$	K_n	T	$n-2\left(1-\cos\frac{2\pi}{n}\right)$	C_n	[10]
$g+a$		$Ki_{n,3}$	[79]	$n+3$	K_n	[10]
g/a	$3/n$	K_n	T		$Lol_{n,\lfloor\frac{n}{2}\rfloor}$	SO
$g\cdot a$		$Ki_{n,3}$	[79]	$3n$	K_n	[10]
$g-\nu$	$4-n$	K_n	T	$n-2$	C_n	T
$g+\nu$	4	S_n^+	T	$n+2$	K_n	[10]
g/ν	$\frac{3}{n-1}$	K_n	T	$n-1$	$Lol_{n,n-1}$	[10]
$g\cdot\nu$	3	S_n^+	T	$3n-3$	K_n	[10]
$g-\kappa$	$4-n$	K_n	T	$n-2$	C_n	T
$g+\kappa$	4	S_n^+	T	$n+2$	K_n	[10]
g/κ	$\frac{3}{n-1}$	K_n	T	$n-1$	$Lol_{n,n-1}$	[10]
$g\cdot\kappa$	3	S_n^+	T	$3n-3$	K_n	[10]
$g-\alpha$	$5-n$	S_n^+	[30]	$\lceil n/2\rceil$	C_n	[30]
$g+\alpha$	4	K_n	T	$n+\lfloor n/2\rfloor$	C_n	[30]
g/α	$3/(n-2)$	S_n^+	[30]	3	K_n	[30]
$g\cdot\alpha$	3	K_n	T	$\begin{cases} (n^2-1)/2 & \text{if } n \text{ is odd,} \\ n^2/2 & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ll_{n,n-1} \\ C_n \end{cases}$	[30]
$g-\beta$	$3-\lfloor n/2\rfloor$	Ur_n	T	$\lfloor 3n/2\rfloor$	C_n	[19]
$g+\beta$	4	K_n	T	$n+\lceil n/3\rceil$	C_n	[10]
g/β	$3/\lfloor n/2\rfloor$	Ur_n	T	3	S_n^+	[19]
$g\cdot\beta$	3	K_n	T	$n\lceil n/3\rceil$	C_n	[10]
$g-\omega$	$3-n$	K_n	T	$n-2$	C_n	T
$g+\omega$	6	S_n^+	T	$n+3$	K_n	[10]
g/ω	$3/n$	K_n	T	$n/2$	C_n	T
$g\cdot\omega$	8	$K_{a,n-a}$ with $a\geq 2$	[10]	$3n$	K_n	[10]
$g-\chi$	$3-n$	K_n	T	$\begin{cases} n-3 & \text{if } n \text{ is odd,} \\ n-2 & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ll_{n,n-1} \\ C_n \end{cases}$	[10]
$g+\chi$	6	S_n^+	T	$n+3$	K_n	[10]
g/χ	$3/n$	K_n	T	$\begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ll_{n,n-1} \\ C_n \end{cases}$	[10]
$g\cdot\chi$	8	$K_{a,n-a}$ with $a\geq 2$	[10]	$3n$	K_n	[10]
$g-\mu$	$3-\lfloor n/2\rfloor$	K_n	T	$\lceil n/2\rceil$	C_n	[10]
$g+\mu$	5	S_n^+	[10]	$n+\lfloor n/2\rfloor$	C_n	T
g/μ	$3/\lfloor n/2\rfloor$	K_n	T	$n/\lfloor n/2\rfloor$	C_n	[10]
$g\cdot\mu$	6	S_n^+	[10]	$n\lfloor n/2\rfloor$	C_n	T
$ecc-\pi$	0	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-\frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n-1}{2}-\frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	[156]

$ecc+\pi$	2	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}+\frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2}+\frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
ecc/π	1	K_n	T			NR
$ecc\cdot\pi$	1	K_n	T	$\begin{cases} \frac{(n-1)(n+1)(3n+1)}{16n} & \text{if } n \text{ is odd,} \\ \frac{n^2(3n-2)}{16(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc-\rho$	0	K_n	[25]	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-\frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4}-\frac{1}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	O
$ecc+\rho$	2	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}+\frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{4}+\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$	P_n	T
ecc/ρ	1	K_n	T		$\begin{cases} K_n-M \\ C_n \end{cases}$	SO
$ecc\cdot\rho$	2	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}\frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{4}\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc-\lambda_1$	$2-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-2\cos\frac{\pi}{n+1} & \text{for } n \text{ odd} \\ \frac{3n-2}{4}-2cos\frac{\pi}{n+1} & \text{for } n \text{ even} \end{cases}$	P_n	T
$ecc+\lambda_1$	$\sqrt{n-1}+2-\frac{1}{n}$	S_n	R		K_n-E	SO
ecc/λ_1	$\frac{1}{n-1}$	K_n	T	$\begin{cases} \frac{\frac{3n+1}{4}\frac{n-1}{n}}{2\cos\frac{\pi}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{\frac{3n-2}{4}}{2\cos\frac{\pi}{n+1}} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot\lambda_1$	$\sqrt{n-1}\cdot(2-\frac{1}{n})$	S_n	R		$PK_{n,x}$	SO
$ecc-Ra$	$(2-n)/2$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-\frac{n-3+2\sqrt{2}}{2} & \text{for } n \text{ odd} \\ \frac{3n-2}{4}-\frac{n-3+2\sqrt{2}}{2} & \text{for } n \text{ even} \end{cases}$	P_n	O
$ecc+Ra$	$\sqrt{n-1}+2-\frac{1}{n}$	S_n	O	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}+\frac{n-3+2\sqrt{2}}{2} & \text{for } n \text{ odd} \\ \frac{3n-2}{4}+\frac{n-3+2\sqrt{2}}{2} & \text{for } n \text{ even} \end{cases}$	P_n	[148]
ecc/Ra	$2/n$	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)}{2n^2-(6+4\sqrt{2})n} & \text{for } n \text{ odd} \\ \frac{3n-2}{2n-6+4\sqrt{2}} & \text{for } n \text{ even} \end{cases}$	P_n	AO
$ecc\cdot Ra$	$\begin{cases} \frac{n}{2} & \text{if } n \leq 13, \\ (2-\frac{1}{n})\cdot\sqrt{n-1} & \text{if } n \geq 14 \end{cases}$	$\begin{cases} K_n \\ S_n \end{cases}$	AO	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}\frac{n-3+2\sqrt{2}}{2} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{4}\frac{n-3+2\sqrt{2}}{2} & \text{if } n \text{ is even} \end{cases}$	P_n	[148]
$ecc-a$	$1-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-2+2\cos\frac{\pi}{n} & \text{for } n \text{ odd} \\ \frac{3n-2}{4}-2+2\cos\frac{\pi}{n} & \text{for } n \text{ even} \end{cases}$	P_n	T
$ecc+a$	$3-\frac{1}{n}$	S_n	[10]	$n+1$	K_n	[10]
ecc/a	$1/n$	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)}{8n(1-cos\frac{\pi}{n})} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{8(1-cos\frac{\pi}{n})} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot a$		$DC_{n,x,y}$	SO	$\begin{cases} 2n-5+\frac{2}{n} & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$	K_n-R	[174]
$ecc-\nu$	$2-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-1 & \text{if } n \text{ is odd,} \\ \frac{3n-6}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc+\nu$	$3-\frac{1}{n}$	S_n	[10]	n	K_n	[10]
ecc/ν	$\frac{1}{n-1}$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot\nu$	$2-\frac{1}{n}$	S_n	[10]	$\begin{cases} 2n-5+\frac{2}{n} & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$	K_n-R	[174]
$ecc-\kappa$	$2-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-1 & \text{if } n \text{ is odd,} \\ \frac{3n-6}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc+\kappa$	$3-\frac{1}{n}$	S_n	[10]	n	K_n	[10]
ecc/κ	$\frac{1}{n-1}$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot\kappa$	$2-\frac{1}{n}$	S_n	[10]	$\begin{cases} 2n-5+\frac{2}{n} & \text{if } n \text{ is odd,} \\ 2n-4 & \text{if } n \text{ is even} \end{cases}$	K_n-R	R
$ecc-\alpha$	$3-n-\frac{1}{n}$	S_n	[10]	$\begin{cases} \frac{3n^2-4n-3}{4n}-\frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n-2}{4} & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} Ki_{n,3} \\ P_n \end{cases}$	O

$ecc+\alpha$	2	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}+\frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{(3n+2)(n-2)}{4n}+\frac{n+2}{2} & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \end{cases}$	[131]
ecc/α	$\frac{2n-1}{n(n-1)}$	S_n	[10]		CPC	SO
$ecc\cdot\alpha$	1	K_n	T			NR
$ecc-\beta$	$\begin{cases} \frac{6-n}{2}-\frac{1}{2n} & \text{if } n \text{ is odd,} \\ \frac{5-n}{2} & \text{if } n \text{ is even} \end{cases}$	Ur_n	O	$\begin{cases} \frac{3n+1}{4}\frac{n}{n-1}-\lfloor\frac{n+1}{3}\rfloor & \text{if} n\equiv 3,5[6] \\ \frac{5n-8}{12}-\frac{3}{4n} & \text{if } n\equiv 1[6] \\ \frac{3n-2}{4}-\lfloor\frac{n+1}{3}\rfloor & \text{if } n\equiv 0,2[6] \\ \frac{5n-8}{12}-\frac{1}{n} & \text{if } n\equiv 4[6] \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \\ P_n \\ Co_{n,3} \end{cases}$	AO
$ecc+\beta$	2	K_n	T	$\begin{cases} \lfloor\frac{n+1}{3}\rfloor+\frac{3n+1}{4}\frac{n}{n-1} & \text{if} n\equiv 3,5[6] \\ \frac{13n-16}{12}-\frac{3}{4n} & \text{if } n\equiv 1[6] \\ \lfloor\frac{n+1}{3}\rfloor+\frac{3n-2}{4} & \text{if } n\equiv 0,2[6] \\ \frac{13n-16}{12}-\frac{1}{n} & \text{if } n\equiv 4[6] \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \\ P_n \\ Co_{n,3} \end{cases}$	AO
ecc/β	$\begin{cases} \frac{5n-1}{n(n-1)} & \text{if } n \text{ is odd,} \\ \frac{5}{n} & \text{if } n \text{ is even} \end{cases}$	Ur_n	AO			NR
$ecc\cdot\beta$	1	K_n	T			NR
$ecc-\omega$	$1-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-2 & \text{if } n \text{ is odd,} \\ \frac{3n-10}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc+\omega$	$4-\frac{1}{n}$	S_n	[10]	$n+1$	K_n	[10]
ecc/ω	$1/n$	K_n	T	$\begin{cases} \frac{3n+1}{8}\frac{n-1}{n} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot\omega$	$4-\frac{2}{n}$	S_n	[10]		$Ki_{n,x}$	[131]
$ecc-\chi$	$1-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}-2 & \text{if } n \text{ is odd,} \\ \frac{3n-10}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc+\chi$	$4-\frac{1}{n}$	S_n	[10]	$n+1$	K_n	[10]
ecc/χ	$1/n$	K_n	T	$\begin{cases} \frac{3n+1}{8}\frac{n-1}{n} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$ecc\cdot\chi$	$4-\frac{2}{n}$	S_n	[10]		$Ki_{n,x}$	SO
$ecc-\mu$	$1-\lfloor n/2\rfloor$	K_n	T	$\begin{cases} \frac{n^2-1}{4n} & \text{if } n \text{ is odd,} \\ \frac{n^2-4}{4n} & \text{if } n \text{ is even} \end{cases}$	$\begin{cases} P_n \\ Co_{n,3} \end{cases}$	[10]
$ecc+\mu$	$3-1/n$	S_n	[10]	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n}+\frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{5n-2}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
ecc/μ	$1/\lfloor n/2\rfloor$	K_n	T	$2-1/n$	S_n	[10]
$ecc\cdot\mu$	$2-1/n$	S_n	[10]	$\begin{cases} \frac{3n+1}{8}\frac{(n-1)^2}{n} & \text{if } n \text{ is odd,} \\ \frac{3n^2-2n}{8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi-\rho$	$\begin{cases} \frac{1-n}{4} & \text{if } n \text{ is odd,} \\ \frac{n}{4n-4}-\frac{n}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	[25]	0	$TReG$	T
$\pi+\rho$	2	K_n	T	$\begin{cases} \frac{3n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{3n}{4}-\frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
π/ρ		$Co_{n,x}$	SO	1	$TReG$	T
$\pi\cdot\rho$	1	K_n	T	$\begin{cases} \frac{n(n+1)}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2}{8}+\frac{n}{8n-8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi-\lambda_1$	$2-n$	K_n	T	$\begin{cases} \frac{n+1}{4}-2\cos\frac{\pi}{n+1} & \text{for } n \text{ odd} \\ \frac{n^2}{4n-4}-2\cos\frac{\pi}{n+1} & \text{for } n \text{ even} \end{cases}$	P_n	T
$\pi+\lambda_1$			NR	n	K_n	[10]
π/λ_1	$1/(n-1)$	K_n	T	$\begin{cases} \frac{n+1}{8\cos\frac{\pi}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{\frac{n+1}{4}+\frac{1}{4n-4}}{2\cos\frac{\pi}{n+1}} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi\cdot\lambda_1$	$\sqrt{n-1}$	S_n	O	$n-1$	K_n	O
$\pi-Ra$	$(2-n)/2$	K_n	T	$1-\sqrt{n-1}$	S_n	O

$\pi + Ra$	$1+\sqrt{n-1}$	S_n	T	$\begin{cases} \frac{3n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)} + \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$	C_n	T
π / Ra	$2/n$	K_n	T	$\begin{cases} \frac{s(s+4t-6)\sqrt{t}}{2(n-1)((s-4)\sqrt{t}+4t+2\sqrt{2}-4)} & \text{for } n \text{ odd} \\ \frac{((s-1)(s-3)+(s+1)(4t-2)-4t)\sqrt{t}}{2(n-1)((s-4)\sqrt{t}+4t+2\sqrt{2}-4)} & \text{for } n \text{ even} \end{cases}$ where $s = n-2t+3$ and $t = \begin{cases} \lceil n/5 \rceil & \text{if } n \equiv 4[5], \\ \lfloor n/5 \rfloor & \text{if } n \not\equiv 4[5] \end{cases}$	$DC_{n,t,t}$	AO
$\pi \cdot Ra$	$\sqrt{n-1}$	S_n	T	$\begin{cases} \frac{n(n+1)}{8} & \text{if } n \text{ is odd,} \\ \frac{n^3}{8n-8} & \text{if } n \text{ is even} \end{cases}$	C_n	T
$\pi - a$	$1-n$	K_n	T	$\begin{cases} \frac{3n+1}{4}\frac{n-1}{n} - 2 + 2\cos\frac{\pi}{n} & \text{for } n \text{ odd} \\ \frac{3n-2}{4} - 2 + 2\cos\frac{\pi}{n} & \text{for } n \text{ even} \end{cases}$	P_n	T
$\pi + a$			NR	$n+1$	K_n	[10]
π / a	$1/n$	K_n	T	$\begin{cases} \frac{(3n+1)(n-1)}{8n(1-\cos\frac{\pi}{n})} & \text{if } n \text{ is odd,} \\ \frac{3n-2}{8(1-\cos\frac{\pi}{n})} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi \cdot a$	$\begin{cases} \frac{3n+1}{2}\frac{n-1}{n}(1-cos\frac{\pi}{n}) & \text{if } n \text{ is odd,} \\ \frac{3n-2}{2}(1-cos\frac{\pi}{n}) & \text{if } n \text{ is even} \end{cases}$	P_n	AO	n	K_n	[10]
$\pi - \nu$	$2-n$	K_n	T	$\begin{cases} \frac{n-3}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)} - 1 & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi + \nu$	2	S_n	T	n	K_n	[10]
π / ν	$1/(n-1)$	K_n	T	$\begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi \cdot \nu$	1	K_n	T	$n-1$	K_n	[10]
$\pi - \kappa$	$2-n$	K_n	T	$\begin{cases} \frac{n-3}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)} - 1 & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi + \kappa$	2	S_n	T	n	K_n	[10]
π / κ	$1/(n-1)$	K_n	T	$\begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi \cdot \kappa$	1	K_n	T	$n-1$	K_n	[10]
$\pi - \alpha$	$2-n$	S_n	T	0	K_n	[10]
$\pi + \alpha$	2	K_n	T	n	S_n	[10]
π / α	$1/(n-1)$	S_n	T	1	K_n	[10]
$\pi \cdot \alpha$	1	K_n	T	$\frac{1}{n-1}\left\lfloor\frac{n+1}{3}\right\rfloor\left\lfloor\frac{2n+1}{3}\right\rfloor^2$ $s = \begin{cases} \left\lfloor\frac{n}{6}\right\rfloor + 3 \text{ if } n \equiv 4[6] \\ \left\lfloor\frac{n}{6}\right\rfloor + 2 \text{ if } n \not\equiv 4[6] \end{cases}$ $t = \begin{cases} s-1 \text{ if } n \equiv 0[2] \\ s \text{ if } n \equiv 1[2] \end{cases}$	$DC_{n,s,t}$	AO
$\pi - \beta$	$\begin{cases} \frac{4-n}{2} - \frac{1}{n-1} & \text{if } n \text{ is odd,} \\ \frac{3-n}{2} - \frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	Ur_n	O		$DC_{n,x,y}$	SO
$\pi + \beta$	2	K_n	T		Ctr	SO
π / β		Ur_n	SO		$Co_{n,x}$	SO
$\pi \cdot \beta$	1	K_n	T			NR
$\pi - \omega$	$1-n$	K_n	T	$\begin{cases} \frac{n-7}{4} & \text{if } n \text{ is odd,} \\ \frac{n-8}{4} + \frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi + \omega$	3	S_n	T	$n+1$	K_n	[10]
π / ω	$1/n$	K_n	T	$\begin{cases} \frac{n+1}{8} & \text{if } n \text{ is odd,} \\ \frac{n}{8} + \frac{n}{8n-8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi \cdot \omega$	2	S_n	T		$PK_{n,x}$	SO
$\pi - \chi$	$1-n$	K_n	T	$\begin{cases} \frac{n-7}{4} & \text{if } n \text{ is odd,} \\ \frac{n-8}{4} + \frac{n}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\pi + \chi$	3	S_n	T	$n+1$	K_n	[10]
π / χ	$1/n$	K_n	T	$\begin{cases} \frac{n+1}{8} & \text{if } n \text{ is odd,} \\ \frac{n}{8} + \frac{n}{8n-8} & \text{if } n \text{ is even} \end{cases}$	P_n	T

$\pi \cdot \chi$	2	S_n	T		$PK_{n,x}$	SO
$\pi - \mu$	$1-\lfloor n/2 \rfloor$	K_n	T	0	S_n	[10]
$\pi + \mu$	2	S_n	T	$\begin{cases} \frac{3n-1}{4} & \text{if } n \text{ is odd,} \\ \frac{3n+1}{4}+\frac{1}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T
π / μ	$1/\lfloor n/2 \rfloor$	K_n	T	1	S_n	[10]
$\pi \cdot \mu$	1	S_n	T	$\begin{cases} \frac{n^2-1}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2+n}{4}+\frac{n}{8n-8} & \text{if } n \text{ is even} \end{cases}$	P_n	T
$\rho - \lambda_1$	$2-n$	K_n	T	$\frac{n}{2}-2\cos(\frac{\pi}{n+1})$	P_n	T
$\rho + \lambda_1$			NR	n	K_n	[10]
ρ / λ_1	$1/(n-1)$	K_n	T	$\frac{n}{4\cos\frac{\pi}{n+1}}$	P_n	T
$\rho \cdot \lambda_1$			NR		$PK_{n,x}$	SO
$\rho - Ra$	$(2-n)/2$	K_n	T		$Co_{n,x}$	SO
$\rho + Ra$	$\begin{cases} \frac{n+2}{2} & \text{if } n \le 6, \\ 2-\frac{1}{n-1}+\sqrt{n-1} & \text{if } n \ge 7 \end{cases}$	$\begin{cases} K_n \\ S_n \end{cases}$	AO	$\begin{cases} n+\sqrt{2}-\frac{3}{2} & \text{if } n \le 56, \\ ? & \text{if } n \ge 57 \end{cases}$	$\begin{cases} P_n \\ PK_{n,x} \end{cases}$	SO
ρ / Ra	$2/n$	K_n	T	$\begin{cases} \frac{n-3+2\sqrt{2}}{n} & \text{if } n \le 7, \\ ? & \text{if } n \ge 8 \end{cases}$	$\begin{cases} P_n \\ Co_{n,x} \end{cases}$	SO
$\rho \cdot Ra$	$\begin{cases} \frac{n}{2} & \text{if } n \le 6, \\ 2\sqrt{n-1}-\frac{1}{\sqrt{n-1}} & \text{if } n \ge 7 \end{cases}$	$\begin{cases} K_n \\ S_n \end{cases}$	AO	$\begin{cases} \frac{n+2\sqrt{2}-3}{2}\frac{n}{2} & \text{if } n \le 57, \\ ? & \text{if } n \ge 58 \end{cases}$	$\begin{cases} P_n \\ PK_{n,x} \end{cases}$	SO
$\rho - a$	$1-n$	K_n	T	$\frac{n}{2}-2+2\cos\frac{\pi}{n}$	P_n	T
$\rho + a$		$K_{\lfloor\frac{n+1}{2}\rfloor,\lceil\frac{n+1}{2}\rceil}$	SO	$n+1$	K_n	[10]
ρ / a	$1/n$	K_n	T	$\frac{n}{4(1-\cos\frac{\pi}{n})}$	P_n	T
$\rho \cdot a$		CPC	SO	n	K_n	[174]
$\rho - \nu$	$2-n$	K_n	T	$\frac{n-2}{2}$	P_n	T
$\rho + \nu$	$\begin{cases} \frac{5}{2} & \text{if } n \text{ is odd,} \\ \frac{5}{2}+\frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor\frac{n+1}{2}\rfloor,\lceil\frac{n+1}{2}\rceil}$	[10]	n	K_n	[10]
ρ / ν	$1/(n-1)$	K_n	T	$n/2$	P_n	T
$\rho \cdot \nu$	$\begin{cases} \frac{3}{2}+\frac{1}{2n-2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor\frac{n+1}{2}\rfloor,\lceil\frac{n+1}{2}\rceil}$	[10]	$n-1$	K_n	[174]
$\rho - \kappa$	$2-n$	K_n	T	$\frac{n-2}{2}$	P_n	T
$\rho + \kappa$	$\begin{cases} \frac{5}{2} & \text{if } n \text{ is odd,} \\ \frac{5}{2}+\frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	S_n	[10]	n	K_n	[10]
ρ / κ	$1/(n-1)$	K_n	T	$n/2$	P_n	T
$\rho \cdot \kappa$	$\begin{cases} \frac{3}{2}+\frac{1}{2n-2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} & \text{if } n \text{ is even} \end{cases}$	S_n	[10]	$n-1$	K_n	R
$\rho - \alpha$	$3-n-1/(n-1)$	S_n	[25]		$PK_{n,x}$	SO
$\rho + \alpha$	2	K_n	T	$\lfloor\frac{4n-1}{5}\rfloor+\frac{1}{n-1}\left(2n-2\lfloor\frac{n}{5}\rfloor-3\right)\lfloor\frac{n+5}{5}\rfloor$ $t=2n-2\lfloor\frac{n}{5}-1\rfloor$	$Co_{n,t}$	AO
ρ / α	$\frac{2n-5}{(n-1)^2}$	S_n	[10]		$PK_{n,x}$	SO
$\rho \cdot \alpha$	1	K_n	T	$\frac{2}{n-1}\lceil\frac{n-1}{3}\rceil\left(n-\lceil\frac{n-1}{3}\rceil-\frac{1}{2}\right)\lceil\frac{2n-1}{3}\rceil$ $t=2n-2\lfloor\frac{n+1}{3}\rfloor+1$	$Co_{n,t}$	AO
$\rho - \beta$	$\begin{cases} \frac{6-n}{2}-\frac{2}{n-1} & \text{if } n \text{ is odd,} \\ \frac{5-n}{2}-\frac{3}{2n-2} & \text{if } n \text{ is even} \end{cases}$	Ur_n	AO		$Co_{n,x}$	SO
$\rho + \beta$	2	K_n	T	$\begin{cases} \frac{5n+6}{6}-\frac{2}{n-1} & \text{if } n \equiv 0[3], \\ \frac{5n+4}{6} & \text{if } n \equiv 1[3], \\ \frac{5n+8}{6}-\frac{6}{n-1} & \text{if } n \equiv 2[3]. \end{cases}$	Ctr	AO
ρ / β	$\begin{cases} \frac{5n-9}{(n-1)^2} & \text{if } n \text{ is odd,} \\ \frac{5n-8}{n(n-1)} & \text{if } n \text{ is even} \end{cases}$	Ur_n	AO		$Co_{n,x}$	SO
$\rho \cdot \beta$	1	K_n	T			NR
$\rho - \omega$	$1-n$	K_n	T	$(n-4)/2$	P_n	T
$\rho + \omega$	$\begin{cases} \frac{7}{2} & \text{if } n \text{ is odd,} \\ \frac{7}{2}-\frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$	[173]	$n+1$	K_n	[173]
ρ / ω	$1/n$	K_n	T	$n/4$	P_n	T

$\rho \cdot \omega$	$\begin{cases} 3 & \text{if } n \text{ is odd,} \\ 3 - \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	[173]		$PK_{n,x}$	[173]
$\rho - \chi$	$1-n$	K_n	T	$(n-4)/2$	P_n	T
$\rho + \chi$	$\begin{cases} \frac{7}{2} & \text{if } n \text{ is odd,} \\ \frac{7}{2} - \frac{1}{2n-2} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	[10]	$n+1$	K_n	[10]
ρ / χ	$1/n$	K_n	T	$n/4$	P_n	T
$\rho \cdot \chi$	$\begin{cases} 3 & \text{if } n \text{ is odd,} \\ 3 - \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$	$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$	[10]		$PK_{n,x}$	SO
$\rho - \mu$	$1 - \lfloor n/2 \rfloor$	K_n	T	$\begin{cases} \frac{n^2}{8n-8} & \text{if } n \equiv 0[4], \\ \frac{(n-2)(n+2)}{8n-8} & \text{if } n \equiv 2[4], \\ \frac{n+1}{8} & \text{otherwise} \end{cases}$	$Co_{n,n-t+1}$ $t = 2\lfloor \frac{n+2}{4} \rfloor$	[25]
$\rho + \mu$	$3 - 1/(n-1)$	S_n	[10]	$\lfloor n/2 \rfloor + n/2$	P_n	T
ρ / μ	$1/\lfloor n/2 \rfloor$	K_n	T	$2 - 1/(n-1)$	S_n	[10]
$\rho \cdot \mu$	$2 - 1/(n-1)$	S_n	[10]	$(n/2)\lfloor n/2 \rfloor$	P_n	T
$\lambda_1 - Ra$	$\begin{cases} 2\cos\frac{\pi}{n+1} - \frac{n-3+2\sqrt{2}}{2} & \text{if } n \leq 9, \\ \frac{4-n}{2} & \text{if } n \geq 10 \end{cases}$	$\begin{cases} P_n \\ C_n \end{cases}$	[37]	$(n-2)/2$	K_n	[37]
$\lambda_1 + Ra$	$2\sqrt{n-1}$	S_n	[37]	$(3n-2)/2$	K_n	T
λ_1 / Ra	$\begin{cases} \frac{4\cos\frac{\pi}{n+1}}{n-3+2\sqrt{2}} & \text{if } n \leq 26, \\ \frac{4}{n} & \text{if } n \geq 27. \end{cases}$	$\begin{cases} P_n \\ C_n \end{cases}$	[37]	$2 - 2/n$	K_n	[37]
$\lambda_1 \cdot R$	$n-1$	S_n	[37]	$n(n-1)/2$	K_n	T
$\lambda_1 - a$	-1	K_n	[10]	$n-3+t$ with $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t = 1$	$Ki_{n,n-1}$	AO
$\lambda_1 + a$	$2 - 2\cos\frac{\pi}{n} + 2\cos\frac{\pi}{n+1}$	P_n	T	$2n-1$	K_n	T
λ_1 / a		$Ki_{n,\lfloor \frac{n}{2} \rfloor}$	SO	$n/(n-1)$	K_n	[10]
$\lambda_1 \cdot a$	$4(1-\cos\frac{\pi}{n})(\cos\frac{\pi}{n+1})$	P_n	T	$n(n-1)$	K_n	T
$\lambda_1 - \nu$	0	K_n	[10]	$n-3+t$ with $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t = 1$	$Ki_{n,n-1}$	[79]
$\lambda_1 + \nu$	$1 + 2\cos\frac{\pi}{n+1}$	P_n	T	$2n-2$	K_n	T
λ_1 / ν	1	K_n	[10]	$n-2+t$ with $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t = 1$	$Ki_{n,n-1}$	[79]
$\lambda_1 \cdot \nu$	$2\cos\frac{\pi}{n+1}$	P_n	T	$(n-1)^2$	K_n	T
$\lambda_1 - \kappa$	0	K_n	[10]	$n-3+t$ with $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t = 1$	$Ki_{n,n-1}$	[79]
$\lambda_1 + \kappa$	$1 + 2\cos\frac{\pi}{n+1}$	P_n	T	$2n-2$	K_n	T
λ_1 / κ	1	K_n	[10]	$n-2+t$ with $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t = 1$	$Ki_{n,n-1}$	[79]
$\lambda_1 \cdot \kappa$	$2\cos\frac{\pi}{n+1}$	P_n	T	$(n-1)^2$	K_n	T
$\lambda_1 - \alpha$	$n - 1 - \sqrt{n-1}$	S_n	O	$n-2$	K_n	T
$\lambda_1 + \alpha$			NR	$\frac{n+\alpha'-1+\sqrt{(n-\alpha'-1)^2+4\alpha'(n-\alpha')}}{2}$; $\alpha' = \begin{cases} \left\lceil \frac{n+1+\sqrt{n^2-n+1}}{3} \right\rceil & \text{if } n \equiv 1[3] \\ \left\lfloor \frac{n+1+\sqrt{n^2-n+1}}{3} \right\rfloor & \text{otherwise} \end{cases}$	$CS_{n,\alpha'}$	[35]
λ_1 / α			NR	$n-1$	K_n	T
$\lambda_1 \cdot \alpha$			NR			NR
$\lambda_1 - \beta$		Ctr	[35]	$n-2$	K_n	T
$\lambda_1 + \beta$		$DC_{n,x,y}$	SO	n	K_n	[35]
λ_1 / β		Ctr	SO	$n-1$	K_n	T

$\lambda_1 \cdot \beta$	$\sqrt{n-1}$	S_n	O		Ur_n	SO
$\lambda_1 - \omega$	-1	K_n	[173]		$T_n^{\lfloor\sqrt{n}\rfloor}$	[173]
$\lambda_1 + \omega$	$2+2cos\frac{\pi}{n+1}$	P_n	T	$2n-1$	K_n	T
λ_1 / ω		$Ki_{n,3}$	[173]	$\frac{1}{2}\sqrt{\lfloor\frac{n}{2}\rfloor\cdot\lceil\frac{n}{2}\rceil}$	$K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$	[173]
$\lambda_1 \cdot \omega$	$4cos\frac{\pi}{n+1}$	P_n	T	$n(n-1)$	K_n	T
$\lambda_1 - \chi$	-1	K_n	K		$T_n^{\lfloor\sqrt{n}\rfloor}$	SO
$\lambda_1 + \chi$	$2+2cos\frac{\pi}{n+1}$	P_n	T	$2n-1$	K_n	T
λ_1 / χ		$\begin{cases} C_n \\ Lol_{n,n-1} \end{cases}$	[35]	$\frac{1}{2}\sqrt{\lfloor\frac{n}{2}\rfloor\cdot\lceil\frac{n}{2}\rceil}$	$K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$	[35]
$\lambda_1 \cdot \chi$	$4cos\frac{\pi}{n+1}$	P_n	T	$n(n-1)$	K_n	T
$\lambda_1 - \mu$	$2\cos\frac{\pi}{n+1} - \lfloor n/2\rfloor$	P_n	T	$n-1-\lfloor n/2\rfloor$	K_n	[178]
$\lambda_1 + \mu$	$\sqrt{n-1}+1$	S_n	R	$n-1+\lfloor n/2\rfloor$	K_n	T
λ_1 / μ	$\frac{2\cos\frac{\pi}{n+1}}{\lfloor\frac{n}{2}\rfloor}$	P_n	T	$\sqrt{n-1}$	S_n	[178]
$\lambda_1 \cdot \mu$	$\sqrt{n-1}$	S_n	[10]	$(n-1)\lfloor n/2\rfloor$	K_n	T
$Ra - a$	$-n/2$	K_n	[36]			NR
$Ra + a$		$Co_{n,x}$	SO	$3n/2$	K_n	T
Ra / a	$1/2$	K_n	[36]	$\frac{n-3+2\sqrt{2}}{4(1-cos\frac{\pi}{n})}$	P_n	AO
$Ra \cdot a$		$DC_{n,x,y}$	SO	$n^2/2$	K_n	T
$Ra - \nu$	$(2-n)/2$	K_n	[36]	$\frac{n-5+2\sqrt{2}}{2}$	P_n	R
$Ra + \nu$	$1+\sqrt{n-1}$	S_n	T	$(3n-2)/n$	K_n	T
Ra / ν	$n/(2n-2)$	K_n	[36]			NR
$Ra \cdot \nu$	$\sqrt{n-1}$	S_n	T	$n(n-1)/2$	K_n	T
$Ra - \kappa$	$(2-n)/2$	K_n	[10]	$\frac{n-5+2\sqrt{2}}{2}$	P_n	R
$Ra + \kappa$	$1+\sqrt{n-1}$	S_n	T	$(3n-2)/n$	K_n	T
Ra / κ	$n/(2n-2)$	K_n	[10]			NR
$Ra \cdot \kappa$	$\sqrt{n-1}$	S_n	T	$n(n-1)/2$	K_n	T
$Ra - \alpha$	$\sqrt{n-1}-(n-1)$	S_n	T	$(n-2)/2$	K_n	T
$Ra + \alpha$	$(n+2)/2$	K_n	O	$\sqrt{n-1}+(n-1)$	S_n	O
Ra / α	$1/\sqrt{n-1}$	S_n	T	$n/2$	K_n	T
$Ra \cdot \alpha$	$n/2$	K_n	O	$\lceil\frac{3n-2}{4}\rceil\sqrt{\lceil\frac{3n-2}{4}\rceil\lfloor\frac{n+2}{4}\rfloor}$	$K_{\lceil\frac{3n-2}{4}\rceil,\lfloor\frac{n+2}{4}\rfloor}$	AO
$Ra - \beta$	$\begin{cases} \frac{n-3}{\sqrt{2n-2}}+\frac{(n-3)(n-5)}{4n-4}+\frac{n-4}{\sqrt{n^2-1}} + \frac{2\sqrt{2}}{\sqrt{n+1}}-\frac{n-1}{2} & \text{if } n \text{ is odd} \\ \sqrt{\frac{n}{2}}-\frac{n-2}{4} & \text{if } n \text{ is even} \end{cases}$	Ur_n	AO	$(n-2)/2$	K_n	T
$Ra + \beta$	$1+\sqrt{n-1}$	S_n	T			NR
Ra / β	$\begin{cases} \frac{2}{n-1}\left(\frac{n-3}{\sqrt{2n-2}}+\frac{2\sqrt{2}}{\sqrt{n+1}}+\frac{(n-3)(n-5)}{4n-4} + \frac{n-4}{\sqrt{n^2-1}}\right) & \text{if } n \text{ is odd} \\ 4\frac{\sqrt{n/2}}{n-2} & \text{if } n \text{ is even} \end{cases}$	Ur_n	AO	$n/2$	K_n	T
$Ra \cdot \beta$	$\sqrt{n-1}$	S_n	T			NR
$Ra - \omega$	$-n/2$	K_n	[119]	$n/2-2$	C_n	T
$Ra + \omega$	$2+\sqrt{n-1}$	S_n	T	$3n/2$	K_n	T
Ra / ω	$1/2$	K_n	[119]	$n/4$	C_n	T
$Ra \cdot \omega$	$2\sqrt{n-1}$	S_n	T	$n^2/2$	K_n	T
$Ra - \chi$	$-n/2$	K_n	[119]	$\sqrt{\lfloor n/2\rfloor\lceil n/2\rceil}-2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[119]
$Ra + \chi$	$2+\sqrt{n-1}$	S_n	T	$3n/2$	K_n	T
Ra / χ	$1/2$	K_n	[119]	$\frac{1}{2}\sqrt{\lfloor n/2\rfloor\lceil n/2\rceil}$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[119]
$Ra \cdot \chi$	$2\sqrt{n-1}$	S_n	T	$n^2/2$	K_n	T
$Ra - \mu$			NR	$\sqrt{\lfloor\frac{n+4}{7}\rfloor\lfloor\frac{6n+2}{7}\rfloor}-\lfloor\frac{n+4}{7}\rfloor$	$K_{n-t,t}$, $t=\lfloor\frac{n+4}{7}\rfloor$	AO
$Ra + \mu$	$1+\sqrt{n-1}$	S_n	T	$\frac{n}{2}+\lfloor\frac{n}{2}\rfloor$	K_n	T
Ra / μ			NR	$\sqrt{n-1}$	S_n	[37]
$Ra \cdot \mu$	$\sqrt{n-1}$	S_n	T	$\frac{n}{2}\lfloor\frac{n}{2}\rfloor$	K_n	T
$a - \nu$			NR	1	K_n	K
$a + \nu$	$3-2\cos\frac{\pi}{n}$	P_n	T	$2n-1$	K_n	T

a/ν	$2-2\cos\frac{\pi}{n}$	P_n	AO	$n/(n-1)$	K_n	K
$a\cdot\nu$	$2-2\cos\frac{\pi}{n}$	P_n	T	$n(n-1)$	K_n	T
$a-\kappa$			NR	1	K_n	K
$a+\kappa$	$3-2\cos\frac{\pi}{n}$	P_n	T	$2n-1$	K_n	T
a/κ	$2-2\cos\frac{\pi}{n}$	P_n	AO	$n/(n-1)$	K_n	K
$a\cdot\kappa$	$2-2\cos\frac{\pi}{n}$	P_n	T	$n(n-1)$	K_n	T
$a-\alpha$	$2-n$	S_n	[10]	$n-1$	K_n	T
$a+\alpha$		$Ke^{\lfloor\frac{n}{2}\rfloor}_{\lceil\frac{n}{2}\rceil}$	SO	$n+1$	K_n	[10]
a/α		$DC_{n,x,y}$	SO	n	K_n	T
$a\cdot\alpha$		CPC	SO	$\lfloor n/2\rfloor\lceil n/2\rceil$	$CS_{n,\lfloor n/2\rfloor}$	[10]
$a-\beta$		Ctr	SO	$n-1$	K_n	T
$a+\beta$	2	S_n	[10]	$n+1$	K_n	[10]
a/β			NR	n	K_n	T
$a\cdot\beta$	1	S_n	[10]	$\lfloor\frac{n}{2}\rfloor-4$	K_n-R	O
$a-\omega$	$2-n$	$Ki_{n,n-1}$	[173]	$\left\lfloor\left(1-\frac{1}{\lfloor\sqrt{n}\rfloor}\right)n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	[173]
$a+\omega$	$4-2\cos\frac{\pi}{n}$	P_n	T	$2n$	K_n	T
a/ω		$PK_{n,\lfloor\frac{n}{2}\rfloor}$	[173]	$\frac{1}{2}\lfloor\frac{n}{2}\rfloor$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[173]
$a\cdot\omega$	$4-4\cos\frac{\pi}{n}$	P_n	T	n^2	K_n	T
$a-\chi$	$2-n$	$Ki_{n,n-1}$	[10]	$\left\lfloor(1-\frac{1}{\lfloor\sqrt{n}\rfloor})n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	AO
$a+\chi$	$4-2\cos\frac{\pi}{n}$	P_n	T	$2n$	K_n	T
a/χ		$PK_{n,\lfloor\frac{n}{2}\rfloor}$	SO	$\frac{1}{2}\lfloor\frac{n}{2}\rfloor$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]
$a\cdot\chi$	$4-4\cos\frac{\pi}{n}$	P_n	T	n^2	K_n	T
$a-\mu$	$2-2\cos\frac{\pi}{n}-\lfloor n/2\rfloor$	P_n	T	$\lceil n/2\rceil$	K_n	[10]
$a+\mu$	2	S_n	[10]	$n+\lfloor n/2\rfloor$	K_n	T
a/μ	$\frac{2-2\cos\frac{\pi}{n}}{\lfloor\frac{n}{2}\rfloor}$	P_n	T	$n/\lfloor n/2\rfloor$	K_n	[10]
$a\cdot\mu$	1	S_n	O	$n\lfloor n/2\rfloor$	K_n	T
$\nu-\kappa$	$\lfloor(3-n)/2\rfloor$	$K^{\lceil\frac{n+1}{2}\rceil}_{\lfloor\frac{n+1}{2}\rfloor}$	[10]	0	K_n	T
$\nu+\kappa$	2	P_n	T	$2(n-1)$	K_n	T
ν/κ	$\lfloor\frac{n-1}{2}\rfloor^{-1}$	$K^{\lceil\frac{n+1}{2}\rceil}_{\lfloor\frac{n+1}{2}\rfloor}$	[10]	1	K_n	T
$\nu\cdot\kappa$	1	P_n	T	$(n-1)^2$	K_n	T
$\nu-\alpha$	$2-n$	S_n	T	$n-2$	K_n	T
$\nu+\alpha$	3	$Ki_{n,n-1}$	T	n	K_n	[10]
ν/α	$1/(n-1)$	S_n	T	$n-1$	K_n	T
$\nu\cdot\alpha$	2	$Ki_{n,n-1}$	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]
$\nu-\beta$	$1-\lfloor n/2\rfloor$	Ur_n	T	$n-2$	K_n	T
$\nu+\beta$	2	S_n	T	n	K_n	[10]
ν/β	$1/\lfloor n/2\rfloor$	Ur_n	T	$n-1$	K_n	T
$\nu\cdot\beta$	1	S_n	T	$4\lfloor n/2\rfloor-4$	K_n-R	O
$\nu-\omega$	$2-n$	$Ki_{n,n-1}$	[173]	$\left\lfloor(1-\frac{1}{\lfloor\sqrt{n}\rfloor})n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	[173]
$\nu+\omega$	3	S_n	T	$2n-1$	K_n	T
ν/ω	$1/(n-1)$	$Ki_{n,n-1}$	[173]	$\lfloor n/2\rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[173]
$\nu\cdot\omega$	2	S_n	T	$n(n-1)$	K_n	T
$\nu-\chi$	$2-n$	$Ki_{n,n-1}$	[10]	$\left\lfloor(1-\frac{1}{\lfloor\sqrt{n}\rfloor})n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	[10]
$\nu+\chi$	3	S_n	T	$2n-1$	K_n	T
ν/χ	$1/(n-1)$	$Ki_{n,n-1}$	[10]	$\lfloor n/2\rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]
$\nu\cdot\chi$	2	S_n	T	$n(n-1)$	K_n	T
$\nu-\mu$	$1-\lfloor n/2\rfloor$	P_n	T	$1-\lceil n/2\rceil$	K_n	[10]
$\nu+\mu$	2	S_n	T	$\lfloor n/2\rfloor+n-1$	K_n	T
ν/μ	$1/\lfloor n/2\rfloor$	P_n	T	$(n-1)/\lfloor n/2\rfloor$	K_n	[10]
$\nu\cdot\mu$	1	S_n	T	$(n-1)\lfloor n/2\rfloor$	K_n	T
$\kappa-\alpha$	$2-n$	S_n	T	$n-2$	K_n	T
$\kappa+\alpha$	3	$Ki_{n,n-1}$	T	n	K_n	[10]

κ/α	$1/(n-1)$	S_n	T	$n-1$	K_n	T
$\kappa\cdot\alpha$	2	$Ki_{n,n-1}$	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]
$\kappa-\beta$	$1-\lfloor n/2\rfloor$	Ur_n	T	$n-2$	K_n	T
$\kappa+\beta$	2	S_n	T	n	K_n	[10]
κ/β	$1/\lfloor n/2\rfloor$	Ur_n	T	$n-1$	K_n	T
$\kappa\cdot\beta$	1	S_n	T	$4\lfloor n/2\rfloor-4$	K_n-R	O
$\kappa-\omega$	$2-n$	$Ki_{n,n-1}$	[173]	$\left\lfloor(1-\frac{1}{\lfloor\sqrt{n}\rfloor})n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	[173]
$\kappa+\omega$	3	S_n	T	$2n-1$	K_n	T
κ/ω	$1/(n-1)$	$Ki_{n,n-1}$	[173]	$\frac{1}{2}\lfloor\frac{n}{2}\rfloor$	$K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$	[173]
$\kappa\cdot\omega$	2	S_n	T	$n(n-1)$	K_n	T
$\kappa-\chi$	$2-n$	$Ki_{n,n-1}$	[10]	$\left\lfloor(1-\frac{1}{\lfloor\sqrt{n}\rfloor})n\right\rfloor-\lfloor\sqrt{n}\rfloor$	$T_n^{\lfloor\sqrt{n}\rfloor}$	[10]
$\kappa+\chi$	3	S_n	T	$2n-1$	K_n	T
κ/χ	$1/(n-1)$	$Ki_{n,n-1}$	[10]	$\lfloor n/2\rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[10]
$\kappa\cdot\chi$	2	S_n	T	$n(n-1)$	K_n	T
$\kappa-\mu$	$1-\lfloor n/2\rfloor$	P_n	T	$1-\lceil n/2\rceil$	K_n	[10]
$\kappa+\mu$	2	S_n	T	$\lfloor n/2\rfloor+n-1$	K_n	T
κ/μ	$1/\lfloor n/2\rfloor$	P_n	T	$(n-1)/\lfloor n/2\rfloor$	K_n	[10]
$\kappa\cdot\mu$	1	S_n	T	$(n-1)\lfloor n/2\rfloor$	K_n	T
$\alpha-\beta$	0	K_n	T	$n-2$	S_n	T
$\alpha+\beta$	2	K_n	T	n	S_n	[10]
α/β	1	K_n	T	$n-1$	S_n	T
$\alpha\cdot\beta$	1	K_n	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	Ur_n	[10]
$\alpha-\omega$	$1-n$	K_n	T	$n-3$	S_n	T
$\alpha+\omega$	$\lceil 2\sqrt{n}\rceil+1$	$Clqs$	[20]	$n+1$	$CS_{n,\alpha}$	K
α/ω	$1/n$	K_n	T	$(n-1)/2$	S_n	T
$\alpha\cdot\omega$			NR	$\left\lfloor\frac{(n+1)^2}{4}\right\rfloor$	$CS_{n,\left\lfloor\frac{n+1}{2}\right\rfloor}$	[10]
$\alpha-\chi$	$1-n$	K_n	T	$n-3$	S_n	T
$\alpha+\chi$	$\lceil(n+1)/3\rceil+1$		R	$n+1$	$CS_{n,\alpha}$	[13]
α/χ	$1/n$	K_n	T	$\frac{n-1}{2}$	S_n	T
$\alpha\cdot\chi$	n		K	$\left\lfloor\frac{n+1}{2}\right\rfloor\left\lceil\frac{n+1}{2}\right\rceil$	$CS_{n,\left\lfloor\frac{n+1}{2}\right\rfloor}$	[13]
$\alpha-\mu$	$1-\lfloor n/2\rfloor$	K_n	T	$n-2$	S_n	T
$\alpha+\mu$	$1+\lfloor n/2\rfloor$	K_n	[13]	n	S_n	[13]
α/μ	$1/\lfloor n/2\rfloor$	K_n	T	$n-1$	S_n	T
$\alpha\cdot\mu$	$\lfloor n/2\rfloor$	K_n	[13]	$\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$	$CS_{n,\left\lfloor\frac{n+1}{2}\right\rfloor}$	[13]
$\beta-\omega$	$1-n$	K_n	T	$\lfloor n/2\rfloor-2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	T
$\beta+\omega$	3	S_n	T	$n+1$	K_n	[10]
β/ω	$1/n$	K_n	T	$\lfloor n/2\rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	T
$\beta\cdot\omega$	2	S_n	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	Ur_n	[10]
$\beta-\chi$	$1-n$	K_n	T	$\lfloor n/2\rfloor-2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	T
$\beta+\chi$	3	S_n	T	$n+1$	K_n	[10]
β/χ	$1/n$	K_n	T	$\lfloor n/2\rfloor/2$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	T
$\beta\cdot\chi$	2	S_n	T	$\lfloor n/2\rfloor\lceil n/2\rceil$	Ur_n	[10]
$\beta-\mu$	0	P_n	[10]	$\lfloor n/2\rfloor-1$	K_n	T
$\beta+\mu$	2	S_n	T	$2\lfloor n/2\rfloor$	P_n	T
β/μ	1	P_n	[10]	$\lfloor n/2\rfloor$	K_n	T
$\beta\cdot\mu$	1	S_n	T	$\lfloor n/2\rfloor^2$	P_n	T
$\omega-\chi$			NR	0	K_n	K
$\omega+\chi$	4	Bipartite	T	$2n$	K_n	T
ω/χ			NR	1	K_n	K
$\omega\cdot\chi$	4	Bipartite	T	n^2	K_n	T
$\omega-\mu$	$2-\lfloor n/2\rfloor$	P_n	T	$\lceil n/2\rceil$	K_n	[10]
$\omega+\mu$	3	S_n	T	$n+\lceil n/2\rceil$	K_n	T
ω/μ	$2/\lfloor n/2\rfloor$	P_n	T	$n/\lfloor n/2\rfloor$	K_n	[10]

$\omega \cdot \mu$	2	S_n	T	$n\lceil n/2\rceil$	K_n	T
$\omega - \mu$	$2-\lfloor n/2\rfloor$	P_n	T	$\lceil n/2\rceil$	K_n	[10]
$\omega + \mu$	3	S_n	T	$n+\lceil n/2\rceil$	K_n	T
ω/μ	$2/\lfloor n/2\rfloor$	P_n	T	$n/\lfloor n/2\rfloor$	K_n	[10]
$\omega \cdot \mu$	2	S_n	T	$n\lceil n/2\rceil$	K_n	T

The next table summarizes the results of AGX Form 1 obtained in [107]. We use the same notations as in Table 8.

Table 9: List of AGX conjectures obtained in [107].

$q_1-\delta$	2	C_n	[107]	$n-\frac{5}{2}+\frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1+\delta$	$3+2\cos\frac{\pi}{n}$	P_n	T	$3n-3$	K_n	T
q_1/δ	2	ReG	K	$n-\frac{3}{2}+\frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1\cdot\delta$	$2+2\cos\frac{\pi}{n}$	P_n	T	$2(n-1)^2$	K_n	T
$q_1-\overline{d}$	2	C_n	[107]			NR
$q_1+\overline{d}$	$4+2\cos\frac{\pi}{n}-\frac{2}{n}$	P_n	T	$3n-3$	K_n	T
$q_1/\overline{d}$	2	ReG	K	$\frac{n^2}{2n-2}$	S_n	[107]
$q_1\cdot\overline{d}$	$4\left(1-\frac{}{n}\right)\left(\cos\frac{\pi}{n}\right)$	P_n	T	$2(n-1)^2$	K_n	T
$q_1-\Delta$	1	S_n	K	$n-1$	K_n	[107]
$q_1+\Delta$	$4+2\cos\frac{\pi}{n}$	P_n	T	$3n-3$	K_n	T
q_1/Δ	$\frac{n}{n-1}$	S_n	[107]	2	ReG	K
$q_1\cdot\Delta$	$4+4\cos\frac{\pi}{n}$	P_n	T	$2(n-1)^2$	K_n	T
$q_1-\overline{l}$	$2+2\cos\frac{\pi}{n}-\frac{n+1}{3}$	P_n	T	$2n-3$	K_n	T
$q_1+\overline{l}$			NR	$2n-1$	K_n	[107]
$q_1/\overline{l}$	$\frac{6+6\cos\frac{\pi}{n}}{n+1}$	P_n	T	$2n-2$	K_n	T
$q_1\cdot\overline{l}$			NR			NR
q_1-D	$2+2\cos\frac{\pi}{n}-(n-1)$	P_n	T	$2n-3$	K_n	T
q_1+D			NR	$\frac{3}{2}n-1+\frac{\sqrt{n^2+4n-12}}{2}$	K_n-e	[107]
q_1/D	$\frac{2+2\cos\frac{\pi}{n}}{n-1}$	P_n	T	$2n-2$	K_n	T
$q_1\cdot D$			ND		Bug	SO
q_1-r	$2+2\cos\frac{\pi}{n}-\lfloor\frac{n}{2}\rfloor$	P_n	T	$2n-3$	K_n	T
q_1+r			NR	$2n-1$	K_n	[107]
q_1/r	$\frac{2+2\cos\frac{\pi}{n}}{\lfloor\frac{n}{2}\rfloor}$	P_n	T	$2n-2$	K_n	T
$q_1\cdot r$			NR		Bag	SO
q_1-ecc	$\begin{cases} 2\cos\frac{\pi}{n}-\frac{3n-10}{4} & \text{if } n \text{ is even} \\ 2\cos\frac{\pi}{n}-\frac{3n^2-10n-1}{4n} & \text{if } n \text{ is odd} \end{cases}$	P_n	T	$2n-3$	K_n	T
q_1+ecc			NR	$2n-1$	K_n	[107]
q_1/ecc	$\begin{cases} \frac{8+8\cos(\pi/n)}{3n-2} & \text{if } n \text{ is even} \\ \frac{8n+8n\cos(\pi/n)}{(3n+1)(n-1)} & \text{if } n \text{ is odd} \end{cases}$	P_n	T	$2n-2$	K_n	T
$q_1\cdot ecc$			NR			NR
q_1-g	$4-n$	C_n	[107]	$2n-5$	K_n	T
q_1+g	$q_1(Ki_{n,3})+3$	$Ki_{n,3}$	[107]			[107]
q_1/g	$\frac{4}{n}$	C_n	[107]	$\frac{2n-2}{3}$	K_n	T
$q_1\cdot g$	$3q_1(Ki_{n,3})$	$Ki_{n,3}$	[107]	$\begin{cases} 6n-6, & 4\leq n\leq 15 \\ q_1(Tu_{n,\lceil\frac{n+2}{2}\rceil})(\lceil\frac{n+2}{2}\rceil) & n\geq 16 \end{cases}$	$Tu_{n,\lceil\frac{n}{2}\rceil+1}$	[107]
$q_1-\pi$	$\begin{cases} 2\cos\frac{\pi}{n}-\frac{3n^2-10n-1}{4n} & \text{if } n \text{ is odd} \\ 2\cos\frac{\pi}{n}-\frac{3n-8}{4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$2n-3$	K_n	T
$q_1+\pi$			NR	$2n-1$	K_n	[107]
q_1/π	$\begin{cases} \frac{8n+8n\cos(\pi/n)}{(3n+1)(n-1)} & \text{if } n \text{ is odd} \\ \frac{8+8\cos(\pi/n)}{3n-2} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$2n-2$	K_n	T
$q_1\cdot\pi$			NR	$2n-2$	K_n	O
$q_1-\rho$	$\begin{cases} 2+2\cos\frac{\pi}{n}-\frac{n+1}{4} & \text{if } n \text{ is odd} \\ 2+2\cos\frac{\pi}{n}-\frac{n^2}{4n-4} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$2n-3$	K_n	T
$q_1+\rho$	$4+\begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd} \\ \frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \end{cases}$	C_n	O	$2n-1$	K_n	[107]
q_1/ρ	$\begin{cases} \frac{8+8\cos(\pi/n)}{n+1} & \text{if } n \text{ is odd} \\ \frac{8(n-1)(1+\cos(\pi/n))}{n^2} & \text{if } n \text{ is even} \end{cases}$	P_n	T	$2n-2$	K_n	T

$q_1 \cdot \rho$			NR		$Ki_{n,\lceil \frac{n+4}{2} \rceil}$	SO
$q_1 - \nu$	2	C_n	[107]	$n - \frac{5}{2} + \frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1 + \nu$	$3 + 2\cos\frac{\pi}{n}$	P_n	T	$3n-3$	K_n	T
q_1/ν	2	C_n	[107]	$n - \frac{3}{2} + \frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1 \cdot \nu$	$2 + 2\cos\frac{\pi}{n}$	P_n	T	$2(n-1)^2$	K_n	T
$q_1 - \kappa$	2	C_n	[107]	$n - \frac{5}{2} + \frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1 + \kappa$	$3 + 2\cos\frac{\pi}{n}$	P_n	T	$3n-3$	K_n	T
q_1/κ	2	C_n	[107]	$n - \frac{3}{2} + \frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	[107]
$q_1 \cdot \kappa$	$2 + 2\cos\frac{\pi}{n}$	P_n	T	$2(n-1)^2$	K_n	T
$q_1 - a$	$2 + 2\cos\frac{2\pi}{n}$	C_n	O	$n - \frac{5}{2} + \frac{\sqrt{4n^2-20n+33}}{2}$	$Ki_{n,n-1}$	O
$q_1 + a$	4	P_n	T	$3n-2$	K_n	T
q_1/a	$2 - \frac{2}{n}$	K_n	[107]		$Ki_{n,\lceil n/3 \rceil+1}$	SO
$q_1 \cdot a$	$4 - 4\cos^2\frac{\pi}{n}$	P_n	T	$2n(n-1)$	K_n	T
$q_1 - \alpha$			NR	$2n-3$	K_n	T
$q_1 + \alpha$	$4 + \lfloor \frac{n}{2} \rfloor$, if n is odd	C_n $DLol_{n,t,t}$; $t = \lfloor \frac{n+6}{6} \rfloor$	SO	$\begin{cases} \frac{3n-2\sqrt{2n^2-4n+4}}{2} & \text{if } n \text{ is even} \\ \frac{3n-2\sqrt{2n^2-6n+3}}{2} & \text{if } n \text{ is odd} \end{cases}$	$CS_{n,\lfloor \frac{n}{2} \rfloor}$	[107]
q_1/α			NR	$2n-2$	K_n	T
$q_1 \cdot \alpha$	$2n-2$	K_n	O	$n(n-1)$	S_n	[107]
$q_1 - \beta$			NR	$2n-3$	K_n	T
$q_1 + \beta$			NR	$2n-1$		[107]
q_1/β			NR	$2n-2$	K_n	T
$q_1 \cdot \beta$	n	S_n	[107]			NR
$q_1 - \omega$	$q_1(Ki_{n,3}) - 3$	$Ki_{n,3}$	[107]		$\overline{\frac{n}{2}K_2}$ $\overline{\frac{n-3}{2}K_2 \cup K_3}$	SO
$q_1 + \omega$	$4 + 2\cos\frac{\pi}{n}$	P_n	T	$3n-2$	K_n	T
q_1/ω	$\frac{q_1(Ki_{n,3})}{3}$	$Ki_{n,3}$	[107]	$\frac{n}{2}$	$K_{p,q}$	O
$q_1 \cdot \omega$	$4 + 4\cos\frac{\pi}{n}$	P_n	T	$2n(n-1)$	K_n	T
$q_1 - \chi$	1 if n is odd	C_n	[107]	$\frac{3}{2}n - 4$ if n is even $q_1(\overline{\frac{n-3}{2}K_2 \cup K_3}) - \lfloor \frac{n}{2} \rfloor$ if n is odd	$\overline{\frac{n-3}{2}K_2 \cup K_3}$	SO
$q_1 + \chi$	$4 + 2\cos\frac{\pi}{n}$	P_n	T	$3n-2$	K_n	T
q_1/χ	$\frac{4}{3}$ if n is odd	C_n	[107]	$\frac{n}{2}$	$K_{p,q}$	[107]
$q_1 \cdot \chi$	$4 + 4\cos\frac{\pi}{n}$	P_n	T	$2n(n-1)$	K_n	T
$q_1 - \mu$	$2 + 2\cos\frac{\pi}{n} - \lfloor \frac{n}{2} \rfloor$	P_n	T	$2(n-1) - \lfloor \frac{n}{2} \rfloor$	K_n	[107]
$q_1 + \mu$			NR	$2(n-1) + \lfloor \frac{n}{2} \rfloor$	K_n	T
q_1/μ	$\frac{2+2\cos\frac{\pi}{n}}{\lfloor \frac{n}{2} \rfloor}$	P_n	T	n	S_n	[107]
$q_1 \cdot \mu$	n	S_n	[107]	$2(n-1)\lfloor \frac{n}{2} \rfloor$	K_n	T
$q_1 - Ra$	$\begin{cases} \frac{7-2\sqrt{2}-n}{2} + 2\cos\frac{\pi}{n} & \text{if } n \le 10 \\ 4 - \frac{n}{2} & \text{if } n \ge 11 \end{cases}$	P_n C_n	[107]	$\frac{3n-4}{2}$	K_n	O
$q_1 + Ra$	$\frac{1}{2} + 2\cos\frac{\pi}{n} + \frac{n}{2} + \sqrt{2}$	P_n	O	$\frac{5}{2}n - 2$	K_n	T
q_1/Ra	$\begin{cases} \frac{4+4\cos(\pi/n)}{n-3+2\sqrt{2}} & \text{if } n \le 14 \\ \frac{8}{n} & \text{if } n \ge 15 \end{cases}$	P_n C_n	[107]	$\begin{cases} \frac{4n-4}{n} & \text{if } 4 \le n \le 12 \\ \frac{n}{\sqrt{n-1}} & \text{if } n \ge 13 \end{cases}$	K_n S_n	O
$q_1 \cdot Ra$	$(1 + \cos\frac{\pi}{n})(n - 3 + 2\sqrt{2})$	P_n	O	$n(n-1)$	K_n	T

The next table summarizes AGX conjectures first studied in [69]. All the inequalities are formulated for a connected graph G, except when we write $q_1(T)$ and $q_1(U)$, which mean the signless Laplacian index os a tree T and that of a unicyclic graph U, respectively. In addion to the notation used for Table 8 and Table 9, BpG denote any connected bipartite graph. We also need to define the following two graphs H_n and H'_n on n vertices each. If n is even, H_n is constructed as follows from two copies of $K_{\frac{n}{2}}$. Delete an edge uv from one copy and an edge $u'v'$ from the other; then add the two edges uu' and vv'. If n is odd, H_n is constructed as follows from two copies of $K_{\frac{n1}{2}}$ and an isolated vertex w. Delete an edge uv from one copy of $K_{\frac{n1}{2}}$ and an edge $u'v'$ from the other; then add the four edges uw, vw, $u'w$ and $v'w$. If n is even, H'_n is obtained from two copies of $K_{\frac{n}{2}}$ by adding a single edge connecting the two cliques. If n is odd, H'_n is obtained from two copies of $K_{\frac{n-1}{2}}$ and an isolated vertex w by adding two edges between w and each clique $K_{\frac{n-1}{2}}$.

Table 10: List of AGX conjectures obtained in [69].

Conjecture	G	st.	Conjecture	G	st.
$q_1 \geq 2+2\cos\frac{\pi}{n}$	P_n	[69]	$q_1(G) \leq 2n-2$	K_n	[69]
$q_1(T) \leq n$	S_n	[69]	$q_1(U) \geq 4$	C_n	[69]
$q_1(U) \leq q_1(S_n^+)$	S_n^+	[69]	$q_1 \geq \Delta+1$	S_n	[69]
$q_1 \geq 2\overline{d} \geq 2\delta$	ReG	[69]	$q_1 \leq 2\Delta$	ReG	[69]
$q_1 - 2\overline{d} \leq n-4+4/n$	S_n	[90]	$q_1 - \overline{d} \geq 2$	C_n	[69]
$q_1 - \overline{d} \leq n-1$	K_n	[90]	$q_1 - \overline{d} - \lambda_1 \geq 0$	ReG	[69]
$q_1 - \overline{d} - \lambda_1 \leq n - \sqrt{n-1} - 2 + 2/n$	S_n	O	$\mu_1 + \lambda_1 - q_1 \geq 1$	K_n	[69]
$\mu_1 + \lambda_1 - q_1 \leq \sqrt{p \cdot q}$, with $p = \lfloor \frac{n}{2} \rfloor$, $q = \lceil \frac{n}{2} \rceil$	$K_{p,q}$	O	$q_1 - \mu_1 \geq 0$	BpG	[69]
$q_1 - \mu_1 \leq n-2$	K_n	[90]	$q_1 - 2\lambda_1 \geq 0$	ReG	[69]
$q_1 - 2\lambda_1 \leq n - 2\sqrt{n-1}$	S_n	O	$q_2 \geq 1$	S_n	[69]
$q_2(T) \leq q_2(DC_{n,p,p})$, with $p = \lfloor \frac{n-1}{2} \rfloor$	$DC_{n,p,p}$	[69]	$q_2 - \overline{d} \geq -1$	K_n	[80]
$q_2 - \overline{d} \leq n-6+8/n$	$K_{n-2,2}$	O	$q_2 - \delta \geq -1$	K_n	[80]
$q_2 - \delta \leq n-3$	$Ki_{n,n-1}$	[80]	$\Delta - q_2 \leq n-2$	S_n	O
$\Delta - q_2 \geq \lfloor \frac{n-2}{2} \rfloor - q_2(H_n)$	H_n	O	$q_2 - \lambda_1 \geq 1 - \sqrt{n-1}$	S_n	[78]
$q_2 - \lambda_1 \leq n-2-\sqrt{2n-4}$	$K_{n-2,2}$	O	$q_2 - a \geq -2$	K_n	[70]
If $G \ncong K_n$, $q_2 - a \geq 0$	S_n	[70]	$q_2 - a \leq q_2(H_n') - a(H_n')$	H_n'	O
$q_1 - q_2 \leq n$	K_n	[80]	If $G \ncong K_n$, $q_1 - q_2 \leq n-1$	S_n	[80]
If G is not bipartite, $q_n \geq q_n(Ki_{n,3})$	$Ki_{n,3}$	[59]	$q_1 - q_n \geq q_1(P_n) - q_n(P_n)$	P_n	O
$q_1 - q_n \leq q_1(Ki_{n,n-1}) - q_n(Ki_{n,n-1})$	$Ki_{n,n-1}$	O	$q_1 + q_n + 2\alpha \leq 3n-2$	$CS_{n,n-\alpha}$	[150]

In addition to the bounds listed in the above table, here are four conjecture obtained with AGX, three of which are proved in [69] and the last one is refuted in the same paper [69].

Conjecture 6.3 ([69]) *Let $e(Q)$ denote the number of distinct eigenvalues of the matrix Q and $m(q_i)$ the multiplicity of the eigenvalue q_i. Then $e(Q) = 2 \Longleftrightarrow m(q_2) = n-1 \Longleftrightarrow G \cong K_n$.*

Conjecture 6.4 ([69]) *If G has k duplicate vertices ($k > 1$), with neighbourhood of size d, then d is an eigenvalue of Q with $m(d) \geq k-1$.*

Conjecture 6.5 ([69]) *If G has k co–duplicate vertices ($k > 1$), with closed neighbourhood of size d, then $d-1$ is an eigenvalue of Q with $m(d-1) \geq k-1$.*

Conjecture 6.6 ([69]) *If G is a connected graph of order $n \geq 4$ with at least two dominating vertices, then $q_2 = \Delta - 1 = n-2$ with multiplicity at most $\lfloor n/2 \rfloor - 2$.*

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