**Trade-Off Between Robust
Risk Measurement and
Market Principals**

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Abstract

Cont et al. [3] recently showed that coherent risk measures are not robust with respect to changes in large data. In this paper we show that robust risk measures always generate pathological financial positions called "Good Deals". We also introduce the minimal distribution invariant modification of risk measures and study their robustness and sensitivity.

Résumé

Cont et al. [3] ont prouvé récemment que les mesures de risque cohérentes ne sont pas robustes par rapport à des variations dans de grands ensemble sde données. Dans ce papier, nous montrons que les mesures de risque robustes génèrent toujours des positions financières extrêmes appelées "bonnes affaires". Nous introduisons aussi la modification minimale invariate de mesures de risque et nous étudions leur robustesse et leur sensibilité.

1   Introduction

In 1999 Artzner et al. [1] introduce a family of sub-additive risk measures, called coherent risk measures. The idea was to measure the capital requirement for financial positions. Due to many good properties this family of risk measures has, it draws many attention in financial literature (see for example Balbas [4] and Schied [12]). However, there are also criticisms on some aspects of coherent risk measures. Among them, Cont et al. [3] and Kou et al. [9] discuss a rather serious criticism on the statistical aspects of coherent risk measures. They show that a distribution invariant coherent risk measures fails to be robust with respect to changes in larger data. Inspired by that, Cont et al. [3] introduce a new family of risk measures which are robust with respect to data entries. They show that the sub-additive risk measures fail to be robust, while the robust ones fail to be sub-additive.

In this paper we take a closer look to the family of robust risk measures introduced in Cont et al. [3], and show that this family of risk measures produce some sort of pathological financial positions, called Good Deals. Inspired by that, we introduce a minimal modification of a risk measure in order to rule out Good Deals. We will see that this minimal modification cannot be robust however, has a better sensitivity behavior compared to coherent risk measures.

The paper is organized as follows. In Section 2 we will define a coherent risk measure and briefly compare it with VaR (Value at Risk). Then, we define distribution invariant risk measures and the notion of a Good Deal. In Section 3 we will show that in a perfect market, a robust risk measure generates Good Deals. In Section 4 we define the minimal modification of a distribution invariant risk measure and analyze its sensitivity.

2   Risk Measures and Good Deals

The concept of a coherent risk measure is introduced in Artzner et al. [1] as follows

Definition 2.1 *Let $\mathcal{K}$ be a cone of random variables each representing the of future value of financial positions. Suppose also that $\mathcal{K}$ contains the set of real numbers $\mathbb{R}$. A coheretn risk measure is a mapping $\rho : \mathcal{K} \to \mathbb{R}$ which satisfies the following properties*

1. *$\rho(\lambda X) = \lambda \rho(X)$ for any $X \in \mathcal{K}$ and $\lambda > 0$ (positive homogeneity);*
2. *$\rho(X + m) = \rho(X) - m$ for any $X \in \mathcal{K}$ and $m \in \mathbb{R}$ (translation-invariance);*
3. *$\rho(X) \leq \rho(Y) \ \forall X, Y \in \mathcal{K}$ and $Y \leq X$ (monotonicity).*
4. *$\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in \mathcal{K}$ (sub-additivity).*

A particularly interesting example is $\mathrm{CVaR}_\alpha : L^p \to \mathbb{R}$ (for some $p \in [1, \infty]$) defined as follows:

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^1 \mathrm{VaR}_s(X) ds, \qquad (2.1)$$

where α is the level of risk tolerance and VaR_s is Value at Risk:

$$\mathrm{VaR}_s(X) = -q_X^-(s) := -\inf\{t | F_X(t) \geq s\}. \qquad (2.2)$$

The increasing function $q_X^-(.)$ is known as left quantile or left inverse of the CDF F_X. The risk measure CVaR_α is average over all values at risk which cannot be tolerated. In the case that X has continuous distribution it is equal to another well-known risk measure, Expected Shortfall, $ES(X) = -\mathbb{E}[X | X \leq \mathrm{VaR}_\alpha(X)]$.

As one can see, a coherent risk measure is sub-additive which is not the case for Value at Risk. There is an example of Pareto distributed random variables X, Y (even independent) such that $\mathrm{VaR}_\alpha(X + Y) > \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$, see Embrechts et al. [6]. This is obviously against the belief that risk is reduced by diversification. It is also worth mentioning that VaR_α is additive when two random variables are co-monotone i.e. $\mathrm{VaR}_\alpha(X + Y) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$ when $X = T(Y)$ for an increasing function T. Another difference between Value at Risk and a coherent riak measure is that Value at Risk is defined for all random variables

while a coherent risk measure is not. Indeed, once one needs to extend the domain of a coherent risk measure to all random variables, there will be a random variable whose risk is $+\infty$, see Delbaen [5].

A distribution invariant risk measure is a risk measure which associates the same value to two random variables with the same distribution. In particular, the most used risk measures, Value at Risk and Expected Shortfall, are distribution invariant. It is also worth mentioning that in the economical approach to risk, using Expected Utility, risk only depends on distribution.

There is a very nice representation for distribution invariant coherent risk measures which is stated as follows from Kusuoka [10]

Theorem 2.1 *Let ρ be a coherent risk measure. Then there exists a set of measures $\mathcal{M}$ on $[0,1]$ such that*

$$\rho(X) = \sup_{m \in \mathcal{M}} \int_0^1 \mathrm{CVaR}_\alpha(X) m(d\alpha). \tag{2.3}$$

Furthermore, if ρ is co-monotone subadditive then

$$\rho(X) = \int_0^1 \mathrm{CVaR}_\alpha(X) m(d\alpha), \tag{2.4}$$

for s given measure m on $[0,1]$.

By changing the order of integrals it is not very difficult to see that

$$\rho(X) = \sup_{\phi \in \mathcal{W}} \int_0^1 \mathrm{VaR}_s(X) \phi(s) ds. \tag{2.5}$$

for some set of decreasing density function $\mathcal{W}$, i.e. $\forall \phi \in \mathcal{W}$, $\phi \geq 0$, $\int_0^1 \phi = 1$.

Good Deal is a concept quite similar to the concept Arbitrage. A Good Deal is a risk free financial position with zero price. To give the exact definition of a Good Deal we first should give the definition of pricing rule.

Definition 2.2 *A continuous mapping $\pi : \mathcal{X} \to \mathbb{R}$ is a sub-linear pricing rule if*

   i) $\pi(X + k) = \pi(X) + k, \forall X \in \mathcal{X}, \forall k \in \mathbb{R}$;
   ii) $\pi(\lambda X) = \lambda \pi(X), \forall X \in \mathcal{X}, \forall \lambda > 0$;
   iii) $\pi(X + Y) \leq \pi(X) + \pi(Y), \forall X, Y \in \mathcal{X}$;
   iv) $\pi(X) \leq \pi(Y), \forall X, Y \in L^p$ and $X \leq Y$.

For the case $\mathcal{X} = L^2$ if we let

$$\mathcal{D} = \left\{ Z \in \mathcal{K} \middle| E[ZY] - \pi(Y) \leq 0, \forall Y \in \mathcal{X} \right\}, \tag{2.6}$$

then

$$\pi(X) = \sup_{m \in \mathcal{D}} E[mX]. \tag{2.7}$$

(for that see [2]). The set $\mathcal{D}$ can be interpreted as the set of all discount factors. When $\mathcal{D}$ is a singleton and also $\mathcal{X} = \mathcal{K}$, we say that the market is perfect. One interesting example is the equivalent martingale measure given by the Black-Scholes model, where $\mathcal{D}$ is a singleton. When the market is not complete $\mathcal{D}$ is the set is no longer a singelton. In an incomplete market for each X let $S(X)$ be the set of all value processes of a viable financial positions at which X is being super hedged i.e. $S_T \geq X$. Let

$$\pi(X) = \inf \left\{ S_0 \middle| (S_t)_{[0,T]} \in S(X) \right\}.$$

Now let us suppose that the market is endowed with a pricing rule π and on the other hand a group of the market participant are endowed with a risk measure (not necessary coherent) ρ. From their point of view, any financial position which has no risk with no price is a good deal. Inspired from this we introduce a Good Deal as follows

Definition 2.3 *With the above notation a financial positionX $\in \mathcal{X}$ is a Good Deal if $\rho(X) < 0$ while also $\pi(X) \leq 0$.*

3 Robustness Risk Measures

If ρ is a distribution invariant risk measure then it could be considered as a function on the CDF of random variables i.e. $\rho(X) = \rho(F_X)$ for all $X \in \mathcal{K}$. Let A be a subset of real numbers $\mathbb{R}$. We denote by $\mathcal{D}_p(A)$ the space of distributions on A with finite p-th moment. We also denote the set of all distributions by $\mathcal{D}(A)$. To simplify the notation if $A = [0,1]$ we write $\mathcal{D}_p$ and $\mathcal{D}$ instead. Now let $\rho : \mathcal{D}_p \to \mathbb{R}$ be a distribution-based risk measure and let $\widehat{\rho_n} : \mathbb{R}^n \to \mathbb{R}$ be a historical risk estimator.We say that a historical estimator of this measure $\widehat{\rho_n}(x_1, \ldots, x_n)$ is robust if a small variation from the distribution F_X results in a small change in the distribution of the estimator. In order to formally state this we need the following notation. We denote by $\mathcal{L}_n(\widehat{\rho_n}, F)$ the law of $\widehat{\rho_n}(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ is a random sample of size $n \geqslant 1$ from F. Moreover, let d_P denote the Prohorov metric for probability measures. In a formal statement we introduce robustness of ρ w.r.t a subset $\mathcal{C} \subseteq \mathcal{D}$ which contains all empirical distribution $\mathcal{D}_{\mathrm{emp}}$:

Definition 3.1 *We say that the historical estimator $\widehat{\rho}$ is $\mathcal{C}$-robust at F if, for any $\epsilon > 0$, there exist $\delta > 0$ and $n_0 \geqslant 1$ such that if $G \in \mathcal{C}$ and $d_P(F, G) < \delta$ then $d_P(\mathcal{L}_n(\widehat{\rho_n}, F), \mathcal{L}_n(\widehat{\rho_n}, G)) < \epsilon$ for all $n > n_0$.*

In this paper we mostly focus our attention to the following type of risk measures

$$\rho(X) = \int_0^1 \mathrm{VaR}_\alpha(X) dm(x) \tag{3.1}$$

where m is a probability measure on $(0, 1)$ We also can show (3.1) with

$$\rho(X) = \int_0^1 \mathrm{VaR}_\alpha(X) \phi(\alpha) d\alpha \tag{3.2}$$

where ϕ is a density function on $(0, 1)$. If m is absolutely continuous w.r.t. the uniform measure λ on $[0, 1]$ then ϕ is a integrable function. In general ϕ doe not need to be a measurable function however when needed we will assert that. Cont et al. [3] prove the following theorem

Theorem 3.1 *Let*

$$\rho(X) = \int_0^1 \mathrm{VaR}_\alpha(X) \phi(\alpha) d\alpha \tag{3.3}$$

where ϕ is a density function on $(0, 1)$. Then the risk measure ρ is robust iff there exists $\beta > 0$ such that $\mathrm{supp}(\phi) \subseteq [\beta, 1 - \beta]$.

From this theorem and (2.5) we have the following corollary:

Corollary 3.1 *There is no distribution invariant coherent risk measure, which is also robust.*

It is somehow understandable because a coherent risk measure penalizes big losses much, rather than the small losses, see Theorem 2.1, relation (2.5). The following proposition is quoted from Cont et al. [3]

Proposition 3.1 *Let ρ be a law invariant risk measure and $F \in \mathcal{C} \subseteq \mathcal{D}_\rho$*

If the historical estimator ρ_n is consistent with ρ at every $G \in \mathcal{C}$, then the following are equivalent:

- the restriction of ρ to $\mathcal{C}$ is continuous (w.r.t. the Lévy distance) at F;
- the historical estimator ρ_n is $\mathcal{C}$-robust at F.

4 Good Deals

Here we start with stating some primary characterizations of Good Deals and then try to show that robust risk measures in a perfect market always produce Good Deals.

Proposition 4.1 *The following conditions are equivalent:*

1. *There is no Good Deal.*
2. $\rho + \pi \geq 0$ *on $\mathcal{X}$.*

Proof. Let $X \in \mathcal{X}$ be a Good Deal. That means $\rho(X) < 0$ and $\pi(X) \leq 0$ which obviously implies $\rho(X) + \pi(X) < 0$. For the other implication let us suppose that there is a member $X \in \mathcal{X}$ such that $\rho(X) + \pi(X) < 0$. Let $Y = X - \pi(X)$. By translation-invariance we have $\rho(Y) = \rho(X) + \pi(X) < 0$ and $\pi(Y) = \pi(X) - \pi(X) = 0$ which shows that Y is a Good Deal.   □

Here we would like to give some interpretation of this latter statement. First, Proposition 4.1 stats that in the absence of Good Deals, not only $\rho(X) < 0$, $\pi(X) \leq 0$, but also $\rho(X) \leq 0$, $\pi(X) < 0$ cannot happen instantaneously. This means that in the presence of a Good Deal, there are risk free positions X $(\rho(X) \leq 0)$ which the buyer will be paid to possess them $(\pi(X) < 0)$. More precisely if X is a Good Deal i.e. $\rho(X) < 0$, $\pi(X) \leq 0$, then the position $X + \rho(X)$ is risk less $(\rho(X + \rho(X)) = 0)$ while $\pi(X + \rho(X)) = \pi(X) + \rho(X) < 0$. Second, from Proposition 4.1 one can find out a resemblence between the concept of Goode Deals and Arbitrages. To see that, let us consider that we want to buy X. For that, we have to pay $\pi(X)$ to possess X. Our global position then is $X - \pi(X)$. If such a position has negative risk then everyone is willing to hold such a position and then X would be in high demand, which is kind of Arbitrage. That means such a position cannot effectively exists which in conclusion implies $\rho(X - \pi(X)) \geq 0$ and by translation-invariance $\rho(X) + \pi(X) \geq 0$.

Now let us be back to our discussion. From now on we suppose that the market is perfect which means the pricing rule is given by $\pi(X) = E(mX)$, for some nonnegative $m \in L^1$ with $E(m) = 1$, and also $\mathcal{X} = \mathcal{K}$. On the other hand, we will work with risk measures of type 3.2, $\rho(X) = \int_0^1 \mathrm{VaR}_s(X)\phi(s)ds$ for some density ϕ. The following proposition shows how we can restrict our space to a smaller one

Proposition 4.2 *If the couple (ρ, π) generates Good Deals then at least there is one Good Deal co-monotone with $-m$ i.e. in the form $T(m)$ for some decreasing function T.*

Proof. Suppose that X is a Good Deal, that means $\rho(X) < 0$ and $\pi(X) \leq 0$. Let $Y = q_X^-(1 - F_m(m))$ where F_m is the commutative density function of m. We know $q_X^-(\alpha) \leq t$ iff $\alpha \leq F_X(t)$ for all $(\alpha, t) \in [0, 1] \times \overline{\mathbb{R}}$ (see Sklar [13]). By knowing that $F_m(m)$ is uniformly distributed, it turns out that Y has the same distribution as X. Since ρ is distribution invariant then $\rho(Y) = \rho(X) < 0$. Since the function $x \mapsto q_X^-(1 - F_m(x))$ is decreasing, by aid of Hardy-Littlewood's Theorem, theorem 2.76 in Fölmer and Schied [8], we have

$$E[mY] \leq E[mX] \leq 0.$$

Now by definition it turns out that Y is a Good Deal.   □

In the proof of the last proposition, the function $s \mapsto q_m^-(1 - s)$ is right continuous, which allows us to restrict ourselves to the set

$$\mathcal{H} = \left\{ T \in \mathcal{D}(\mathbb{R}) \,\middle|\, T(m) \in \mathcal{K}, \, T \text{ decreasing and right continuous} \right\}. \tag{4.1}$$

We now do some simple calculation. First

$$E[mT(m)] = \int_{-\infty}^{\infty} xT(x)dF_m(x) = \int_0^1 q_m^-(s)T(q_m^-(s))ds. \tag{4.2}$$

Second, for a decreasing function T we have

$$\rho(T(m)) = \int_0^1 -q_{T(m)}^-(s)\phi(s)ds$$

$$= -\int_0^1 T(q_m^-(s))\phi(s)ds$$

$$= -\int_{-\infty}^\infty T(x)(\phi \circ F_m)(x)dF_m(x). \quad (4.3)$$

Adding up (4.2) and (4.3) we get

$$\rho(T(m)) + \pi(T(m))$$

$$= \int_0^1 T(q_m^-(s))\Big(q_m^-(s) - \phi(s)\Big)ds$$

$$= \int_{-\infty}^\infty T(s)\Big(s - \phi \circ F_m(s)\Big)dF_m(s) \quad (4.4)$$

In the absence of Good Deals we have $\rho + \pi \geq 0$ which by Proposition 4.2 is equivalent to $\rho(T(m)) + \pi(T(m)) \geq 0$ for any decreasing and right continuous function T. So, by (4.4) the function $s \mapsto (\phi \circ F_m(s) - s)$ belongs to the following set

$$\mathcal{C}_1 = \left\{ S \in \mathcal{D}(\mathbb{R}) \;\middle|\; \begin{array}{c} \forall T \text{ decreasing and right continuous} \\ \int_{-\infty}^\infty TSdF_m \leq 0, \end{array} \right\} \quad (4.5)$$

Lemma 4.1 $\mathcal{C}_1$ *is equal to the following*

$$\mathcal{C} = \left\{ S \in \mathcal{D}(\mathbb{R}) \;\middle|\; \int_{-\infty}^{x-} SdF_m \leq 0 \,, \forall x \in \mathbb{R} \text{ and } \int_{-\infty}^\infty SdF_m = 0 \right\}. \quad (4.6)$$

Proof. Pick $S \notin \mathcal{C}$. This means either there is x_0 such that $\int_{-\infty}^{x_0-} SdF_m(x) > 0$ or $\int_{-\infty}^\infty SdF_m \neq 0..$ In the first case, if we let $T = 1_{(-\infty,x_0)}$ then $\int_{-\infty}^\infty STdF_m = \int_{-\infty}^{x_0-} SdF_m > 0$, which shows that $S \notin \mathcal{C}_1$. In the second case if $\int_{-\infty}^\infty SdF_m \neq 0$, then putting $T \equiv (\int_{-\infty}^\infty SdF_m)$ yields

$$\int_{-\infty}^\infty STdF_m(s) = (\int_{-\infty}^\infty SdF_m)^2 > 0.$$

This shows that $S \notin \mathcal{C}_1$. Now, for the other implication suppose that $S \in \mathcal{C}$. Let T be a decreasing and right continuous function. Since T is decreasing and right continuous, for a given $\epsilon > 0$ there is a sequence of real numbers $\{a_n\}_{n \in \mathbb{Z}}$, such that $a_n \to \pm\infty$ if $n \to \pm\infty$, and for the step function $T_\epsilon = \sum_{n \in \mathbb{Z}} (T(a_i) - T(a_{i+1}))1_{(-\infty,a_{i+1})}$ we have

$$|T_\epsilon(x) - T(x)| < \epsilon, \forall x \in \mathbb{R}. \quad (4.7)$$

By our assumption $= \int_{-\infty}^{a_i} TdF_m \leq 0$ which gives $\int_{-\infty}^\infty T_\epsilon SdF_m \leq 0$. Letting $\epsilon \to 0$ we have that $\int_{-\infty}^\infty TSdF_m \leq 0$. Now if we let $T = h$ for an arbitrary real number h we also conclude that $\int_{-\infty}^\infty STdF_m = 0$, which completes the proof of $S \in \mathcal{C}_1$. $\qquad\square$

Now we are in a position to state our main result:

Theorem 4.1 *Let ρ be a risk measure of type (3.2), and that for some $\beta > 0$, $supp(m) \subseteq [\beta, 1-\beta]$. Then in a perfect market, the couple (ρ, m) generates Good Deals. In particular, no robust risk measure of type (3.2), does exists.*

Proof. Suppose there is no Good Deal. Since F_m is a commutative distribution function then there is a number a_0 such that $F_m(a_0) > 1 - \beta$. Note that $\phi(F_m(s)) = 0$ for $s \geq a_0$. Since there is no Good Deal, the function $s \mapsto (\phi \circ F_m(s) - s)$ is a member of $\mathcal{C}$ and then we have:

$$0 = \int_{-\infty}^{\infty} \left(\phi \circ F_m(s) - s \right) dF_m(s) \tag{4.8}$$

$$= \int_{-\infty}^{a_0-} \left(\phi \circ F_m(s) - s \right) dF_m(s) + \int_{a_0}^{\infty} -s dF_m(s) \tag{4.9}$$

$$\leq \int_{a_0}^{\infty} -s dF_m(s) \tag{4.10}$$

But since $F_m(a_0) > 1 - \beta$ we have $\int_{a_0}^{\infty} -s dF_m(s) < 0$ which is contradiction $\qquad\square$

Now let us review the idea of the proof. What happens in the proof of Theorem 4.1 is, producing a Good Deal upon information in the event $\{m < \text{VaR}_\beta(m)\} \cup \{m > \text{VaR}_{1-\beta}\}$. Indeed, $\{m < \text{VaR}_\beta(m)\} \cup \{m > \text{VaR}_{1-\beta}\}$ is invisible to the ρ-user, therefore any financial position in this set is ignored by ρ-user eventhogh is is very risky. Based on this idea we make the following simple example:

Example Suppose that an event happens tomorrow, with probability 1 percent. There is a simple lottery, which penalizes for 100 dollar if the event happens, and nothing otherwise. The price of this lottery is simply $-100 \times \frac{1}{100} + 0 \times \frac{99}{100} = -1$ dollar.. It shows that if someone enters the lottery, he/she should be payed by one or if he/she is risk averse by more than one dollar. So let us consider that the price is something $c \leq -1$ dollar. Suppose that there is a financial practitioner who is endowed with the risk measure $\text{VaR}_{0.05}$. From his/her point of view the risk associated with this lottery is zero while he/she will be paid by $-c > 0$ dollar. Denoting this lottery by X it is clear that $X - c$ is a Good Deal.

On the other hand, in the previous example if the financial practitioner was endowed with $\text{CVaR}_{0.05}$ then he/she would assess the risk $\frac{1}{0.05} \int_0^{0.01} 100 ds = 20$ dollar, which is quite risky. This shows that in modifying $\text{CVaR}_{0.05}$ to $\text{CVaR}_{0.05}$ Good Deals will be ruled out. Now a natural question is what is the minimal way to come up with a risk measure whcih rules out Good Deals. That motivates us to give the following definition.

5 Minimal Distribution Invariant Modification

In this section we propose a way to overcome the situation that there Good Deals exists in a perfect market. For that we define a minimal distribution invariant modification of risk. Then we try to study the robustness and the sensitivity of this minimal modification.

Before stating the definition of the minimal modification we give a sufficient condition for existence of Good Deals, showing that in many cases Good Deals exist. To this end we need the following proposition taken from [2]

Proposition 5.1 *Let ρ be a coherent risk measure which has a representation as follows*

$$\rho(X) = \sup_{Z \in \mathcal{Q}} \mathbb{E}[XZ],$$

where $\mathcal{Q}$ is a weak-star closed convex set of nonnegative r.v. in L^∞ with $\mathbb{E}[Z] = 1$ for all $Z \in \mathcal{Q}$. In a perfect market with pricing rule $\pi(X) = \mathbb{E}[mX]$ the following two statement are equivalent

  1. *there is no Good Deal;*
  2. *$m \in \mathcal{Q}$.*

It has been shown in Filipovic and Svindland [7]that the canonical domain of any law invariant coherent risk measure ρ can be extended to L^1. This shows a representation as given in the previous proposition for the risk measure ρ exists.

Theorem 5.1 *Let $\rho, \tilde{\rho}$ be two distribution invariant risk measure such that $\rho \leq \tilde{\rho}$. Assume that $\tilde{\rho}$ is coherent and has a representation as $\tilde{\rho}(X) = \sup\limits_{Z \in \mathcal{Q}} \mathbb{E}[XZ]$. Furthermore, assume that the market is perfect with pricing rule given by $\pi(X) = \mathbb{E}[mX]$. With the above notation if $m \notin \mathcal{Q}$ then the couple (ρ, π) produces Good Deals.*

Proof. It is clear that since $m \notin \mathcal{Q}$ then the couple $(\tilde{\rho}, \pi)$ produce a Good Deal like X. So then $\rho(X) < 0$ and $\pi(X) \leq 0$. Since $\rho \leq \tilde{\rho}$ this yields $\rho(X) < 0$ and $\pi(X) \leq 0$ which shows that X is a Good Deal for the couple $(\tilde{\rho}, \pi)$. $\qquad\square$

As a particular interesting example one can consider the pricing rule given by the Black-Scholes model and VaR_α for a confidence level α. Indeed, $\mathrm{VaR} \leq \mathrm{CVaR}_\alpha$ where CVaR_α is conditional VaR defined as follows

$$\mathrm{CVaR}_\alpha(X) = \sup_{\{f \geq 0 | f \leq \frac{1}{\alpha}, \mathbb{E}[f]=1\}} \mathbb{E}[fX]. \tag{5.1}$$

Since the m, density of equivalent Martingale measure, provided by the Black-Scholes model is always unbounded, it cannot belong to $\{f \geq 0 | f \leq \frac{1}{\alpha}, \mathbb{E}[f] = 1\}$ and hence by previous theorem it turns out that there is a Good Deal for (VaR_α, m).

Definition 5.1 *Let π be a pricing rule and ρ a law invariant risk measure. A minimal law invariant modification, denoted by ρ_{min}, is a risk measure such that:*

a) $\rho \leq \rho_{min}$ and $\rho_{min} + \pi \geq 0$;

b) ρ_{min} is minimal, i.e. for any law invariant risk measure $\tilde{\rho}$ such that $\tilde{\rho} + \pi \geq 0$ and $\rho \leq \tilde{\rho}$ we have that $\rho_{min} \leq \tilde{\rho}$.

We have the following theorem

Theorem 5.2 *Suppose that ρ is a law invariant risk measure and the market is perfect i.e. the pricing rule $\pi : \mathcal{K} \to \mathbb{R}$ is given by $\pi(X) = \mathbb{E}(mX)$ for a positive random variable m with $\mathbb{E}[m] = 1$. Then the minimal modification of ρ is given as*

$$\rho_{min}(X) = \max\left\{\rho(X), -\int_0^1 \mathrm{VaR}_{1-s}(m)\mathrm{VaR}_s(X)ds\right\}. \tag{5.2}$$

Proof. First of all it is very easy to see that ρ_{min} is a law invariant risk measure. It is also obvious that $\rho \leq \rho_{min}$. Now let us consider that there is another law invariant risk measure $\tilde{\rho}$ such that $\tilde{\rho} + \pi \geq 0$ and $\rho \leq \tilde{\rho}$. By our assumption on $\tilde{\rho}$, it is clear that $-\mathbb{E}[mX] = -\pi(X) \leq \tilde{\rho}(X)$ which gives that $\max\{\rho(X'), \mathbb{E}[-mX']\} \leq \tilde{\rho}(X)$ for all $X \in \mathcal{K}$. Since the inequality holds for all $X \in \mathcal{K}$ one can take supremum over all X' which has the same distribution as X

$$\sup_{\{X' \sim X\}} \max\{\rho(X'), \mathbb{E}[-mX']\} \leq \sup_{\{X' \sim X\}} \tilde{\rho}(X')$$

By law-invariance of ρ and $\tilde{\rho}$ it yields

$$\max\{\rho(X), -\inf_{\{X' \sim X\}} \mathbb{E}[mX]\}.$$

By Hardy-Littlewood's Theorem it is known that

$$\inf_{\{X' \sim X\}} \mathbb{E}[mX] = \int_0^1 \mathrm{VaR}_{1-s}(m)\mathrm{VaR}_s(X)ds$$

which implies $\rho_{min} \leq \tilde{\rho}$ and the proof is complete. $\qquad\square$

As an interesting byproduct of Theorem 5.2 is that the short price must be seen in the risk assessment of a financial position, which is out of the scope of this paper. For more interesting discussions we refer the reader to [2]. It is also worth mentioning that if the market is imperfect then it is not very easy to give a minimal modification as (5.2). Indeed for a general risk measure $X \to \max\{\rho(X), \pi(-X)\}$ is a modification of ρ bot it is no longer minimal. Finding a minimal modification for a general risk measure is still an open problem, but for an interesting discussion on minimal modification for coherent risk measure one can see [2].

Now the following proposition shows that even this minimal modification cannot be robust.

Theorem 5.3 *Suppose that ρ is a risk measure of type* (3.2) *and the market is perfect with a positive discount factor. Then the minimal law invariant modification ρ_{min} is non-robust on any continuous r.v. X.*

Proof. Let $\rho(X) = \int_0^1 \mathrm{VaR}_s(X)\phi(s)ds$ for a distribution ϕ on $[0,1]$, and $\pi(X) = \mathbb{E}[-mX]$ for a nonnegative r.v. m with $\mathbb{E}[m] = 1$. Denote $s \to -\mathrm{VaR}_{1-s}(m)$ by $\tilde{\phi}$. First of all we must mention that the consistency of the historical estimator of ρ_{min} follows easily from the structure given in (5.2) and the consistency of each component in the maximum. Since m is positive then there exist $\epsilon > 0$ and $\delta > 0$ such that $\tilde{\phi} > \delta$ on $[0, \epsilon)$. We want to show in the following lines that there is a sequence of CDFs $\{F_n\}_{n\in\mathbb{N}}$ such that $F_n \xrightarrow{p.w.} F_X$ while $\rho_{min}(F_n) \to +\infty$. Let $b \in \mathbb{R} \cup \{-\infty\}$ be such that $F_X(b) = 0$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive decreasing sequence converging to 0. Let b_n be such that $F_X(b_n) = a_n$. Now pick a positive sequence $\epsilon_n > 0$ converging to zero, and such that $[-\epsilon_n + a_n, \epsilon_n + a_n] \subseteq (0, 1)$. For each $n \in \mathbb{N}$, let F_n be the CDF of a random variable given as follows

$$F_n(r) = \begin{cases} F_X(r) \;,\; r \in (-\infty, b_n - \frac{1}{\epsilon_n^2}) \cup (b_n, \infty) \\ a_n \;,\; r \in [b_n - \frac{1}{\epsilon_n^2}, b_n] \end{cases} \tag{5.3}$$

Let n_0 be a natural number such that $n \geq n_0$, $a_n < \epsilon$. Now by our assumptions and discussion above we have that

$$\int_0^1 \mathrm{VaR}_s(F_n)\tilde{\phi}(s) \geq \delta\epsilon_n \frac{1}{\epsilon_n^2} \;,\; n \geq n_0. \tag{5.4}$$

By Theorem 5.2 the above inequality implies $\rho_{min}(F_n) \geq \frac{\delta}{\epsilon_n}$ for $n \geq n_0$ which shows that $\lim_{n\to\infty} \rho_{min}(F_n) = +\infty$. Since convergence in the Lévy distance is equivalent to convergence in distribution then it yields that ρ_{min} is not continuous w.r.t. the Lévy distance and consequently by Proposition 3.1 the historical estimator of ρ_{min} is not robust.

$\square$

6 Sensitivity Analysis of Minimal Modification

Next, we want to analyze the sensitivity function of the historical estimator of the minimal extension ρ_{min}.

Let us start with the following definition

Definition 6.1 *Let ρ be a risk measure. For any real number $z \in \mathbb{R}$ and CDF F, define the sensitivity function $S(z, F)$ as follows*

$$S(z, F) = \lim_{\epsilon \downarrow 0} \frac{\rho(\epsilon\delta_z + (1-\epsilon)F) - \rho(F)}{\epsilon}. \tag{6.1}$$

Hence we use $S(z)$ instead of $S(z, F)$ if there is no confusion. In continuation we want to give an upper bound for the sensitivity function of the historical estimator of ρ_{min}. But before that we state the following proposition and lemma

Proposition 6.1 *Let ϕ and F and $f > 0$ have the following properties*

1. $\int_0^1 \frac{u}{f(\mathrm{VaR}_u(F))}\phi(u)du < \infty$;

2. $\phi \in L^1[0,1]$;

3. f is increasing in the left and decreasing in the right;

then

$$S(z) = -\int_0^{F(z)} \frac{u}{f(\mathrm{VaR}_u(F))}\phi(u)du + \int_{F(z)}^1 \frac{1-u}{f(\mathrm{VaR}_u(F))}\phi(u)du \qquad (6.2)$$

Proof. We make a dominating convergence argument as follows. Let f be increasing on $(-\infty, x_L)$ for $x_L < 0$ and decreasing for (x_R, ∞) for $x_R > 0$. Let $u_L, u_R \in (0,1)$ be numbers such that $u_L < \inf\{F(x_L) \, F(z), 1/3\}$ and $u_R > \max\{F(x_R), F(z), 2/3\}$. We know that

$$\mathrm{VaR}_u(F_\epsilon) = \begin{cases} \mathrm{VaR}_{\frac{u}{1-\epsilon}}(F) & , u < (1-\epsilon)F(z) \\ \mathrm{VaR}_{\frac{u-\epsilon}{1-\epsilon}}(F) & , u \geq (1-\epsilon)F(z) + \epsilon \\ z & , \text{otherwise} \end{cases} \qquad (6.3)$$

Keeping in mind that $\frac{d}{du}\mathrm{VaR}_u(F) = \frac{1}{f(\mathrm{VaR}_u(F))}$ relation (6.3) gives that

$$\frac{d}{d\epsilon}\mathrm{VaR}_u(F_\epsilon)|_{\epsilon=0} = \begin{cases} -\frac{u}{(1-\epsilon)^2 f(\mathrm{VaR}_{\frac{u}{1-\epsilon}}(F))} & , F(z) < u \\ \frac{1-u}{(1-\epsilon)^2 f(\mathrm{VaR}_{\frac{u-\epsilon}{1-\epsilon}}(F))} & , F(z) > u \\ 0 & , F(z) = u \end{cases} \qquad (6.4)$$

Now from this and mean value theorem one can see for any $\epsilon_0 > 0$ and $\epsilon \in (0, \epsilon_0)$ there exist $\eta \in (0, \epsilon)$ such that

$$\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F_0) = -\epsilon \frac{u}{(1-\eta)^2 f(\mathrm{VaR}_{\frac{u}{1-\eta}}(F))}. \qquad (6.5)$$

Since f is increasing over $(-\infty, x_L)$ it turns out that for $u \leq u_L$

$$\left|\frac{\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F_0)}{\epsilon}\right| \leq \frac{u}{(1-\epsilon_0)^2 f(\mathrm{VaR}_u(F))} \qquad (6.6)$$

which by our assumption is in $L^1(\phi)$. On the other hand from mean value theorem one can see also that for any $\epsilon_0 > 0$ and $\epsilon \in (0, \epsilon_1)$ there exist $\eta \in (0, \epsilon)$ such that

$$\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F_0) = \epsilon \frac{1-u}{(1-\eta)^2 f(\mathrm{VaR}_{\frac{u}{1-\eta}}(F))}. \qquad (6.7)$$

Since f is decreasing over (x_R, ∞) it turns out that for $u \geq u_R$

$$\left|\frac{\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F_0)}{\epsilon}\right| \leq \frac{1}{(1-\epsilon_1)^2 f(\mathrm{VaR}_{\frac{u}{1-\epsilon_1}}(F))} = \text{const} \qquad (6.8)$$

The right hand side is in $L^1(\phi)$, since by our assumption $\phi \in L^1[0,1]$. This makes us able to exchange the integral with the integral in the following expressions

$$\begin{aligned} S(z) &= \lim_{\epsilon \to 0^+} \int_0^1 \frac{\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F)}{\epsilon}\phi(u)du \\ &= \int_0^1 \lim_{\epsilon \to 0^+} \frac{\mathrm{VaR}_u(F_\epsilon) - \mathrm{VaR}_u(F)}{\epsilon}\phi(u)du \\ &= -\int_0^{F(z)} \frac{u}{f(\mathrm{VaR}_u(F))}\phi(u)du + \int_{F(z)}^1 \frac{1-u}{f(\mathrm{VaR}_u(F))}\phi(u)du. \end{aligned}$$

$\square$

The following lemma is a simple exercise from calculus which the proof is left to the reader

Lemma 6.1 *Let $h, g : [0,1] \to \mathbb{R}$ be two right differentiable functions. Let $k = \max\{h, g\}$. Then we have*

1. k is right differentiable on $\{g > h\} \cup \{h > g\}$;

2.

$$\frac{d^+ k}{dx}(x) = \begin{cases} \frac{d^+ g}{dx}(x) & , x \in \{g > h\} \\ \frac{d^+ h}{dx}(x) & , x \in \{h > g\} \end{cases} \tag{6.9}$$

3. $\limsup_{\epsilon \to 0^+} \frac{k(x+\epsilon) - k(x)}{\epsilon} \leq \max\{\frac{d^+ g}{dx}(x), \frac{d^+ h}{dx}(x)\}$

Let us denote the sensitivity function of the historical estimator of a risk of type (3.2) by S_ϕ. We have the following theorem on sensitivity of the historical estimator of ρ_{min}

Theorem 6.1 *With previous notation, let ρ be a risk measure of type (3.2) and the market be a perfect market. Assume ϕ and $s \to \mathrm{VaR}_{1-s}(m)$ satisfy the assumption of Proposition 6.1. Denote $\tilde{\phi}(s) = \mathrm{VaR}_{1-s}$ and $\rho_m(X) = -\int_0^1 \tilde{\phi}(s)\mathrm{VaR}_s(F_X)ds$. Then we have*

$$S_{\rho_{min}}(z) = \begin{cases} -\int_0^{F(z)} \frac{u}{f(\mathrm{VaR}_u(F))}\phi(u)du + \int_{F(z)}^1 \frac{1-u}{f(\mathrm{VaR}_u(F))}\phi(u)du & , \{\rho(X) > -\int_0^1 \mathrm{VaR}_{1-s}(m)\mathrm{VaR}_s(X)ds\} \\ -\int_0^{F(z)} \frac{u}{f(\mathrm{VaR}_u(F))}\tilde{\phi}(u)du + \int_{F(z)}^1 \frac{1-u}{f(\mathrm{VaR}_u(F))}\tilde{\phi}(u)du & , \{\rho(X) < -\int_0^1 \mathrm{VaR}_{1-s}(m)\mathrm{VaR}_s(X)ds\}\} \end{cases}$$

and on the set $\{\rho(X) = -\int_0^1 \mathrm{VaR}_{1-s}(m)\mathrm{VaR}_s(X)ds\}$ we have

$$S_{\rho_{min}}(z) \leq \max\{S_\phi(z), S_{\tilde{\phi}}(z)\}.$$

Proof. It is easily proven by using proposition 6.1 and Lemma 6.1. $\qquad\square$

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