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Abstract

The computational intractability of the dynamic programming (DP) equations associated with optimal admission and routing in stochastic loss networks of any non-trivial size (Ma et al., 2006, 2008) leads one to consider suboptimal decentralized game theoretic formulations of the problem.

The special class of radial networks with a central core of infinite capacity is considered, and it is shown (under adequate assumptions) that an associated decentralized call admission control problem becomes tractable asymptotically, as the size of network grows to infinity. This is achieved by following a methodology first explored by M. Huang et al. (2003, 2006-2008) in the context of large scale dynamic games for sets of weakly coupled linear stochastic control systems. At the established Nash equilibrium, each agent reacts optimally with respect to the average trajectory of the mass of all other agents; this trajectory is approximated by a deterministic infinite population limit (associated with the mean field or ensemble statistics of the random agents) which is the solution of a particular fixed point problem. This framework has connections with the mean field models studied by Lasry and Lions (2006, 2007) and close connections with the notion of oblivious equilibrium proposed by Weintraub, Benkard, and Van Roy (2005, 2008) via a mean field approximation.

Résumé

La complexité des calculs liés à la résolution des équations de la programmation dynamique pour la construction de stratégies d'admission et de routage optimales pour tout réseau stochastique de communication avec pertes de taille raisonnable (Ma et al. 2006, 2008), nous conduit à envisager une classe de solutions sous optimales décentralisées, fondées sur un formalisme de théorie des jeux.

La classe particulière des réseaux de type radial avec un noyau central de capacité de communication infinie est considérée, et il est établi, sous des hypothèses adéquates, qu'un problème d'admission décentralisée qui lui est associé devient asymptotiquement gérable au niveau des calculs, lorsque la taille du réseau tend vers l'infini. Ce résultat est obtenu en s'inspirant de la méthodologie explorée à l'origine dans Huang et al. (2003, 2006, 2008) dans le contexte des jeux dynamiques à grande échelle pour des systèmes linéaires stochastiques avec couplage faible. Lorsqu'un équilibre de Nash est établi, chaque agent doit réagir de manière optimale par rapport à la trajectoire moyenne de la masse des autres agents; cette dernière est approchée par une trajectoire déterministe représentant le comportement d'une population infinie (associée avec le champ moyen ou les statistiques d'ensemble des agents aléatoires); elle est calculée grâce à la résolution d'un problème de point fixe particulier. Cette méthodologie est en rapport avec les champs moyens étudiés dans Lasry et Lions (2006, 2007), et elle est intimement liée à la notion d'équilibres grossiers proposée par Weintraub, Benkard, et Van Roy (2005-2008), par le biais de champs moyens.

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1 Introduction

In the 1960s Benes [3] pioneered routing control in telephone networks, providing a general mathematical structure and deriving fundamental properties for such systems. Since that time call admission and routing control problems for the (stochastic) loss (or circuit-switched) networks have been topics of active research, examples of recent work in the area are given by [17, 19, 7, 27, 1, 9] and the references therein.

Loss networks can be viewed as systems of multi-server queues with zero internal buffering capacity whereby one customer can simultaneously occupy or release several servers or links along a given route. More precisely, a link of capacity c is equivalent to c parallel servers with zero waiting room each and, as a result, a call request which cannot be admitted instantaneously upon arrival and placed on a route, is immediately lost. The associated optimal admission and routing problems are strongly tied with problems of optimal control of queues with the specific difference that in classical queuing problems servers cannot be chosen simultaneously by the controller. In recent work [24], we have developed a state space representation and the relevant dynamic programming equations for multi-class, general call request arrivals and general connection durations loss networks. They correspond to systems of coupled partial differential equations which reduce to the piecewise linear algebraic equations of Markov decision problems when arrivals are Poisson and connection durations are exponential. See [2, 1, 10, 29] for related dynamic programming papers in a queuing context. While the availability of a formal system of equations characterizing optimal admission and routing decisions in loss networks under fairly general conditions of interest, one cannot escape the computational intractability of these equations as network size increases; this problem occurs even under the most favorable assumptions of Poisson call request arrivals and exponential connection durations [27].

In this paper, following the work in [26, 23] we employ non-cooperative dynamic game theory for the analysis of networks of large populations of weakly coupled agent subnetworks. More specifically, in this formulation, the original large network is split into a collection of small, self-optimizing agent network systems feeding into a large core network. For the purposes of the paper, each local agent network system is modelled as a peripheral node of a radial network together with its link (each of these pairs termed an *agent*) to the central core, which itself is abstracted as a central node (with infinite capacity). Due to the assumed radial network topology and the infinite capacity of the core, the analysis in this paper is effectively limited to issues of admission (not routing) control.

Our analysis is motivated by the relationship in a large population system between an individual agent and the mass of agents which is fundamental to the so-called Nash certainty equivalence (NCE)(or mean field games) methodology developed by Huang et al., see e.g. [14, 13, 16, 15, 11]; this has been applied to the construction of explicit distributed control laws in large scale linear quadratic regulator games and to the analysis of large classes of multi-agent non-linear stochastic dynamic games. A closely related approach has recently been independently developed by Lasry and Lions [21, 20], while Weintraub, Benkard and Van Roy [32, 31, 30] proposed the notion of oblivious equilibrium for models of many firm industry dynamics by use of a mean field approximation.

Call requests are assumed to randomly appear at each local network and can be local, or destined for distant networks. A call request mobilizes actual link resources only if it is admitted by both origin and destination nodes. In an effort to view each agent as a separate decision maker, we focus on the outgoing and incoming subsets of external aggregated call request streams respectively originating at or destined for the local agent, and which have already secured the distant node part of the admission process. These streams are called the *filtered streams/processes* of the local network. A notion which is central to our analysis and optimization of the distributed control of radial networks with uniform subnetworks is that of a *network decentralized state (NDS)*. An NDS is characterized by filtered processes which constitute a pair of Poisson processes, which are mutually independent, independent from one agent to another, and in general have a pair of distinct *deterministic intensities*, common to all agents. The NDS property implies that under local controls, the states of individual agents become independent, and for uniform controls, a uniformity relationship holds in that the probabilities of admission by a local network of either an outgoing call request or an incoming call request respectively are common to all networks at any given time. Thus under NDS, major simplifications of the analysis become possible.

(i) Given a specified class of local cost functions for each agent, and (ii) if it is the case that each of a set of local admission control laws is optimal for its associated decentralized state boundary streams, then there results in an NDS, called a *network decentralized equilibrium (NDE)*, for which the Nash equilibrium property holds. The existence of precisely such a set of local admission control laws is established in this paper within the class of randomized feedback admission rules.

A variety of approaches distinct from that taken in this paper has been proposed in the literature to mitigate the computational complexity of search procedures for optimal admission and routing decisions in loss networks. More specifically, the analysis in [7] is based upon the assumption of the statistical independence of each network link; an approximation result is obtained in [27] using reinforcement learning techniques [5]; and in [18] a game theoretic analysis is developed and uses the notion of shadow prices from decentralized optimization; while using a technique of polynomial cost approximation a suboptimal control is implemented in [28].

The paper is organized as follows. In Section 2, we formulate call admission control problems for a class of loss networks. Section 3 then specifies the class of agent subnetworks which are connected into the radial mass network under study. Then network decentralized states (NDSs) for distributed network systems are defined and their existence established. In Section 4, based upon the decentralized model developed in Section 3, control problems for the global networks under consideration are formulated as distributed control problems and the hybrid dynamic programming (DP) equation systems for each of the agent systems (developed in [24]) are then presented. It is subsequently established that there always exist randomized admissible local feedback rules which induce an NDS of the type introduced above with the Nash equilibrium property with respect to each agent's loss function. Section 6 contains the conclusions and outlines future work. Some of the proofs in this paper are given in the Appendices.

2 Call admission control of radial loss networks

The conceptual model of a communications network in this paper is that of a radial network consisting of a set of subnetworks which are mutually connected through a central hub of unlimited capacity (see Figure 2.1 (i)); this configuration is then abstracted into a radial network of connected vertices corresponding to individual agents.

Definition 2.1 A capacitated *network* of size M, $M \geq 2$, denoted $Net^M(\mathbb{V}, \mathbb{L}, \mathbb{C})$, is a graph with a set of vertices $\mathbb{V} = \{v_0, v_1, \cdots, v_M\}$, and a set of (bidirectional) links $\mathbb{L} = \{(v_0, v_i);\ i \in \mathcal{M}\}$, $\mathcal{M} = \{1, \cdots, M\}$, where the capacities of the links are denoted $\mathbb{C} = \{c_l = c; l \in \mathbb{L}\}$ and where $c \geq 1$. Any link (v_0, v_k), $k \in \mathcal{M}$, is called an *agent network*. □

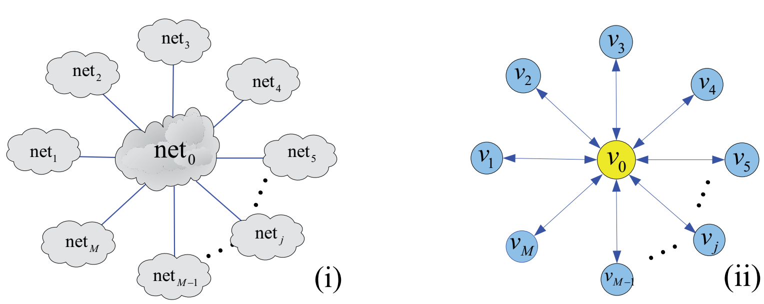

Figure 2.1: (i) Conceptual representation of a radial network; (ii) The abstraction $Net^M(\mathbb{V}, \mathbb{L}, \mathbb{C})$

2.1 Call request, connection and termination process specification

We assume that the call request, connection and termination process for a given infinite sequence $Net^M(\mathbb{V}, \mathbb{L}, \mathbb{C})$, $M \geq 2$, satisfy the specifications (S1)-(S3) given below:

(S1) The (local) call request process, i.e. a call request process between a peripheral vertex v_i and the central vertex v_0, denoted $Rq^M_{\{j,0\}}$, with call request events denoted $e^+_{\{j,0\}}$, is a homogeneous Poisson process with rate $\lambda_1 < \infty$;
The duration of the m-th established (local) connection between v_j and v_0, denoted $\eta_m^{\{j,0\}}$, with termination event denoted $e^-_{\{j,0\}}$, is exponentially distributed with parameter $1/\mu_1 < \infty$. The counting termination process of the events $e^-_{\{j,0\}}$ process is denoted $Dp^M_{\{j,0\}}$.

(S2) The (external) call request process, i.e. a call request from v_j to v_k, denoted $Rq^M_{\langle j,k\rangle}$, with events denoted $e^+_{\langle j,k\rangle}$, is a homogeneous Poisson process with rate $\frac{1}{M-1}\lambda_2 < \infty$. In other words, the rate of the process $Rq^M_{\langle j,k\rangle}$ is inversely proportional to the M.
The duration of the m-th established (external) connection from v_j to v_k, denoted $\eta_m^{\langle j,k\rangle}$ with termination event $e^-_{\langle j,k\rangle}$, is exponentially distributed with fixed parameter equal to $1/\mu_2 < \infty$. The counting termination process of the events $e^-_{\langle j,k\rangle}$is denoted $Dp^M_{\langle j,k\rangle}$.

Throughout the paper, whenever convenient and when no ambiguity is engendered, the occurrence of a given binary point process event will be identified with the characteristic function of the event taking the value 1, which otherwise takes the value 0.

(Note the use of $\{.\}$ for local (necessarily bidirectional) events and $\langle . \rangle$ for external (necessarily directional) events.)

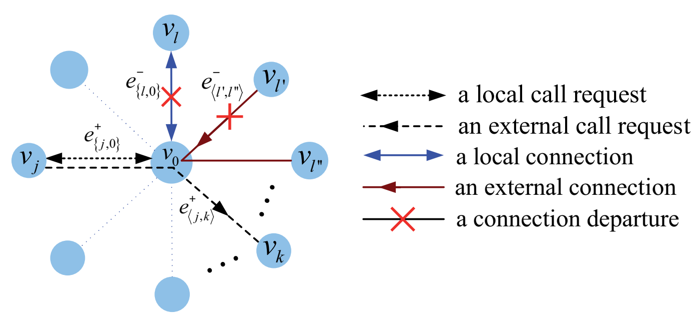

Figure 2.2: An illustration of events in a radial network

(S3) The set of the stochastic processes and random variables of each network $Net^M(\mathbb{V},\mathbb{L},\mathbb{C})$, specified in (S1) and (S2) are mutually independent.

Henceforth, we use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a probability space which will carry all the stochastic processes and random variables in the paper. Furthermore, all control processes are taken to be measurable with respect to the σ-fields generated by their argument processes.

2.2 Fundamental call request and connection activation model

An *activated connection* occupies a link resource in either an origin agent network only, or in both an origin network and a destination agent network.

Connection activation can only occur in response to a *call request event*. Call request events are of two types:

(i) *Local call requests*, these are such that connection activation decisions are entirely local (to an agent).

(ii) *External call requests*, these are simultaneously detected by an origin agent and a destination agent.
If the origin agent does not reject the call request, it is said to be *released*; likewise, if the destination network does not reject the call request, it is said to be *accepted*.
An external call request which is both released and accepted becomes a *connection activation event*.

A call request resulting in a connection activation is said to be *admitted*.

It is assumed that all of the decisions above take place instantaneously upon detection of a call request. The activated connection is of random duration, it ends with a *connection termination event* at which the occupied link resources in both the origin and destination networks are released.

2.3 The decentralized nature of generic agent j's decision making process

For the purposes of developing a decentralized control model at a local agent for what is, in essence, a sequence of bilateral agents' transactions over time, we conduct the analysis in terms of a stream of labeled incoming call requests and labeled outgoing call requests.

We define the jth stream of incoming and outgoing call requests respectively as the stream of external call requests destined for, or that originating from, agent j. In addition, we define the jth stream of local call requests, as those local call requests originating in the network of agent j.

We shall develop the decentralized formulation of the decision making process by extracting the following three substreams from the streams above, the first two of which are called *filtered processes*:

- The jth stream of incoming call requests labeled as released by their originating agent, (also referred to as the jth incoming released stream of call requests) designed $e^{j,+}_{\mathrm{RI}}$.
- The jth stream of outgoing call requests labeled as accepted by their destination agent, (also referred to as the jth outgoing accepted stream of call requests) designed $e^{j,+}_{\mathrm{AO}}$.
- The jth stream of local call requests designed $e^{j,+}_{\mathrm{Loc}}$.
  The action by an agent of labeling an incoming, respectively, outgoing, call request $e^{+}_{\langle j,k\rangle}$ as released, or accepted, an incoming, respectively, outgoing, call request is enacted by the local multiplication of the $\{0,1\}$ valued event $e^{+}_{\langle j,k\rangle}$ by a $\{0,1\}$ valued random variable ϕ_j, respectively ϕ_k, which in general depend only upon local state, the event type and time.

A call request between local networks i and j is activated if and only if $\phi_j.\phi_k = 1$.

Note that in the above external call request streams, the identity of agents other than j is suppressed, see Figure 3.5 for an illustration.

We denote $Rq^{M,j}_e(t)$ and $Dp^{M,j}_e(t)$, where $e = Loc, RI$, or AO as the counting process of the call request event and the departure event of activated connections respectively.

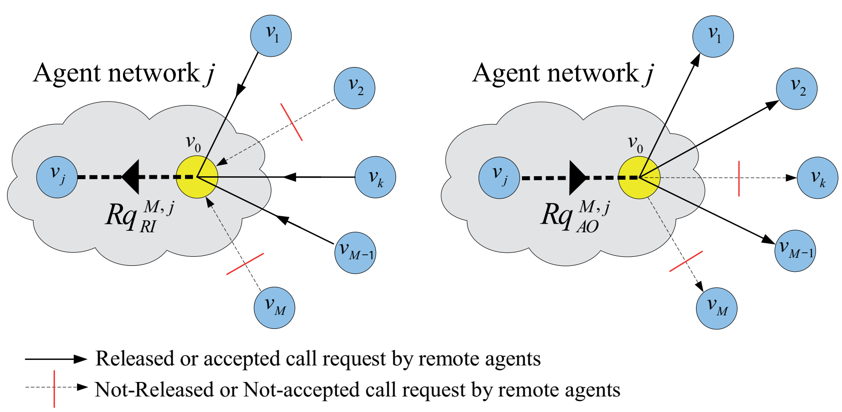

Figure 2.3: Left: Released incoming call request stream of agent network j; Right: Accepted outgoing call request stream of agent network j

Evidently, the request and termination counting processes increment by 1 if and only if the corresponding events occur; for example, $Rq^{M,j}_{\mathrm{RI}}(t) = Rq^{M,j}_{\mathrm{RI}}(t^-) + 1$ *if and only if* the call request event $e^{j,+}_{\mathrm{RI}}$ occurs at instant t, and so on.

Furthermore, note that given the above three point processes, all admission decisions can be carried out by the local agent in a decentralized manner, without violating system wide admission rules. Specifically, in order to complete admission an incoming released call request must be *accepted* by j, while an outgoing accepted call request must be *released* by j.

We shall call the above external call request streams the *filtered processes* of agent j. Although local agents need only be concerned with their own filtered processes to make their (optimal) control decisions, it is important to note that they must qualify *all* external call requests of which they are either a source or a destination with a 1 (release or accept) or 0 (neither release nor accept)label obtained as an outcome of their randomized local state feedback control laws (see Section 3.1 below). This is in order that all agents be able to identify their individual filtered processes.

More precisely, any external call request involving agents j and k will produce simultaneous randomized decisions (which may be pictured as coin tossing with local state dependent probability) qualifying the call as released or not by the source, and accepted or not by the destination With this information, agents can construct their individual filtered processes.

3 Linkwise decentralized admission control analysis

3.1 Randomized control laws of agent networks

We denote $\mathcal{N}^j$ as the *set of (admissible) connection vectors* of agent network j, such that:

$$\mathcal{N}^j = \{\mathbf{n}^j \equiv (n_1^j, n_2^j, n_3^j); \;\; n_b^j \in \{0, 1, \cdots\}, \sum_{b=1}^{3} n_b^j \leq c\} \tag{3.1}$$

with n_b^j, $b = 1, 2, 3$, denoting respectively the number of active local, released incoming and accepted outgoing connections of agent network j, (see Figure 3.4 for an example of $\mathbf{n}^j$).

Note that by the symmetry property of the mass system, $\mathcal{N}^j = \mathcal{N}^k$, $j \neq k \leq M$; then for notational simplicity we shall use $\mathcal{N}$ to refer to any $\mathcal{N}^j$.

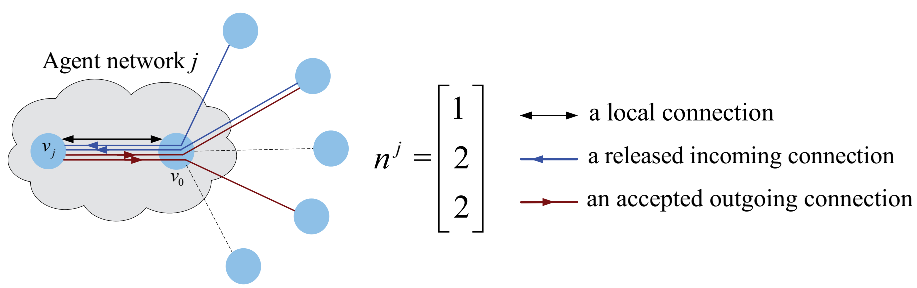

Figure 3.4: Example of a connection vector of agent network j

We denote E^j as the *set of events* of agent network j, such that:

$$E^j = \{\emptyset, e^{j,+}_{\text{Loc}}, e^{j,+}_{\text{RI}}, e^{j,+}_{\text{AO}}, e^{j,-}_{\text{Loc}}, e^{j,-}_{\text{RI}}, e^{j,-}_{\text{AO}}\}, \tag{3.2}$$

see Section 2.3 for the interpretations of components of E^j.

Denote $\mathbf{n}_j^M : [0, \infty) \times \Omega \to \mathcal{N}$ and $e_j^M : [0, \infty) \times \Omega \to E^j$ as the *process of connection vectors* and *event process* of agent network j respectively.

The *event transition equation* of agent network j is specified in the following:

$$e_j^M(t) = \begin{cases} e, & \text{in case } N_e^{M,j}(t) = N_e^{M,j}(t^-) + 1 \\ \phi, & \text{otherwise} \end{cases} \tag{3.3}$$

where $N_e^{M,j}(t)$ represents the counting process of event e of agent network j.

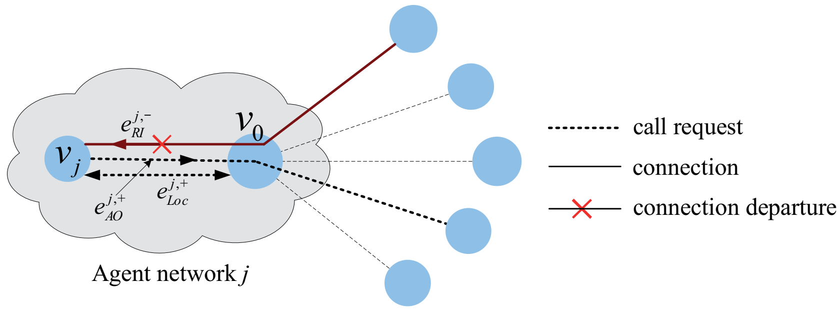

Figure 3.5: Illustration of events of agent network j

We denote $U(\mathbf{n}, e)$ as the *(admissible) control set*, of agent network j, with respect to $(\mathbf{n}, e) \in \mathcal{N} \times E^j$, such that

$$U(\mathbf{n}, e) = \begin{cases} \{\mathbf{0}, a_b\}, & \text{in case } e = e_b^{j,+} \text{ and } \mathbf{n} + a_b \in \mathcal{N} \\ \{-a_b\}, & \text{in case } e = e_b^{j,-} \\ \{\mathbf{0}\}, & \text{otherwise} \end{cases}, \qquad \text{with } b = 1, 2, 3, \tag{3.4}$$

where a_b denotes respectively the b-th unit vector in $\mathbb{R}^3$, and for notational simplicity, we denote $e_b^{j,+}$, with $b = 1, 2, 3$, as $e_{\text{Loc}}^{j,+}$, $e_{\text{RI}}^{j,+}$ and $e_{\text{AO}}^{j,+}$ respectively, and denote $e_b^{j,-}$, with $b = 1, 2, 3$, as $e_{\text{Loc}}^{j,-}$, $e_{\text{RI}}^{j,-}$ and $e_{\text{AO}}^{j,-}$, respectively.

Here, the control actions are interpreted as follows: (i) $u^j(\mathbf{n}, e_1^{j,+}) = \mathbf{0}$ (a_1, respectively) denotes that the call request $e_1^{j,+} \equiv e_{\text{Loc}}^{j,+}$ is rejected (accepted, resp.) by agent j; (ii) $u^j(\mathbf{n}, e_3^{j,+}) = \mathbf{A0}$ (a_3, resp.) denotes that the call request $e_3^{j,+} \equiv e_{\text{AO}}^{j,+}$ is rejected (released, resp.) by agent j; (iii) $u^j(\mathbf{n}, e_2^{j,+}) = \mathbf{0}$ (a_2, resp.) denotes that the call request $e_2^{j,+} \equiv e_{\text{RI}}^{j,+}$ is rejected (accepted, resp.) by agent j; and (iv) $u^j(\mathbf{n}, e_b^{j,-}) = -a_b$, with $b = 1, 2, 3$, denotes the deletion of an active local, released incoming or accepted outgoing connection in agent network j respectively which necessarily occurs upon its termination.

We define a set of *(randomized) control laws* of agent network j, denoted $\mathcal{U}^j[0, \infty)$, as follows:

$$\mathcal{U}^j[0, \infty) = \big\{u^j : [0, \infty) \times \mathcal{N} \times E^j \times \Omega \to U; \\ \text{such that } u^j(t) \text{ is } \sigma(\mathbf{n}_j^M(t^-), e_j^M(t)) \times \mathcal{B}_t^j(\Omega) \text{ measurable}\big\}, \tag{3.5}$$

where $\mathcal{B}_t^j(\Omega)$ is a sigma field on the probability space Ω, such that $\mathbb{P}\big(u^j(t) \,\big|\, \mathbf{n}_j^M(t^-), e_j^M(t)\big)$, for all j, are mutually independent and independent of all random variables in $\{s; s < t\}$.

Furthermore, we call $\{u^j; j \leq M\}$ as a collection of *uniform controls*, if the following holds

$$\mathbb{P}\big(u^j(t) = u \,\big|\, (\mathbf{n}, e)\big) = \mathbb{P}\big(u^k(t) = u \,\big|\, (\mathbf{n}, e)\big), \quad \text{for any } u \text{ and } (t, \mathbf{n}, e). \tag{3.6}$$

for any pair of agent networks j and k.

Subject to a control law u^j, the *state transition equation* of agent network j is specified as follows:

$$\mathbf{n}_j^M(t) = \mathbf{n}_j^M(t^-) + u^j(t, \mathbf{n}_j^M(t^-), e_j^M(t)) \tag{3.7}$$

We may now formally state agent and mass networks as follows:

Definition 3.1 A family of local state processes $\begin{bmatrix} \mathbf{n}_j^M \\ e_j^M \end{bmatrix}$, $1 \leq j \leq M$, induced by the transition equations (3.3,(3.7) subject to (S1) - (S3), with the set of control laws $\mathcal{U}^j[0, \infty), 1 \leq j \leq M$, is called an *agent network (system)* $S^j, 1\ lej \leq M$, while the collection of agent network $S^{\mathcal{M}} = \{S^j; j \leq M\}$ is called a *mass agent network* (with size M) and a *sequence of mass systems* $S^\infty = \{S^{\mathcal{M}}; M \geq 2\}$ is referred to as an *infinite mass network* S^∞. □

3.2 Asymptotic mutual independence of the set of state processes $\mathbf{n}^{\mathcal{M}}$, and asymptotic Poisson property of filtered streams

We now specify the following critical initial hypothesis $\mathrm{H}(t_0)$ at the initial instant t_0:

$\mathrm{H}(t_0)$: The set of initial local connection vector values $\mathbf{n}^{\mathcal{M}}(t_0) \equiv \{\mathbf{n}_j^M(t_0); j \leq M\}$ is asymptotically i.i.d. as M goes to infinity. □

It will be established in Lemma 3.1 below that given the hypothesis $\mathrm{H}(t_0)$, as M goes to infinity, the probability of either a direct or indirect call request between any given pair of agents j and k over an interval $[t_0, T]$, with $T - t_0 < 1/(2\lambda_2)$, goes to zero. We observe that the analysis in the lemma is independent of the value of t_0.

Then in Theorem 3.1, under the hypothesis $\mathrm{H}(t_0)$, and by use of Lemma 3.1 and the total probability theorem, it is shown that asymptotic identical distribution and independence holds for the set $\{\mathbf{n}_j^M(t); j \leq M\}$ for any $t \geq t_0$.

Consider any pair of distinct agent networks S^j and S^k, in the mass system $S^{\mathcal{M}}$, and $K \in \{0, \cdots, M-2\}$, a set of ordered sequence of vertices denoted $A_K^M(j,k)$ is specified as follows:

$$A_K^M(j,k) \triangleq \left\{\langle v_1, \cdots, v_K\rangle;\ v_i \in \mathcal{M} \setminus \{j,k\} \text{ and } v_i \neq v_l, \forall i, l\right\}, \tag{3.8}$$

with $K = 1, \cdots, M-2$, and $A_0^M(j,k) \triangleq \{\emptyset\}$, where $\langle v_1, \cdots, v_K\rangle$ denotes the ordered sequence of vertices.

Note: $\emptyset$ in the above is considered as a component of the set.

For any $\langle v_1, \cdots, v_K\rangle \in A_K^M$, we define $B_{K,T}^M(j,k;\langle v_1, \cdots, v_K\rangle)$ as the member of the sigma field, on Ω, on which each call request in the set $\left\{e^+_{\{j,v_1\}}, e^+_{\{v_i,v_{i+1}\}}, e^+_{\{v_K,k\}};\ i \in \{1, \cdots, K-1\}\right\}$ occurs during $[t_0, T+t_0]$ at least once, with $0 < T < \infty$(see Figure 3.6). Furthermore, we specify B_K^M as follows:

$$B_{K,T}^M(j,k) \quad \triangleq \bigcup_{\langle v_1,\cdots,v_K\rangle \in A_K^M(j,k)} B_{K,T}^M(j,k;\langle v_1, \cdots, v_K\rangle) \tag{3.9}$$

Note that $e^+_{\{v_i,v_{i+1}\}}$ denotes the (bidirectional) call request between v_i and v_{i+1}.

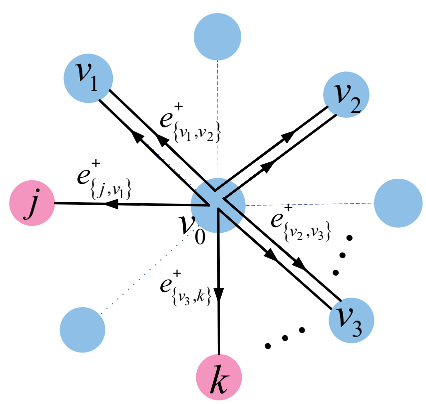

Figure 3.6: Illustration of $B_{3,T}^M(j,k;\langle v_1, v_2, v_3\rangle)$

Also given that by (3.8) the set $A_K^M(j,k)$, with $K = 0, \cdots, M-2$, is composed of components each of which is an ordered sequence of K distinct vertices belonging to the set $\mathcal{M} \setminus \{j,k\}$, with cardinality equal to $M-2$, the cardinality of $A_K^M(j,k)$, denoted $|A_K^M(j,k)|$, is given by:

$$|A_K^M(j,k)| = P_K^{M-2} = \frac{(M-2)!}{(M-2-K)!} = \begin{cases} 1, & \text{in case } K = 0 \\ \prod_{i=0}^{K-1}(M-2-i), & \text{in case } K > 0 \end{cases} \tag{3.10}$$

For notational simplicity in Lemma 3.1 below we consider

$$A_K^M \equiv A_K^M(j,k),\ B_{K,T}^M(v_1,\cdots,v_K) \equiv B_{K,T}^M(j,k;\langle v_1,\cdots,v_K\rangle),\ \text{ and }\ B_{K,T}^M \equiv B_{K,T}^M(j,k).$$

It will now be established that as M goes to infinity, the probability of either a direct or indirect connection between any given pair of agents j and k over any interval $[0,T]$ with $T < 1/(2\lambda_2)$ goes to zero.

Lemma 3.1 Under assumptions (S1)-(S3),

$$\lim_{M\to\infty} \mathbb{P}\big(\cup_{K=0}^{M-2} B_{K,T}^M\big) = 0, \quad \text{for any } T < 1/(2\lambda_2). \tag{3.11}$$

Remark: We observe that $B_{K,T}^M(v_1,\cdots,v_K)$ does not exclude the occurrence of any other call requests, so (i) $B_{K,T}^M(v_1,\cdots,v_K)\cap B_{K,T}^M(\widehat{v}_1,\cdots,\widehat{v}_K) \neq \emptyset$ for any $\langle v_1,\cdots,v_K\rangle \neq \langle \widehat{v}_1,\cdots,\widehat{v}_K\rangle \in A_K^M$, and (ii) $B_{K,T}^M \cap B_{\widehat{K},T}^M \neq \emptyset$ for any $K \neq \widehat{K}, 0 \leq K, \widehat{(K)} < \infty$.

Hence, because the individual sets in event $\cup_{K=0}^{M-2} B_{K,T}^M$ have so many overlaps, it is a complex task to calculate exact expressions for its probability. Instead, in the following, we obtain an upper bound on that probability for any fixed M.

Proof. For any pair of vertices v_l and $v_{\widehat{l}}$, under assumptions (S2) and (S3), the call request process from v_l to $v_{\widehat{l}}$, denoted $R_{\langle l,\widehat{l}\rangle}^M$, and the call request process from $v_{\widehat{l}}$ to v_l, denoted $R_{\langle \widehat{l},l\rangle}^M$, are mutually independent Poisson processes with rates equal to $\lambda_2/(M-1)$.

Consequently the bidirectional call request process between v_l and $v_{\widehat{l}}$, denoted $R_{\{l,\widehat{l}\}}^M$, which is the superposition of $R_{\langle l,\widehat{l}\rangle}^M$ and $R_{\langle \widehat{l},l\rangle}^M$, is a Poisson process with rate equal to $2\lambda_2/(M-1)$. Hence the probability that at least one (bidirectional external) call request $e_{\{l,\widehat{l}\}}^+$ occurs during $[t_0, T+t_0]$ is equal to $1-e^{-\frac{2\lambda_2 T}{M-1}}$. By the mutual independence property of all of the call request processes given by assumption (S3) we have

$$\mathbb{P}(B_{K,T}^M(v_1,\cdots,v_K)) = \big(1-e^{-\frac{2\lambda_2 T}{M-1}}\big)^{K+1} \tag{3.12}$$

Furthermore, $f(b) := be^b - e^b + 1 \geq 0$ for any $b \geq 0$ since $f(0) = 0$ and $f'(b) = e^b + be^b - e^b = be^b \geq 0$ for any $b \geq 0$. Then for any $b \equiv a_M \equiv \frac{2\lambda_2 T}{M-1} > 0$ and $K \in \{0,\cdots,M-2\}$ we have

$$\big(1-e^{-b}\big)^{K+1} \ \leq\ b^{K+1} \tag{3.13}$$

Hence by (3.9), (3.12) and (3.13) the following holds:

$$\begin{aligned}\mathbb{P}(B_{K,T}^M) &= \bigcup_{\langle v_1,\cdots,v_K\rangle\in A_K^M} B_{K,T}^M(v_1,\cdots,v_K) \leq \sum_{\langle v_1,\cdots,v_K\rangle\in A_K^M} \mathbb{P}\big(B_{K,T}^M(v_1,\cdots,v_K)\big) \\ &= \quad P_K^{M-2}\,\big(1-e^{-\frac{2\lambda_2 T}{M-1}}\big)^{K+1} \leq P_K^{M-2}\,\big(\frac{2\lambda_2 T}{M-1}\big)^{K+1},\end{aligned} \tag{3.14}$$

with the cardinality of set A_K^M is denoted $|A_K^M|$ which is specified in (3.10).

Then by (3.14)

$$\mathbb{P}\big(\cup_{K=0}^{M-2} B_{K,T}^M\big) \leq \sum_{K=0}^{M-2} \mathbb{P}(B_{K,T}^M) \leq \sum_{K=0}^{M-2} \Big\{P_K^{M-2}\,\big(\frac{2\lambda_2 T}{M-1}\big)^{K+1}\Big\} \tag{3.15}$$

By the definition of $|A_K^M|$ given in (3.10), we have $|A_K^M| \leq (M-1)^K$ for any $K \in \{0,\cdots,M-2\}$, then with $2\lambda_2 T \neq 1$, the following holds

$$\sum_{K=0}^{M-2} \Big\{|A_K^M|\,\big(\frac{2\lambda_2 T}{M-1}\big)^{K+1}\Big\} \leq \frac{1}{M-1}\sum_{K=0}^{M-2} \big(2\lambda_2 T\big)^{K+1} = \frac{1}{M-1}\Big(\frac{1-\big(2\lambda_2 T\big)^M}{1-\big(2\lambda_2 T\big)} - 1\Big) \tag{3.16}$$

In case $T < 1/(2\lambda_2)$ by (3.15) and (3.16),

$$\lim_{M\to\infty} \mathbb{P}\big(\cup_{K=0}^{M-2} B_{K,T}^M \big) \le \lim_{M\to\infty} \frac{1}{M-1}\Big(\frac{1-\big(2\lambda_2 T\big)^M}{1-\big(2\lambda_2 T\big)} - 1 \Big) = 0, \tag{3.17}$$

which implies that $\lim_{M\to\infty} \mathbb{P}\big(\cup_{K=0}^{M-2} B_{K,T}^M \big) = 0$. □

Theorem 3.1 Under assumptions (S1)-(S3), given hypotheses $\mathrm{H}(t_0)$, and subject to a collection of uniform local control laws $\{u^j;\, 1 \le j \le M\}$, the set of local processes $\{\mathbf{n}_j^M(.); 1 \le j \le M\}$ is asymptotically i.i.d as M goes to infinity.

Proof. For simplicity of presentation, we establish the property for pairs of agent networks; it is readily verified that the proof method extends to any finite group of agent networks.

Consider any pair of agent networks S^j and S^k in the mass system $S^{\mathcal{M}}$. Then, given the initial independence hypothesis $\mathrm{H}(t_0)$, $\mathbf{n}_j^M(s)$ and $\mathbf{n}_k^M(t)$ are dependent on the interval $[t_0, t_0+T]$ only if during the interval $[t_0, t_0+T]$ either (i) there occurs at least one (bidirectional external) call request $e^+_{\{j,k\}}$, or (ii) each of the external call requests in the set of $\{e^+_{\{j,v_1\}}, e^+_{\{v_i,v_{i+1}\}}, e^+_{\{v_K,k\}}; i \in \{1,\cdots,K-1\}\}$, occurs at least once, with distinct $v_1,\cdots,v_K, \{1,\cdots,k\} \subset \{1,\cdots,M\} \setminus \{j,k\}$ for some $K \in \{1,\cdots,M-2\}$.

Given the independence of the initial conditions posited in $\mathrm{H}(t_0)$, we consider below a total probability theorem statement, wherein we further condition on the complement of the union of the events in (i) and (ii) above and on the union of (i) and (ii), respectively. Now to establish independence over $[t_0, t_0+T]$, all finite dimensional distributions of $\mathbf{n}_j^M$ and $\mathbf{n}_k^M$ over all possible time instants in the interval should be considered. Here, however, we make the extreme simplification of only considering the event $C_{M,s,t} \equiv \{\omega; \mathbf{n}_j^M(s) = \mathbf{n}, \mathbf{n}_k^M(t) = \widehat{\mathbf{n}}\}$, since the general argument follows from this base case.

For notational simplicity we consider $D_M \equiv \{\omega; \mathbf{n}^{\mathcal{M}}(t_0) = \mathbf{n}_0^{\mathcal{M}}\}$, $E_{M,s,t} \equiv \Omega \setminus \cup_{K=0}^{M-2} B_{K,t_0,max(s,t)}^M$ and $F_{M,s,t} \equiv \cup_{K=0}^{M-2} B_{K,t_0,max(s,t)}^M$; then $E_{M,s,t} \dot\cup F_{M,s,t} = \Omega$.

$$\begin{aligned}
&\lim_{M\to\infty} \mathbb{P}(\mathbf{n}_j^M(s) = \mathbf{n}, \mathbf{n}_k^M(t) = \widehat{\mathbf{n}} \,|\, D_M) \\
&= \lim_{M\to\infty} \Big\{ \mathbb{P}(\mathbf{n}_j^M(s) = \mathbf{n} \,|D_M, E_{M,s,t})\, \mathbb{P}(\mathbf{n}_k^M(t) = \widehat{\mathbf{n}} \,|D_M, E_{M,s,t})\, \mathbb{P}(E_{M,s,t}|D_M) \Big\} \\
&\quad + \lim_{M\to\infty} \Big\{ \mathbb{P}(\mathbf{n}_j^M(s) = \mathbf{n}, \mathbf{n}_k^M(t) = \widehat{\mathbf{n}} \,|D_M, F_{M,s,t})\, \mathbb{P}(F_{M,s,t}|D_M) \Big\}, \\
&\qquad \text{(by the total probability theorem and the initial independence hypothesis } H(t_0)) \\
&= \lim_{M\to\infty} \Big\{ \mathbb{P}(\mathbf{n}_j^M(s) = \mathbf{n} \,|D_M, E_{M,s,t})\, \mathbb{P}(\mathbf{n}_k^M(t) = \widehat{\mathbf{n}} \,|D_M, E_{M,s,t}) \mathbb{P}(E_{M,s,t}|D_M) \Big\} \\
&= \lim_{M\to\infty} \mathbb{P}(\mathbf{n}_j^M(s) = \mathbf{n} \,|D_M) \, \lim_{M\to\infty} \mathbb{P}(\mathbf{n}_k^M(t) = \widehat{\mathbf{n}} \,|D_M),
\end{aligned}$$

for any $\mathbf{n}, \widehat{\mathbf{n}} \in \mathcal{N}$, where the penultimate equality holds by Lemma 3.1, $\lim_{M\to\infty} \mathbb{P}(F_{M,s,t}) = 0$, with $s,t \in [t_0, t_0+T], T \le 1/(2\lambda_2)$, and the last equality holds since $\lim_{M\to\infty} \mathbb{P}\big(E_{M,s,t}\big) = 1 - \lim_{M\to\infty} \mathbb{P}\big(F_{M,s,t}\big) = 1$ with $s,t \in [t_0, t_0+T]$, $T \le 1/(2\lambda_2)$). Hence, $\mathbf{n}_j^M(s)$ and $\mathbf{n}_k^M(t)$ are asymptotically independent under the initial independence hypothesis $\mathrm{H}(t_0)$.

The argument above may now be repeated on a sequence of intervals $[t_0+T, t_0+2T], [t_0+2T, t_0+3T]$, and so on, of length less than or equal to $\frac{1}{2\lambda_2}$, to yield the desired result. □

Corollary 3.1 Under assumptions (S1)-(S3), hypotheses $\mathrm{H}(t_0)$, and subject to a collection of uniform local control laws $\{u^j;\, j \le M\}$, the set of filtered call request streams $\big\{Rq_{\mathrm{RI}}^{M,j}, Rq_{\mathrm{AO}}^{M,j}; j \le M\big\}$ is asymptotically mutually independent and each converges in distribution to a Poisson process with rate specified as follows:

$$\begin{aligned}
\lambda_{\mathrm{RI}}^j =& \lambda_2 \lim_{M\to\infty} \frac{1}{M} \Sigma_1^M \big\{ \mathbf{I}(u_k(t; \mathbf{n}_k^M(t^-), e_{\mathrm{O}}^{k,+}) = a_3) \big\} \\
=& \lambda_2 \lim_{M\to\infty} \mathbb{E}_{P_0} \big\{ \mathbf{I}(u_k(t; \mathbf{n}_k^M(t^-), e_k^M(t)) = a_3) \mid e_k^M(t) = e_{\mathrm{O}}^{k,+} \big\}
\end{aligned} \tag{3.18}$$

$$\lambda_{\mathrm{O}}^{j} = \lambda_2 \lim_{M\to\infty} \frac{1}{M}\Sigma_1^M \{\mathbf{I}(u_k(t;\mathbf{n}_k^M(t^-), e_{\mathrm{I}}^{k,+}) = a_2)\} \\ = \lambda_2 \lim_{M\to\infty} \mathbb{E}_{P_0}\{\mathbf{I}(u^k(t;\mathbf{n}_k^M(t^-), e_k^M(t)) = a_2) \mid e_k^M(t) = e_{\mathrm{I}}^{k,+}\} \tag{3.19}$$

for any k, where $\mathbb{E}_{P_0}\{.|.\}$ displays the (parametric) dependence of $\mathbb{E}_{P_0}\{.|.\}$ on P_0, the initial distribution of local connection vectors. □

Proof Outline of Corollary 3.1:

(i) Under hypothesis H(t_0), based upon the results given in Lemma 3.1 and Theorem 3.1, we can show that each of the released incoming process $Rq_{\mathrm{RI}}^{M,j}$ and accepted outgoing process $Rq_{\mathrm{AO}}^{M,j}$, for all j, is a superposition of mutually independent sparse point processes. Then by applying Theorem 3.10 in [6], we can show that $Rq_{\mathrm{RI}}^{M,j}$ and $Rq_{\mathrm{AO}}^{M,j}$ converge to Poisson processes respectively.

(ii) The claim on the relation of rates and limits above needs no proof since it is self evident from an appropriate analysis of the firing of acceptances given the input events, subject to the limiting behaviour of the population. Use SLLN and defination of conditional probabilities and the properties of indicator functions.
The rate of limit Poisson released incoming process of an agent network is proportional to the statistical releasing behaviour of the outgoing call request among the rest of the agent network population; and the rate of limit Poisson accepted outgoing process of an agent network is proportional to the statistical accepting behaviour of the incoming call request among the rest of the agent network population.

The complete proof of Corollary 3.1 is tedious, see [25] for details.

3.3 Fixed point rate parameters for isolated single Poisson agent systems

We develop the notion of a (generic) isolated single agent network which is essential to the computation of the probability parameters associated with the particular global network state specified in Definition 3.3.

Definition 3.2 ((Generic) isolated single agent network)

We specify a class of *(parameterized generic) isolated single agent networks* S^{a} with (time dependent) release and acceptance probability parameters ($p_{\mathrm{RI}}^0(t)$, $p_{\mathrm{AO}}^0(t)$) and rate parameter vector $\xi = (\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-)$, where the state set is the local connection vector set $\mathcal{N}$ (see (3.1)) where it is assumed that the local, released incoming and accepted outgoing (call request) point processes (associated with events $e_1^+ \equiv e_{Loc}^{\mathrm{a},+}$, $e_2^+ \equiv e_{\mathrm{RI}}^{\mathrm{a},+}$, $e_3^+ \equiv e_{\mathrm{AO}}^{\mathrm{a},+}$), denoted Rq_b^+, $b = 1, 2, 3$, respectively, are mutually independent Poisson processes. Furthermore

(1) The rates of the call request processes Rq_b^+, $b = 1, 2, 3$, at time t are respectively equal to ξ_1^+, $p_{\mathrm{RI}}^0(t)\xi_2^+$ and $p_{\mathrm{AO}}^0(t)\xi_2^+$.
The durations of the b-th class of connections with $b = 1, 2, 3$, are exponentially distributed with rates respectively equal to ξ_1^-, ξ_2^- and ξ_3^-, where $\xi_3^- = \xi_2^-$ by assumption. Let $Dp_b^-(t)$, $b = 1, 2, 3$, denote respectively the associated connection termination counting processes.

(2) The stochastic dynamics of S^{a} (see Figure 3.7) with a randomized local feedback control law u^{a} (see Section 3.1) are given by the local state transition equation:

$$e^{\mathrm{a}}(t) = \begin{cases} e_b^+, & \text{in case } Rq_b^+(t) = Rq_b^+(t^-) + 1 \\ e_b^-, & \text{in case } Dp_b^-(t) = Dp_b^-(t^-) + 1, \quad b = 1,2,3 \\ \emptyset_{\mathrm{a}}, & \text{otherwise} \end{cases} \tag{3.20}$$

$$\mathbf{n}^{\mathrm{a}}(t) = \mathbf{n}^{\mathrm{a}}(t^-) + u^{\mathrm{a}}(t;\mathbf{n}^{\mathrm{a}}(t^-), e^{\mathrm{a}}(t)) \tag{3.21}$$

□

Theorem 3.2 (Existence of isolated agent rate parameters compatible with any given local feedback control law)

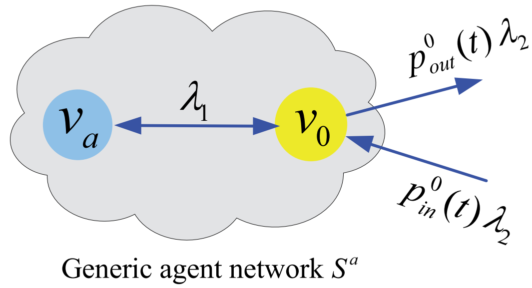

Figure 3.7: Poisson call request processes of an isolated single agent network $S^{\rm a}$. Filtered call request streams model the impact of acceptance or release decisions by the mass of distant agent networks on $S^{\rm a}$.

For any (parameterized) isolated single agent network $S^{\rm a}$, given any initial state distribution P_0 and any local randomized control law $u^{\rm a}$, there exists a unique two component vector of probabilities $\mathbf{p}^0_t = (p^0_{\rm RI}(t), p^0_{\rm AO}(t))$ such that

$$p^0_{\rm RI}(t) = \mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_3 \,|\, e^{\rm a}(t) = e^{\rm a,+}_{\rm AO}), \tag{3.22}$$
$$p^0_{\rm AO}(t) = \mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_2 \,|\, e^{\rm a}(t) = e^{\rm a,+}_{\rm RI}), \tag{3.23}$$

with $u^{\rm a}_t \equiv u^{\rm a}(t; \mathbf{n}^{\rm a}(t^-), e^{\rm a}(t))$, where a_2 corresponds to acceptance and a_3 to release, $\mathbb{P}_{P_0,\mathbf{p}^0_t}(.|\,.)$ displays the (parametric) dependence of $\mathbb{P}(.|\,.)$ on P_0 and $\mathbf{p}^0_t$, i.e. there exist released incoming and accepted outgoing Poisson processes, with *release and acceptance rate parameters* respectively equal to $p^0_{\rm RI}(t)\xi^+_2$ and $p^0_{\rm AO}(t)\xi^+_2$, such that the fixed point equations (3.22) and (3.23) hold.

Proof. The initial condition $P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t)$ at the instant $t = 0$ is equal to the initial occupation probability $P_{P_0}(\mathbf{n}; 0) = P_{P_0}(\mathbf{n}; 0)$; which specifies $\mathbf{p}^0_0$ via (3.22) and (3.23).

(i) Imposing the relations (3.22) and (3.23), and by the (conditional) total probability theorem, the following hold:

$$p^0_{\rm RI}(t) = \mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_3 \,|\, e^{\rm a,+}_{\rm AO}) = \sum_{\mathbf{n}\in\mathcal{N}} \mathbb{P}\big(u^{\rm a}_t = a_3 \,|\, \mathbf{n}, e^{\rm a,+}_{\rm AO}\big)\, P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t), \tag{3.24}$$
$$p^0_{\rm AO}(t) = \mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_2 \,|\, e^{\rm a,+}_{\rm RI}) = \sum_{\mathbf{n}\in\mathcal{N}} \mathbb{P}\big(u^{\rm a}_t = a_2 \,|\, \mathbf{n}, e^{\rm a,+}_{\rm RI}\big)\, P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t), \tag{3.25}$$

where $\mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_3 \,|\, e^{\rm a,+}_{\rm AO})$ and $\mathbb{P}_{P_0,\mathbf{p}^0_t}(u^{\rm a}_t = a_2 \,|\, e^{\rm a,+}_{\rm RI})$ are given by the specification of the feedback control law u^a, and where $P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t)$ denotes the probability that $\mathbf{n}^{\rm a}(t)$ is equal to $\mathbf{n} \in \mathcal{N}$, i.e. $P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t) = \mathbb{P}_{P_0,\mathbf{p}^0_t}\big(\omega;\ \mathbf{n}^{\rm a}(t, \omega) = \mathbf{n}\big)$.

(ii) Under the specifications of an isolated single agent network $S^{\rm a}$ given in Definition 3.2, the call request processes are Poisson processes and each of the active connection durations is exponentially distributed; then subject to the local randomized control law $u^{\rm a}$, the local connection vector process $\mathbf{n}^{\rm a}$ of $S^{\rm a}$ is a *birth-death Markov process.*

As a result, the probability processes $P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; .)$, for any $\mathbf{n} \in \mathcal{N}$, satisfy the coupled (forward) Kolmogorov first order ODE system (3.26), (3.27), together with the feedback equations (3.24) and (3.25):

$$\frac{d}{dt} P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t) = -\Big(\sum_{e\in E^{\rm a}(\mathbf{n})} \lambda^e(\mathbf{n}, t; \mathbf{p}^0_t)\Big)\, P_{P_0,\mathbf{p}^0_t}(\mathbf{n}; t) \tag{3.26}$$
$$+ \sum_{\substack{\mathbf{m}\in\mathcal{N} \\ \exists\, e\in E^{\rm a}(\mathbf{m}) \\ \text{s.t. } \mathbf{m}+u=\mathbf{n}}} \Big(\lambda^e(\mathbf{m}, t; \mathbf{p}^0_t)\, \mathbb{E}\big\{\mathbf{I}(u^{\rm a}(t; \mathbf{m}, e) = u)\big\}\, P_{P_0,\mathbf{p}^0_t}(\mathbf{m}; t)\Big),$$

for any $\mathbf{n} \in \mathcal{N}$, with an initial distribution $\mathbb{P}(.;0)$, where

$$\lambda^e(\mathbf{n}, t; \mathbf{p}_t^0) = \begin{cases} \xi_1^+ \\ p_{\text{RI}}^0(t)\xi_2^+ \\ p_{\text{AO}}^0(t)\xi_2^+ \\ n_1\, \xi_1^- \\ n_2\, \xi_2^- \\ n_3\, \xi_2^- \end{cases} \quad \text{in case} \quad e = \begin{cases} e_{\text{Loc}}^{\text{a},+} \\ e_{\text{RI}}^{\text{a},+} \\ e_{\text{AO}}^{\text{a},+} \\ e_{\text{Loc}}^{\text{a},-} \\ e_{\text{RI}}^{\text{a},-} \\ e_{\text{AO}}^{\text{a},-} \end{cases}, \tag{3.27}$$

for any $(\mathbf{n}, t) \in \mathcal{N} \times [0, \infty)$.

To show the existence and uniqueness of the solution to $\mathbf{p}^0$ to (3.22) and (3.23), with a given set of initial conditions, it is sufficient to prove that there exists a unique solution $P_{P_0,\mathbf{p}_t^0}(\mathbf{n}; t.)$, for the given initial conditions, to the closed loop state space equations (3.26), (3.27), (3.24), (3.25).

However the existence of such a unique solution $P_{P_0,\mathbf{p}_t^0}(\mathbf{n}; t.)$, is proven by verifying the hypotheses of Caratheodory's Theorem (see Theorem 4.2 in [8]) as carried out in Appendix A. □

In Theorem 3.2: the RHS of (3.22) captures the identical statistical behavior of the mass system to the local agent network in terms of rate of release of external call requests (by taking action a_3) to the agent network S^j; while the RHS of (3.23) is its counterpart in terms of rate of acceptance by the mass system (by taking action a_2) of external call requests from S^j. The fact that the RHSs of (3.22) and (3.23) depend upon the vector $\mathbf{p}_t^0$ (defined for an isolated agent network S^{a}) corresponds to the mass-individual symmetry of the global system's behaviour under assumptions of (i) radial network symmetry and (ii) uniform randomized control laws implemented by all agent networks.

3.4 The network decentralized state (NDS)

Definition 3.3 (The network decentralized state (NDS))

Consider the infinite mass system $S^\infty = \{S^{\mathcal{M}}; M \geq 2\}$, subject to the local transition equations (3.3) - (3.7) with rate parameters $(\lambda_1, \lambda_2, \mu_1, \mu_2)$. We say that the infinite mass system is in an *(asymptotic) network decentralized state (NDS)* with the pair of time dependent probabilities $\mathbf{p}_t = (p_{\text{RI}}(.), p_{\text{AO}}(.))$, if, as M goes to infinity, (i) the set of filtered call request streams $\{Rq_{\text{RI}}^{M,j}, Rq_{\text{AO}}^{M,j}; j \leq M\}$ converge to mutually independent streams of Poisson processes with rates respectively equal to $p_{\text{RI}}(.)\lambda_2$ and $p_{\text{AO}}(.)\lambda_2$, with $p_{\text{RI}}(.)$ and $p_{\text{AO}}(.)$, as given in (3.24) and (3.25), and (ii) the set of individual agent states $\{\mathbf{n}_j^M, ; j \leq M\}$ is asymptotically independent. □

Theorem 3.3 (Existence of network decentralized states for uniform control laws)

Subject to a collection of uniform local control laws $\mathbf{u} \equiv \{u^j; j < \infty\}$, and under assumptions (S1)-(S3) and hypothesis H(t_0), the infinite mass system S^∞ with random initial network state distribution P_0, is in an NDS state with the unique (u-induced)pair of parametric release and acceptance rate parameters $(p_{\text{RI}}^0(.), p_{\text{AO}}^0(.))$ generated by (3.26), (3.27), (3.24), (3.25).

Proof. By Theorem 3.1, subject to the collection of uniform local control laws $\mathbf{u}$ and under assumptions (S1)-(S3) and the hypothesis H(t_0), the set of local states $\{\mathbf{n}_j^M(t); 1 \leq j \leq M\} = \mathbf{n}(\mathbf{t})^{\mathcal{M}}$ is asymptotically i.i.d. in the population limit as M goes to infinity. Furthermore, by Corollary 3.1, the set of filtered streams $\{Rq_{\text{RI}}^{M,j}, Rq_{\text{AO}}^{M,j}; 1 \leq j \leq M\}$ is asymptotically mutually independent and each converges in distribution to a Poisson process with parameters at t equal to $p_{\text{RI}}(t)\lambda_2$ and $p_{\text{AO}}(t)\lambda_2$ respectively.

By virtue of the asymptotic (in M)independence of the individual agent states $\mathbf{n}_j^M, ; j \leq M\}$ and the asymptotic independence of the asymptotically Poisson filtered streams $\{Rq_{\text{RI}}^{M,j},\ Rq_{\text{AO}}^{M,j};\ 1 \leq j \leq M\}$, the defining properties of an NDS state are satisfied with the (u-induced)pair of rate parameters $(p_{\text{RI}}^0(.), p_{\text{AO}}^0(.))$ generated by (3.26), (3.27), (3.24), (3.25) and with the unique common parameters $(p_{\text{RI}}(.), p_{\text{AO}}(.), \lambda_1, \lambda_2, \mu_1,$ $\mu_2)$ respectively, as expressed in the RHS of the equations in Corollary 3.1. □

Theorem 3.3 is a statement that:

1. There exist released incoming and accepted outgoing Poisson processes with rates equal respectively to $p^0_{\mathrm{RI}}(.)\lambda_2$ and $p^0_{\mathrm{AO}}(.)\lambda_2$ such that the fixed point equations (3.22) and (3.23) hold;
2. The NDS state specified in Theorem 3.3 is neither necessarily stationary nor asymptotically stationary as t goes to infinity;
3. The RHS of (3.22) captures the statistical behavior of the infinite mass system S^∞ in terms of rate of release of external call requests (by taking action a_3) to agent network S^j; while the RHS of (3.23) is its counterpart in terms of rate of acceptance by the infinite mass system (by taking action a_2) of external call requests from S^j. The fact that the RHSs of (3.22) and (3.23) depend upon the vector $\mathbf{p}^0_t$ (defined for S^j) corresponds to the mass-individual symmetry of the mass system's behaviour under assumptions of (i) radial network symmetry and (ii) uniform randomized control laws for all agent networks.

4 Decentralized control and the PPNCE principle

In this section we analyse the decentralized optimization of network decentralized states (NDS) for infinite mass systems S^∞ with respect to a given class of local cost functions.

Consider the following performance specifications for an agent network S^j:

(S4) Agent network S^j pays a connection fee equal to $\alpha e^{-\beta t} b_2$ ($(1-\alpha)e^{-\beta t}b_2$ respectively), with $\alpha \in [0,1]$, in case that a call request $e^{j,+}_{\mathrm{RI}}$ ($e^{j,+}_{\mathrm{AO}}$ resp.) is admitted at instant t. Subsequently S^j pays a cost per unit time equal to $\alpha e^{-\beta t} g_2$ ($(1-\alpha)e^{-\beta t} g_2$ resp.) over the duration of each active external incoming (outgoing resp.) connection.

(S5) Agent network S^j pays an instantaneous cost equal to

$$\varepsilon e^{-\beta t}\mathbb{P}(u^j(t;\mathbf{n}^j_{t-},e^j_t) \in U^+ \big| e^j_t = e), \qquad \text{with a fixed } \varepsilon \in [0,\infty), \tag{4.28}$$

upon admission of the call request $e \in E^{j,+} \equiv \{e^{j,+}_{\mathrm{Loc}}, e^{j,+}_{\mathrm{RI}}, e^{j,+}_{\mathrm{AO}}\}$ is admitted at instant t.

The motivation for the introduction of the cost $\varepsilon e^{-\beta t}\mathbb{P}(u^j(t;\mathbf{n}^j_{t-},e^j_t) \in U^+ \big| e)$ in (S5) is that it acts as a regularization device yielding (via a fixed point proof) the existence of a randomized control which results in a decentralized equilibrium (see the definition of NDE specified in Section 4.3) for the infinite system S^∞.

4.1 Optimal control problems for isolated single agent systems

Consider an isolated single agent network S^{a} as specified in Definition 3.2, with (consolidated) parameter $\rho(t) \equiv (p_{\mathrm{RI}}(t), p_{\mathrm{AO}}(t), \xi^+_1, \xi^+_2, \xi^-_1, \xi^-_2) \in \mathbb{R}^6_+, t \in [0,\infty)$ satisfying assumptions (S4) and (S5), with parameters (α,ε), and subject to a local randomized control law u^{a}, where we note that $\rho(.)$ for the agent S^{a} does not in general satisfy the fixed point equations of 3.2. Then the cost function for S^{a} is given by:

$$\begin{aligned} J^{\mathrm{a}}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n};u^{\mathrm{a}}) = \mathbb{E}_{|(s,\mathbf{n})}\Big\{ \int_t^\infty e^{-\beta t} G(\mathbf{n}^{\mathrm{a}}_s) ds \\ + \sum_{k=1}^\infty e^{-\beta t_k}\big(B(e^{\mathrm{a}}_{t_k}) + \varepsilon \mathbb{P}(u^{\mathrm{a}}_{t_k} \in U^+)\big)\mathbf{I}(u^{\mathrm{a}}_{t_k} \in U^+)\Big\}, \end{aligned} \tag{4.29}$$

where, corresponding to assumptions (S4),(S5),

$$G(\mathbf{n}) = g_1 n_1 + \alpha g_2 n_2 + (1-\alpha) g_2 n_3, \text{ and } B(e) = \begin{cases} b_1, & \text{in case } e = e^{\mathrm{a},+}_{\mathrm{Loc}} \\ \alpha b_2, & \text{in case } e = e^{j,+}_{\mathrm{RI}} \\ (1-\alpha) b_2, & \text{in case } e = e^{j,+}_{\mathrm{AO}} \\ 0, & \text{otherwise} \end{cases}.$$

A *local optimal stochastic control (OSC) problem* (for the local agent agent $S^{\rm a}$ with parameter ρ), with local cost function (4.29), is specified by the infimization problem:

$$V^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n}^{\rm a}) \;=\; \inf_{u^{\rm a}\in\mathcal{U}^{\rm a}[t,\infty)} J^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n}^{\rm a};u^{\rm a}), \tag{4.30}$$

with value function $V^{\rm a}_{(\rho,\alpha,\varepsilon)} : [t,\infty)\times\mathcal{N}\to\mathbb{R}$, s with $\mathcal{N}$ defined in (3.1). An infimizing function $u^{{\rm a},*}\in\mathcal{U}^{\rm a}[t,\infty)$, $u^{{\rm a},*}$ shall be called an *optimal control law* for the given isolated single agent OSC problem.

The following lemma is then obtained directly from Corollary 3.3, [24].

Lemma 4.1 The HJB equation for the OSC problem of an isolated single agent network. $S^{\rm a}$ with time invariant probability rate parameters $(p_{\rm RI}(t),p_{\rm AO}(t)) = (p_{\rm RI},p_{\rm AO})$ and hence time invariant consolidated parameter $(\rho,\alpha,\varepsilon)$, is given by the set of coupled piecewise linear equations:

$$\begin{aligned}\beta V^{\rm a}_{\mathbf{n}} = G(\mathbf{n}) + \sum_{b=1}^{3}\lambda_b^-(\mathbf{n})\big(V^{\rm a}_{\mathbf{n}-a_b}-V^{\rm a}_{\mathbf{n}}\big) + \sum_{b=1}^{3}\lambda_b^+ \inf_{u^{\rm a}\in\mathcal{U}^{\rm a}}\Big\{\varepsilon\big(\mathbb{P}(u^{\rm a}=a_b|\,\mathbf{n},e_b^+)\big)^2 \\ +\big(B(e_b^+)+V^{\rm a}_{\mathbf{n}+a_b}-V^{\rm a}_{\mathbf{n}}\big)\mathbb{P}(u^{\rm a}=a_b|\,\mathbf{n},e_b^+)\Big\},\end{aligned} \tag{4.31}$$

for any $\mathbf{n}\in\mathcal{N}$, with $V^{\rm a}\equiv V^{\rm a}(\rho,\alpha,\varepsilon)$, where λ_b^-, with $b=1,2,3$, denotes the rate of the counting processes $Dp^{\rm a}_{\rm Loc}$, $Dp^{\rm a}_{\rm RI}$ and $Dp^{\rm a}_{\rm AO}$ respectively, and λ_b^+, with $b=1,2,3$, denotes the rate of the counting processes $Rq^{\rm a}_{\rm Loc}$, $Rq^{\rm a}_{\rm RI}$ and $Rq^{\rm a}_{\rm AO}$ respectively. □

4.2 Fixed points of isolated single agent optimal stochastic control problems

Before establishing Theorem 4.1, we give the technical results in Lemmas 4.2 and 4.3 below.

Lemma 4.2 The value function $V^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n})$ is continuous in the parameter ρ for fixed $(\alpha,\varepsilon;t,\mathbf{n})$, with $\varepsilon>0$.

Proof sketch. First, using the argument in the proof of the continuity of the cost function in Proposition 3.1, [22] (wrt the system age parameter ζ), we establish that subject to a randomized local control law $u^{\rm a}$, and for fixed $(\alpha,\varepsilon)\in[0,1]\times(0,\infty)$, the function $J^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n};u^{\rm a})$, is continuous in ρ. (We note that the fact that the controls here are randomized, while those in [22] are deterministic merely adds an additional expectation operator in the present case.)

Second, since the estimate

$$\big|V^{\rm a}_{\rho+\delta}(t,\mathbf{n})-V^{\rm a}_{\rho}(t,\mathbf{n})\big| \le \sup_{u^{\rm a}\in\mathcal{U}^{\rm a}[t,\infty)}\big|J^{\rm a}_{(\rho+\delta,\alpha,\varepsilon)}(t,\mathbf{n};u^{\rm a})-J^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n};u^{\rm a})\big| \tag{4.32}$$

holds for any pair $(t,\mathbf{n})$ and δ, and since,$\mathbf{n})$ and δ, the continuity of $J^{\rm a}_{(\rho,\alpha,\varepsilon)}(t,\mathbf{n};u^{\rm a})$ is continuous in ρ UNIFORMLY IN $u^{\rm a}\in\mathcal{U}^{\rm a}[t,\infty)$ the conclusion follows.

The indicated issue above concerning uniformity needs to be explicitly checked. □

Lemma 4.3 Denote by $u^{{\rm a},*}_{(\rho,\alpha,\varepsilon)}$ the local optimal randomized control law for the local optimal stochastic control problem for a generic agent system with time invariant parameters $(\rho,\alpha,\varepsilon)$, then for any $(\mathbf{n},e_b^+)\in\mathcal{N}\times E^{\rm a}$,

$$\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{{\rm a},*}_{(\rho,\alpha,\varepsilon)}(\mathbf{n}^{\rm a}_{t-},e^{\rm a}_t)=a_b) \text{ is continuous on } \rho, \text{ for a fixed } (\alpha,\varepsilon),\ \text{with } \varepsilon>0.$$

Proof ¡¡ TO GO IN APPENDIX¿¿. To analyse the properties of the local randomized optimal control law $u^{{\rm a},*}(\rho,\alpha,\varepsilon)$ with respect to time invariant parameters $(\rho,\alpha,\varepsilon)$, we need to analyze the *inf* operator in the hybrid HJB equation (4.31), subject to a $u^{\rm a}\in\mathcal{U}^{\rm a}[0,\infty)$, that is to say

$$\begin{aligned}&\varepsilon\big(\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{{\rm a},*}_{(\rho,\alpha,\varepsilon)}=a_b)\big)^2 - y_{(\mathbf{n},e_b^+)}(\rho,\alpha,\varepsilon)\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{{\rm a},*}_{(\rho,\alpha,\varepsilon)}=a_b) \\ &= \varepsilon\Big(\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{{\rm a},*}_{(\rho,\alpha,\varepsilon)}=a_b)-\frac{y_{(\mathbf{n},e_b^+)}(\rho,\alpha,\varepsilon)}{2\varepsilon}\Big)^2-\frac{y^2}{4\varepsilon},\end{aligned} \tag{4.33}$$

for any $b \in \{1,2,3\}$, with $y_{(\mathbf{n},e_b^+)}(\rho,\alpha,\varepsilon) \equiv -B(e_b^+) - V^{\mathrm{a}}_{(\rho,\alpha,\varepsilon)}(\mathbf{n}+a_b) + V^{\mathrm{a}}_{(\rho,\alpha,\varepsilon)}(\mathbf{n})$.

Then by (4.33) together with the HJB equation (4.31), it is clear that that the optimal randomized control law with respect to $(\rho,\alpha,\varepsilon)$, denoted $u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)}$, satisfies

$$\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_b) = \begin{cases} 1, & \text{in case} \quad y/(2\varepsilon) > 1 \\ y/(2\varepsilon), & \text{in case} \quad 0 \le y/(2\varepsilon) \le 1 \\ 0, & \text{otherwise} \end{cases}, \tag{4.34}$$

i.e. $\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_b)$ is continuous in $y/(2\varepsilon)$, (see Figure 4.8).

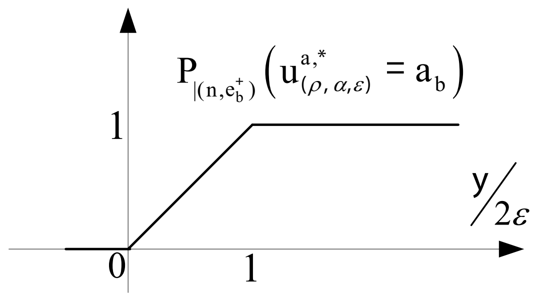

Figure 4.8: Continuity property of $\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_b)$ with respect to $y/(2\varepsilon)$

Since, by Lemma 4.2, for any $(\alpha,\varepsilon;\mathbf{n})$, the value function $V^{\mathrm{a}}_{(\rho,\alpha,\varepsilon)}(\mathbf{n})$ is continuous in ρ, SO we obtain the continuity of $y_{(\mathbf{n},e_b^+)}(\rho,\alpha,\varepsilon)$ in ρ for any $(\alpha,\varepsilon;\mathbf{n},e_b^+)$.

Hence by (4.34) and Lemma 4.2, we immediately obtain that $\mathbb{P}_{|(\mathbf{n},e_b^+)}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_b)$ is continuous in ρ for any $(\alpha,\varepsilon;\mathbf{n},e_b^+)$. □

We now specify:

H$_\mathrm{s}$: Subject to a local (time invariant) optimal randomized control law $u^{\mathrm{a},*} \in \mathcal{U}^{\mathrm{a}}$ for the OSC problem of an isolated agent network S^{a} with time invariant parameters $(\rho,\alpha,\varepsilon)$, there exists a stationary state distribution $P(\mathbf{n};u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)})$ which is continuous in the parameters $(\rho,\alpha,\varepsilon)$. □

Theorem 4.1 (Existence of NDE parameters compatible with local optimal feedback randomized control laws)

Under hypothesis H$_\mathrm{s}$ and considering the local OSC problem given in (4.29) for an isolated single agent network specified in Definition 3.2, there exists a *fixed point* pair of time invariant $\mathbf{p}^* = (p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$, such that

$$p^*_{\mathrm{RI}} = \mathbb{P}_{\rho^*}(u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)} = a_3 \,|\, e^{\mathrm{a}}(t) = e^{\mathrm{a},+}_{\mathrm{AO}}), \tag{4.35}$$
$$p^*_{\mathrm{AO}} = \mathbb{P}_{\rho^*}(u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)} = a_2 \,|\, e^{\mathrm{a}}(t) = e^{\mathrm{a},+}_{\mathrm{RI}}), \tag{4.36}$$

where (i) $\rho^* \equiv (p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}}, \xi_1^+, \xi_1^+, \xi_2^-, \xi_2^-) \in \mathbb{R}^+_6$, (ii) $u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)}$ denotes the local optimal randomized control law for the local OSC problems with time invariant parameters $(\rho^*,\alpha,\varepsilon)$; (iii) $\mathbb{P}_{\rho^*}(.|\,.)$ displays the (parametric) dependence of $\mathbb{P}(.|\,.)$ on ρ^*.

Proof. By the (conditional) total probability theorem we have

$$\mathbb{P}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_3 \,|\, e^{\mathrm{a},+}_{\mathrm{AO}}) = \sum_{\mathbf{n}\in\mathcal{N}} \mathbb{P}\big(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_3 \,|\, \mathbf{n}, e^{\mathrm{a},+}_{\mathrm{AO}}\big)\, P(\mathbf{n}; u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)}) \tag{4.37}$$

$$\mathbb{P}(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_2 \,|\, e^{\mathrm{a},+}_{\mathrm{RI}}) = \sum_{\mathbf{n}\in\mathcal{N}} \mathbb{P}\big(u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)} = a_2 \,|\, \mathbf{n}, e^{\mathrm{a},+}_{\mathrm{RI}}\big)\, P(\mathbf{n}; u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)}) \tag{4.38}$$

where $\mathbb{P}(.;u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)})$ denotes the stationary probability of $\mathbf{n}^{\mathrm{a}}_t$ subject to the randomized control law $u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)}$ which exists by the hypothesis $\mathrm{H_s}$.

Hence to show the existence of a *fixed point* pair of stationary probabilities $\mathbf{p}^* = (p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$, satisfying (4.37) and (4.38) it is sufficient to show that there exists a solution to the vector valued equation $\mathbf{p}^* = h(\mathbf{p}^*)$ where the first and second components of the function h are given by the RHS of (4.37) and (4.38) respectively.

By Lemma 4.3 together with the continuity property of $P(\mathbf{n};u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)})$ with respect to ρ for any given parameters (α,ε), we obtain that h is continuous in its argument $\mathbf{p}$ (a subset of the components of ρ) on the unit simplex in R^2, which is evidently a compact set.

Then by Brouwer's fixed point theorem there exists a fixed point $\mathbf{p}^*$ for the continuous function h which consequently satisfies $\mathbf{p}^* = h(\mathbf{p}^*)$. □

4.3 The network decentralized equilibrium (NDE)

Definition 4.1 Consider the infinite mass system $S^\infty = \{S^{\mathcal{M}}; M \geq 2\}$, with the transition equations (3.3) and (3.7) and the local cost function (4.29), subject to a collection of uniform local randomized control laws $\mathbf{u}^* \equiv \{u^{j,*};\, j \leq M\}$.

We say the infinite system S^∞ is in a *(stationary) network decentralized equilibrium (NDE)* with a pair of time invariant parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$ if: (i) the system S^∞ is in an NDS state with time invariant parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$; and (ii) $\mathbf{u}^*$ is a collection of optimal control laws for the local OSC problems of the set of agent networks, each with cost functions of the form (4.29) and the time invariant rate parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}}, \lambda_1, \lambda_1, \mu_2, \mu_2; \alpha, \varepsilon)$. □

Theorem 4.2 (Existence of NDE states generated by local feedback controls)

Under the hypotheses $\mathrm{H}(t_0)$ and $\mathrm{H_s}$, there exists a collection of uniform local randomized control laws $\mathbf{u}^* \equiv \left\{u^{j,*}; j < \infty\right\}$ and a pair of associated time invariant parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$, such that the infinite mass system S^∞ is in an NDE state with the time invariant parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$.

Proof. By Theorem 4.1, for an isolated single agent network S^{a} as given in Definition 3.2, there exists a local randomized control law $u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)}$ and a pair of time invariant parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$ satisfying:

$$p^*_{\mathrm{RI}} = \mathbb{P}_{\rho^*}(u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)} = a_3 \,|\, e^{\mathrm{a},+}_{\mathrm{AO}}), \tag{4.39}$$

$$p^*_{\mathrm{AO}} = \mathbb{P}_{\rho^*}(u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)} = a_2 \,|\, e^{\mathrm{a},+}_{\mathrm{RI}}), \tag{4.40}$$

where $u^{\mathrm{a},*}_{(\rho^*,\alpha,\varepsilon)}$ is a (time invariant) local optimal randomized control law for the local OSC problem with time invariant parameter $(\rho^*,\alpha,\varepsilon)$, $\rho^* = (p^*_{\mathrm{RI}},\, p^*_{\mathrm{AO}},\, \lambda_1,\, \lambda_2,\, \mu_1,\, \mu_2)$.

Under the hypothesis $\mathrm{H_s}$, when each single agent network is subject to the optimal randomized control law $u^{\mathrm{a},*} \in \mathcal{U}^{\mathrm{a}}$, there exists a stationary state distribution $P(\mathbf{n};u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)})$.

Now consider the infinite mass system S^∞; for each member of the family $\{S^{\mathcal{M}}; M \geq 2\}$ let the initial state distribution for each single agent network be taken to be equal to $P(\mathbf{n};u^{\mathrm{a},*}_{(\rho,\alpha,\varepsilon)})$. Then by Theorem 3.3, the infinite mass system S^∞ is in an NDS state and is such that the parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$ are time invariant.

□

Corollary 4.1 (Nash property of an NDE state)

Consider the infinite mass system S^∞, with uniform exogenous network and parameters $\lambda_1, \lambda_2, \mu_1, \mu_2)$, for which the agent networks $S^j, k < \infty$, have uniform cost parameters (α, ε).

Assume S^∞ is in an NDE state where the uniform time invariant rate parameters for each agent network $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$ result from the collection of uniform local control laws $\mathbf{u}^* \equiv \left\{u^{j,*}; 1 \leq j < \infty\right\}$.

Then S^∞ is in a Nash equilibrium NDE state subject to the control laws $\mathbf{u}^*$.

Proof. By definition, an NDE state is a stationary NDS state such that (i) the reciprocity requirements (3.22) and (3.23) hold, and (ii) the local optimality property with respect to the mass behaviour holds, (see Figure 4.9).

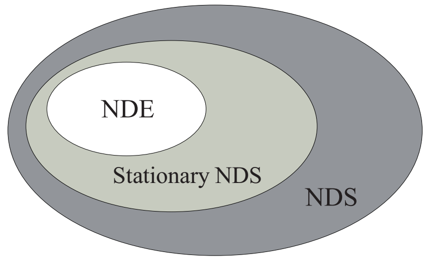

Figure 4.9: Network decentralized states and network decentralized equilibria

Suppose that under a collection of uniform local control laws $\mathbf{u}^* \equiv \{u^{j,*}; j < \infty\}$, the infinite system S^∞ is in an NDE state with an associated time invariant parameter $(p^*_{\text{RI}}, p^*_{\text{AO}})$.

Consider the case where all agent networks except the agent network S^j implement the collection of control laws $\mathbf{u}^{-j,*} \equiv \{u^{k,*}; k \neq j, k < \infty\}$, while the agent network S^j implements a local randomized control law $u^j \in \mathcal{U}^j$. In the population limit, as M goes to infinity, under the collection of (nonuniform) local control laws $\{u^j, \mathbf{u}^{-j,*}\}$, the infinite system $S^\infty_{[j]}$ is in a stationary NDS state with the parameter $(p^*_{\text{RI}}, p^*_{\text{AO}})$.

Then consider the control problem for the agent network S^j with the parameter $(\rho^*, \alpha, \varepsilon)$, where $\rho^* = (p^*_{\text{RI}}, p^*_{\text{AO}}, \lambda_1, \lambda_2, \mu_1, \mu_2)$. By the optimality property of $u^{j,*}$ with respect to the parameter $(\rho^*, \alpha, \varepsilon)$, with any $u^j \in \mathcal{U}^j$, it is the case that

$$J^j_{(\rho^*,\alpha,\varepsilon)}(s, \mathbf{n}; u^{j,*}; \mathbf{u}^{-j,*}) \leq J^j_{(\rho^*,\alpha,\varepsilon)}(s, \mathbf{n}; u^j; \mathbf{u}^{-j,*}), \tag{4.41}$$

which establishes the Nash equilibrium property with respect to the collection of local randomized control laws $\mathbf{u}^*$. □

Hence by Theorem 4.1 we may claim that:

(i) If the system is started along an NDE state then it necessarily remains on that state (by the definition of optimizing agents and the existence of the solution);

(ii) Possible convergence to such an NDE state from initially non-Nash local control laws for all agents is not studied in this paper but in this connection we observe that a finite number of deviant (i.e. non-Nash) players will asymptotically (in population size) not disturb the equilibrium. Furthermore, in the NCE framework initial results on adaptive convergence have been obtained in [12].

In summary, we have formulated call admission problems for global radial network systems as decentralized suboptimal control problems in Sections 3 and 4. This methodology whereby one aims at simulating mass effects on an individual through independent filtered processes of statistical characteristics assumed to be known, has been called the Mean Field (MF) (or Nash certainty equivalence (NCE)) principle, in the context of linear quadratic regulator large scale games [15]. We shall call it here the *point process MF (PPMF) principle.*

This is an extension to the network point process context of the NCE Principle originally formulated in the LQG framework by M. Huang et al., [13], [16] and [15].

5 Computation of the NDE parameters and the system performance

In this section we present a conceptual algorithm for the computation of the pair of time invariant NDE parameters $(p^*_{\text{RI}}, p^*_{\text{AO}})$ for a given system and give an example illustrating its implementation.

(1) *Initialization* The algorithm is initialized with the uniform exogenous network and cost parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$), and (α, ε), respectively , and the time invariant nominal initial rate parameters $(p_{\mathrm{RI}}, p_{\mathrm{AO}}) \in [0,1]^2$; these isolated agent network quantities parameterize the incoming and outgoing filtered streams which are assumed to be Poisson processes with rates equal to $p_{\mathrm{RI}}\lambda_2$ and $p_{\mathrm{AO}}\lambda_2$ respectively, (see Figure 5.10). A δ norm tolerance level is also specified.

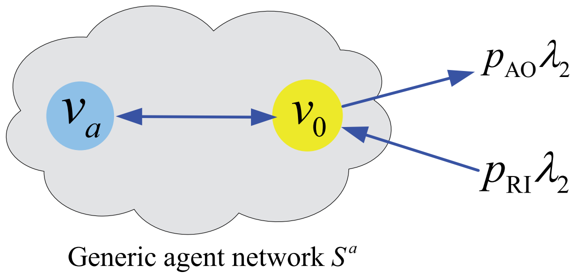

Figure 5.10: Poisson external call request streams of generic agent network S^{a}

(2) *Computation of optimal control $u^{a,*}(p_{RI}, p_{AO})$* applying *policy iteration [4]*, to the HJB equations (4.31), the optimal stationary local randomized control law $u^{\mathrm{a},*}(p_{\mathrm{RI}}, p_{\mathrm{AO}})$ is computed for an isolated single agent network with exogenous parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$), cost parameters (α, ε), respectively , and nominal or most recently computed time invariant rate parameters $(p_{\mathrm{RI}}, p_{\mathrm{AO}}) \in [0,1]^2$; $(p_{\mathrm{RI}}, p_{\mathrm{AO}})$.

(3) *Computation of the stationary distribution under the control law $u^{a,*}(p_{RI}, p_{AO})$*
The stationary state distribution denoted $P(\mathbf{n})$, $\mathbf{n} \in \mathcal{N}$, is computed for the agent system is under the optimal control law $u^{\mathrm{a},*}(p_{\mathrm{RI}}, p_{\mathrm{AO}})$ by solving the stationary form of the forward Kolmogorov equations (3.26) with respect to the time invariant parameters nominal or most recently computed time invariant rate parameters $(p_{\mathrm{RI}}, p_{\mathrm{AO}}) \in [0,1]^2$; $(p_{\mathrm{RI}}, p_{\mathrm{AO}})$ and the time invariant exogenous parameters.
The updated $u^{\mathrm{a},*}$ feedback induced parameters denoted $(\widehat{p}_{\mathrm{RI}}, \widehat{p}_{\mathrm{AO}})$ are given by the formula (3.24) and (3.25).

(4) *Stopping rule* In case the current computed value of $(p_{\mathrm{RI}}, p_{\mathrm{AO}})$ is within the given δ norm tolerance of the previously computed value. $(\widehat{p}_{\mathrm{RI}}, \widehat{p}_{\mathrm{AO}})$, $(p_{\mathrm{RI}}, p_{\mathrm{AO}})$ is a pair of NDE parameters for the underlying decentralized OSC problems; otherwise set the feedback induced parameters $(\widehat{p}_{\mathrm{RI}}, \widehat{p}_{\mathrm{AO}})$, stop, else return to 2.

□

Example 5.1 We consider the decentralized control problem for the radial network whose uniform exogenous network parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$), (see (S1)-(S5)), capacity c, and cost parameter ε are specified in Table 5.1. For simplicity we suppose there does not exist an internal call request process for any of the agent networks, i.e. $\lambda_1 = 0$.

Table 5.1: Parameters for decentralized control of a radial network

c	λ_1	λ_2	μ_1	μ_2	β	g_2	b_2	ε
15	0	10	1	1	0.3	-1	-2	0.3

In the case under consideration, the cardinality $|\mathcal{N}|$ of the discrete state set of the system is of the order of 100. The resulting state dependent control u^{a} is consequently a vector of 100 probabilities.

An implementation of the conceptual algorithm (1)-(4) above yields a sequence of iterates converging to the limiting Nash NDE parameter values $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$; Figure 5.11 displays such converging sequences in the cases where α takes the values 0.4 and 0.5 where α is the relative cost of outgoing to incoming accepted calls (see Section 3.1).

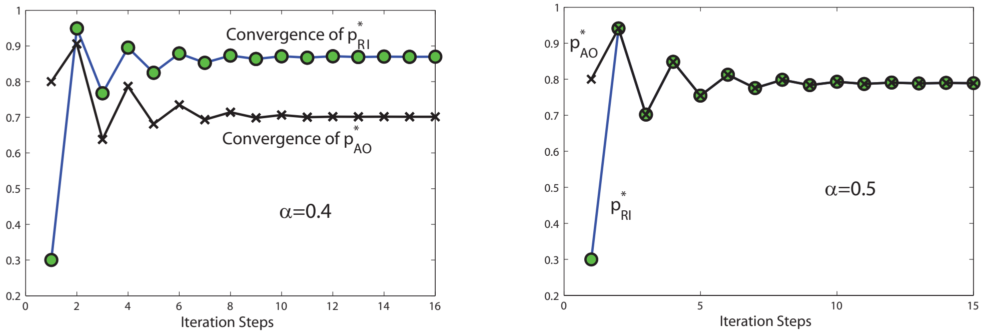

Figure 5.11: Computation of the NDE parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$

Furthermore, the simulations reveal the dependence of the NDE rate parameters $\begin{bmatrix} p^*_{\mathrm{RI}} \\ p^*_{\mathrm{AO}} \end{bmatrix}$, (see Figure 5.12), and the vector of performance indices $\begin{bmatrix} p^*_{\mathrm{RI}} p^*_{\mathrm{AO}} \\ V \end{bmatrix}$, (see Figure 5.13), at the Nash equilibrium with the pair of parameters μ_2 and α) varying over $[0.5, 5.5]$ and $[0, 1]$, respectively.

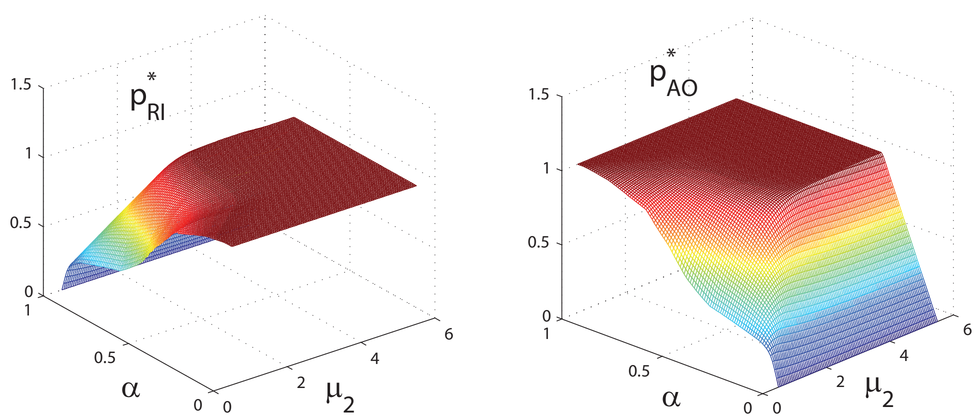

Figure 5.12: Dependence of the NDE parameters $(p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}})$ on (μ_2, α)

More specifically, in Figure 5.14, we illustrate the system performance in detail with $\mu_2 = 1$ and α varying over $[0, 1]$. Concerning this figure we make the following observations:

(i) Consider the case $\alpha > 0.5$, implying that incoming call requests are more lucrative than outgoing ones. Then, as shown in the right half region of the left diagram in Figure 5.14, $p^*_{\mathrm{RI}} \le p^*_{\mathrm{AO}}$ corresponding to the intuitively plausible result that at the Nash equilibrium each each agent's control law is such that the probability of admitting an incoming call request is greater than that of releasing an outgoing one.

(ii) With $\alpha < 0.5$, by the symmetry property of the mass systems, we have an analysis exactly corresponding to that for $\alpha > 0.5$ in (i) above (see the left hand diagram in Figure 5.11 with $\alpha = 0.4$).

(iii) The case of $\alpha = 0.5$ corresponds to the equality of the rewards for the acceptance of incoming and outgoing call requests, hence by the symmetry of the components of the NDE state with $\alpha = 0.5$ one obtains $p^*_{\mathrm{RI}} = p^*_{\mathrm{AO}}$; the numerical results obtained are consistent with this fact as is shown in the right hand diagram in Figure 5.11 and the corresponding intersection point of $p^*_{\mathrm{RI}}(.)$ and $p^*_{\mathrm{AO}}(.)$ curves at $\alpha = 0.5$ in the left hand diagram in Figure 5.14.

(iv) The $p^*_{\mathrm{RI}}(\alpha)p^*_{\mathrm{AO}}(\alpha)$ curve in the left hand diagram in Figure 5.14 depicts the variation of the product of $p^*_{\mathrm{RI}}(.)$ and $p^*_{\mathrm{AO}}(.)$ with respect to α, where $2p^*_{\mathrm{RI}}(.)p^*_{\mathrm{AO}}(.)$ is the average number of active external

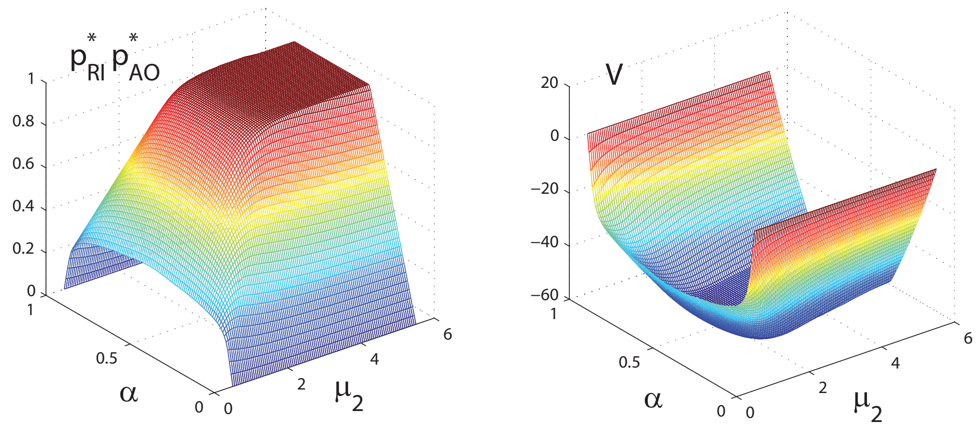

Figure 5.13: Variation of $p^*_{\text{RI}}.p^*_{\text{AO}}$ and V w.r.t. (μ_2, α)

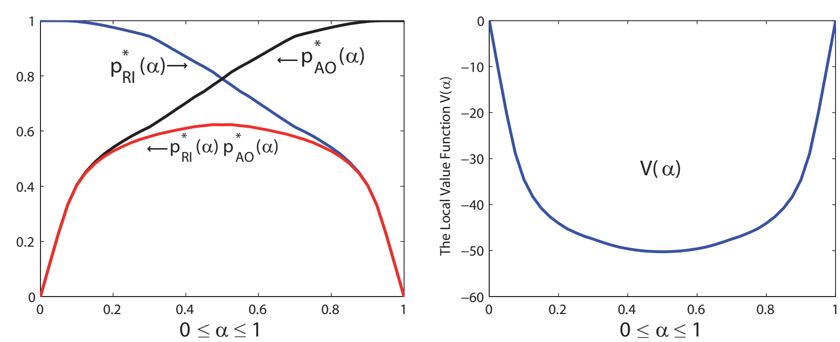

Figure 5.14: Dependence of the NDE parameters and the value function on $\mu_2 = 1$ and $\alpha \in [0, 1]$

connection in each agent network, while the right hand diagram in Figure 5.14 displays the dependence on α of the value function $V(\alpha) \equiv V^{\text{a}}_{(p^*_{\text{RI}}, p^*_{\text{AO}}, \alpha)}$, associated with the NDE parameters $(p^*_{\text{RI}}, p^*_{\text{AO}})$. These curves clearly reveal the symmetry of the system behaviour with respect to values of α symmetrically distributed around $\alpha = 0.5$, and, moreover, the optimality of the value of the Nash equilibrium at $\alpha = 0.5$. □

5.1 Summary

(1) ***Mass (radial) network systems*** Consider a class of call admission control problems of large loss network systems specified in Section 2 where each pair of agent networks (see Definition 3.1) are weakly coupled with each other under assumptions (S1)-(S3), (see Figure 2.1).

(2) ***Derivation of the network decentralized states (NDSs)*** Subject to a collection of uniform local randomized control laws $\mathbf{u} \equiv \{u^j; j < \infty\}$, by Theorem 3.3 the infinite system S^∞ is in an *NDS* with a unique pair of parameters $(p^0_{\text{RI}}, p^0_{\text{AO}})$ equal to:

$$\Big(\sum_{\mathbf{n} \in \mathcal{N}} P_{\mathbf{n}}(t) \mathbb{P}(u^{\text{a}} = a_3 \big| \mathbf{n}, e^{\text{a},+}_{\text{AO}}), \quad \sum_{\mathbf{n} \in \mathcal{N}} P_{\mathbf{n}}(t) \mathbb{P}(u^{\text{a}} = a_2 \big| \mathbf{n}, e^{\text{a},+}_{\text{RI}}) \Big)$$

where $P_{\mathbf{n}}(t)$, denoting the probability that $\mathbf{n}^{\mathrm{a}}_{t-}$ is equal to $\mathbf{n}$, is the unique absolutely continuous solution to the following collection of *quadratic differential equations:*

$$\frac{dP_{\mathbf{n}}(t)}{dt} = f_{\mathbf{n}}(P(t),t), \qquad \text{with } P(t) \equiv \big(P_1(t);\cdots;P_{|\mathcal{N}|}(t)\big),$$

for any $\mathbf{n} \in \mathcal{N}$, with the quadratic form function $f_{\mathbf{n}}$ specified in (A.44).

(3) *Derivation of stationary network decentralized equilibrium (NDE)* Consider the local cost function of isolated single agent network S^{a} with a time invariant parameter $(\rho,\alpha,\varepsilon)$ where $\rho = (p_{\mathrm{RI}}(t), p_{\mathrm{AO}}(t), \lambda_1, \lambda_2, \mu_1, \mu_2)$:

$$J^{\mathrm{a}}_{(\rho,\alpha,\varepsilon)}(s,\mathbf{n};u^{\mathrm{a}}) = \mathbb{E}_{|(s,\mathbf{n})}\Big\{\int_s^\infty e^{-\beta t}G(\mathbf{n}^{\mathrm{a}}_t)dt + \sum_{k=1}^{\infty} e^{-\beta t_k}\big(B(e^{\mathrm{a}}_{t_k}) + \varepsilon\mathbb{P}(u^{\mathrm{a}}_{t_k}\in U^+)\big)\mathbf{I}(u^{\mathrm{a}}_{t_k}\in U^+)\Big\}$$

Then by Theorem 4.2 the infinite system S^∞ is in a *stationary NDE* with some pair of time invariant parameters $\mathbf{p}^* \equiv (p^*_{\mathrm{RI}};p^*_{\mathrm{AO}})$ which is a time invariant *fixed point* of the following pair of equations:

$$p^*_{\mathrm{RI}} = \sum_{\mathbf{n}\in\mathcal{N}} P^*_{\mathbf{n}}\mathbb{P}(u^{\mathrm{a},*}_{\rho^*} = a_3 \big| \mathbf{n}, e^{\mathrm{a},+}_{\mathrm{AO}}),$$
$$p^*_{\mathrm{AO}} = \sum_{\mathbf{n}\in\mathcal{N}} P^*_{\mathbf{n}}\mathbb{P}(u^{\mathrm{a},*}_{\rho^*} = a_2 \big| \mathbf{n}, e^{\mathrm{a},+}_{\mathrm{RI}}),$$

where (i) $\rho^* \equiv (p^*_{\mathrm{RI}}, p^*_{\mathrm{AO}}, \lambda_1,\lambda_2,\mu_1,\mu_2)$, (ii) $u^{\mathrm{a},*}_{\rho^*} \equiv u^{\mathrm{a},*}(\rho^*;\mathbf{n}^{\mathrm{a}}_{t-}, e^{\mathrm{a}}_t)$ is a stationary local optimal randomized control law with respect to parameters ρ^*, and (iii) $P(.;u^{\mathrm{a},*}_{\rho^*})$ denotes the stationary probability of local connection vector process $\mathbf{n}^{\mathrm{a}}_t$ subject to $u^{\mathrm{a},*}_{\rho^*}$.

6 Conclusion and Future Work

This paper has presented an analysis of distributed call admission control problems for a class of global loss networks each of which is composed of a group of weakly coupled individual systems. Asymptotically, under an initial independence of states hypothesis, and for uniform local control laws, agent network state processes and their boundary filtered call request processes (accepted outgoing and released incoming call requests) remain mutually independent. Furthermore, moving from centralized OSC problems to a distributed OSC paradigm whereby agents apply local control laws to optimize their individual costs, it is shown that there exist boundary filtered call request processes and uniform randomized local control law pairs such that the local control laws are optimal with respect to the very boundary processes they collectively induce. In other words, the Nash certainty equivalence principle holds and in this context we call it the point process NCE (PPNCE) principle. Future work will include the study of the simultaneous solution of the problems of call admission and routing control within the NCE framework, and the introduction of non uniform randomized topologies of local networks. Note that a solution to the centralized optimal control problem is given in [24].

7 Appendices

A Verification of the hypotheses of Caratheodory's theorem

For notational simplicity, $P_{P_0,\mathbf{p}^0_t}(\mathbf{n},t)$ in (3.26) is written $P_{\mathbf{n}}(t)$. By (3.26) and (3.27),

$$\frac{dP(t)}{dt} = f(P(t),t), \qquad \text{with } P(t) = \big(P_{\mathbf{n}_1}(t);\cdots;P_{\mathbf{n}_{|\mathcal{N}|}}(t)\big) \in [0,1]^{|\mathcal{N}|}, \tag{A.42}$$

where $f : [0,1]^{|\mathcal{N}|}\times[0,\infty) \to \mathbb{R}^{|\mathcal{N}|}$, such that for any $t\in[0,\infty)$

$$f_{\mathbf{n}}(P(t),t) = -\Big(\sum_{e\in E(\mathbf{n})}\lambda^e(\mathbf{n},t)\Big)P_{\mathbf{n}}(t) + \sum_{\substack{\mathbf{m}\in\mathcal{N}\\ e\in E(\mathbf{m})\\ \mathbf{m}+u=\mathbf{n}}}\Big(\lambda^e(\mathbf{m},t)\,\mathbb{P}_{|\mathbf{m},e}\big(u^{\mathrm{a}}(t)=u\big)P_{\mathbf{m}}(t)\Big), \tag{A.43}$$

with $\lambda^e(\mathbf{n},t)$ specified in (3.27).

Then by (3.24), (3.25), (3.27) and (A.43), the function $f_{\mathbf{n}}$ has a quadratic form with respect to P as the follows, such that for any $t \in [0,\infty)$ and $P \in [0,1]^{|\mathcal{N}|}$,

$$\begin{aligned} f_{\mathbf{n}}(P,t) = &-\xi_2^+ \sum_{\mathbf{m}\in\mathcal{N}} \mathbb{P}_{|\mathbf{m},e^{\mathrm{a},+}_{\mathrm{AO}}}\big(u^{\mathrm{a}}(t)=a_3\big)\,\mathbb{P}_{|\mathbf{n},e^{\mathrm{a},+}_{\mathrm{RI}}}\big(u^{\mathrm{a}}(t)=a_2\big)\,P_{\mathbf{m}}P_{\mathbf{n}} \\ &-\xi_2^+ \sum_{\mathbf{m}\in\mathcal{N}} \mathbb{P}_{|\mathbf{m},e^{\mathrm{a},+}_{\mathrm{RI}}}\big(u^{\mathrm{a}}(t)=a_2\big)\,\mathbb{P}_{|\mathbf{n},e^{\mathrm{a},+}_{\mathrm{AO}}}\big(u^{\mathrm{a}}(t)=a_3\big)\,P_{\mathbf{m}}P_{\mathbf{n}} \\ &-\xi_1^+\,\mathbb{P}_{|\mathbf{n},e^{\mathrm{a},+}_{\mathrm{Loc}}}\big(u^{\mathrm{a}}(t)=a_1\big)\,P_{\mathbf{n}} \\ &-\Big(\sum_{b=1}^{3} n_b\xi_b^-\Big)\,P_{\mathbf{n}} \\ &+\xi_2^+ \sum_{\mathbf{m}\in\mathcal{N}} \mathbb{P}_{|\mathbf{m},e^{\mathrm{a},+}_{\mathrm{AO}}}\big(u^{\mathrm{a}}(t)=a_3\big)\,\mathbb{P}_{|\mathbf{n},e^{\mathrm{a},+}_{\mathrm{RI}}}\big(u^{\mathrm{a}}(t)=a_2\big)\,P_{\mathbf{m}}P_{\mathbf{n}-a_2} \\ &+\xi_2^+ \sum_{\mathbf{m}\in\mathcal{N}} \mathbb{P}_{|\mathbf{m},e^{\mathrm{a},+}_{\mathrm{RI}}}\big(u^{\mathrm{a}}(t)=a_2\big)\,\mathbb{P}_{|\mathbf{n},e^{\mathrm{a},+}_{\mathrm{AO}}}\big(u^{\mathrm{a}}(t)=a_3\big)\,P_{\mathbf{m}}P_{\mathbf{n}-a_3} \\ &+\xi_1^+\,\mathbb{P}_{|\mathbf{m},e^{\mathrm{a},+}_{\mathrm{Loc}}}\big(u^{\mathrm{a}}(t)=a_1\big)\,P_{\mathbf{m}-a_1} \\ &+\sum_{b=1}^{3}(\mathbf{n}_b+1)\xi_b^-\,P_{\mathbf{n}+a_b} \end{aligned} \tag{A.44}$$

By (A.44) and $\mathbb{P}_{|\mathbf{n},e}(u^{\mathrm{a}}(t)=u) \in [0,1]$, we have, for any $P,\widehat{P} \in [0,1]^{|\mathcal{N}|}$

$$\begin{aligned} |f_{\mathbf{n}}(P,t) - f_{\mathbf{n}}(\widehat{P},t)| \le\; & 2\xi_2^+ \sum_{\mathbf{m}\in\mathcal{N}} |P_{\mathbf{m}}P_{\mathbf{n}} - \widehat{P}_{\mathbf{m}}\widehat{P}_{\mathbf{n}}| \\ &+ (\xi_1^+ + \sum_{b=1}^{3} n_b\xi_b^-)|P_{\mathbf{n}} - \widehat{P}_{\mathbf{n}}| \\ &+ \xi_2^+ \sum_{b=2}^{3}\sum_{\mathbf{m}\in\mathcal{N}} |P_{\mathbf{m}}P_{\mathbf{n}-a_b} - \widehat{P}_{\mathbf{m}}\widehat{P}_{\mathbf{n}-a_b}| \\ &+ \xi_1^+|P_{\mathbf{n}-a_1} - \widehat{P}_{\mathbf{n}-a_1}| \\ &+ \sum_{b=1}^{3}(\mathbf{n}_b+1)\xi_b^-|P_{\mathbf{n}+a_b} - \widehat{P}_{\mathbf{n}+a_b}| \end{aligned} \tag{A.45}$$

Furthermore, we suppose that $P = \widehat{P} + \epsilon$, then by the fact of $P,\widehat{P} \in [0,1]^{|\mathcal{N}|}$, we have $|\epsilon_{\mathbf{n}}| \in [0,1]$, for any $\mathbf{n} \in \mathcal{N}$. Hence

$$\begin{aligned} |P_{\mathbf{m}}P_{\mathbf{n}} - \widehat{P}_{\mathbf{m}}\widehat{P}_{\mathbf{n}}| &= |\epsilon_{\mathbf{m}}\widehat{P}_{\mathbf{n}} + \epsilon_{\mathbf{n}}\widehat{P}_{\mathbf{m}} + \epsilon_{\mathbf{m}}\epsilon_{\mathbf{n}}| \\ &\le 3\max_{\widehat{\mathbf{n}}\in\mathcal{N}}|\epsilon_{\widehat{\mathbf{n}}}| \equiv 3\max_{\widehat{\mathbf{n}}\in\mathcal{N}}|(P-\widehat{P})_{\widehat{\mathbf{n}}}| \le 3|P-\widehat{P}| \end{aligned} \tag{A.46}$$

By (A.45) and (A.46), for any $(P,\widehat{P},t) \in [0,1]^{|\mathcal{N}|}\times[0,1]^{|\mathcal{N}|}\times[0,\infty)$, there exists a finite valued $k_1 \in [0,\infty)$, such that

$$|f(P,t) - f(\widehat{P},t)| < k_1|P-\widehat{P}|, \tag{A.47}$$

i.e. f is uniformly Lipschitz continuous with respect to P.

Furthermore, for any $\mathbf{n} \in \mathcal{N}$, by (A.44), we have, there exists some finite valued $k_2 \in [0,\infty)$, such that

$$|f_{\mathbf{n}}(P^j,t)| \le 2\,\xi_2^+|\mathcal{N}| + \big(\xi_1^+ + \sum_{b=1}^{3} n_b\xi_b^-\big) + 2\xi_2^+|\mathcal{N}| + \xi_1^+ + \sum_{b=1}^{3}(n_b+1)\xi_b^- < k_2, \tag{A.48}$$

i.e. f is bounded with respect to t for any fixed $P \in [0,1]^{|\mathcal{N}|}$.

This completes the verification of the hypotheses of Caratheodory's Theorem for Theorem 3.2.

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