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Abstract

In this paper, we introduce the notion of dynamic copulas to model serial dependence as well as interdependence between several time series. The proposed methodology is totally different from the usual time-varying copula modeling of time series in which only the dependence between the serially independent innovations is taken into account and time series must be modeled individually. Here no modeling of univariate time series in necessary and we tackle at the same time serial dependence and interdependence. We discuss issues related to parameter estimation as well as tests of goodness-of-fit. We treat in greater detail two families, specifically the meta-elliptic copulas and Archimedean copulas. The methodology is then applied to model the dynamic dependence between the Canadian/US exchange rate and value of oil futures during the last ten years.

Key Words: Copulas, Markov processes, multivariate time series, serial dependence.

Résumé

Dans cet article, nous introduisons la notion de copule dynamique pour modéliser la dépendance sérielle ainsi que l'interdépendance entre plusieurs séries chronologiques. La méthodologie proposée est totalement différente de celles habituellement proposée, où la copule varie dans le temps et modélise seulement l'interdépendance entre les bruits blancs de séries chronologiques, chacune devant être modélisée. Ici, aucun modèle n'est nécessaire pour les chroniques univariées et nous modélisons à la fois l'interdépendance et la dépendance sérielle. Nous abordons aussi l'estimation et les tests d'adéquation de ces modèles. Deux types de copules retiennent notre attention, soient les copules méta-elliptiques et les copules archimédiennes. La méthodologie est ensuite appliquée à la modélisation de la dépendance dynamique entre le taux de change Canada/US et la valeur de contrats à terme sur le pétrole, pour les dix dernières années.

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1 Introduction

The understanding and proper modeling of dependence between financial assets is an important issue. The 2008 crisis, and specifically the structured products that were collateralized by pools of sub-prime mortgages, provided us with a very tangible example of the devastating financial and economic repercussions that can result from overly naive and simplistic assumptions about default ”contagion”. Given that financial institutions were clearly more concerned with generating fees that calculating risk, a better understanding of dependence might not necessarily have been enough to avert the recent crisis. Nonetheless, as market recover from the recent debacle and we move forward, it is fundamental that we develop tools that can help mitigate the possibility of a similar future meltdown.

Dependence is not just an “academic” fantasy, and is by no means limited to complex financial products. Let’s consider the following question, which was recently posted by a reader of the financial section of the *Globe and Mail*:

> The Canadian dollar always seems to go up [with respect to the US dollar] when oil prices rise. Is there a direct correlation between the two?

Although the reader probably assumed there would be a simple answer to his question, the reality is that there are many factors that drive the relationship between the two variables. The journalist’s response[1] identified several economic and financial factors underlying the apparent relationship between the two variables. These included the simple fact that an increase demand for a commodity naturally leads to an increase demand for the currency of the country that produces the commodity. However, citing the economist Dale Orr, he also indicated that the causal relation can go the other way: weakness in the US dollar can lead to demand for commodities such as oil as a hedge against exchange rate risk, pushing up oil prices. Another theory suggests that countries that purchase oil will buy more when the U.S. dollar is down, because oil is priced in U.S. currency.

Irrespective of the economic or financial factors underlying the relationship, can we at least prove that the answer to this question is yes? In order to provide a clear answer, we need to analyze the relationship from a quantitative perspective. The mathematical translation of our reader’s question can be stated as follows: Given two time series X_t and Y_t, does there exist a positive dependence between their returns $R_t^X = \log(X_t/X_{t-1})$ and $R_t^Y = \log(Y_t/Y_{t-1})$? And perhaps even more importantly, what is the strength and stability of this relationship?

There is a plethora of articles in Economics, Finance and Actuarial sciences literature dealing with dependence between time series that can help address these questions. One thing that is certain, however, is that one cannot simply rely on the traditional (Pearson) correlation measure. After all, it is well-known that outside the realm of Gaussian models, correlation is a poor measure of dependence; see e.g., Embrechts et al. (2002). In fact, the best way to measure or quantify dependence between two variables is the so-called copula (also called dependence function). As opposed to the fully parametric approaches which model the marginal distributions along with the dependence, the semi-parametric copulas (a precise definition of copula will be presented in the next section) focus exclusively on the dependence. The major advantage, therefore, in using copulas is that one does not require information about the marginal distributions of the variables in order to study the dependence. After all, the marginal distributions provide no useful information regarding the dependence and are simply nuisance parameters. Erroneous assumptions as to the nature of the distributions as well as estimation errors of their parameters will feed through to the estimation of the dependence. Eliminating the marginal distributions from the analysis renders the study of the dependence much more reliable and robust.

When copulas are used, they are usually fitted to observations or even to residuals of time series (e.g., GARCH models). The next issue is the evaluation of the goodness-of-fit. This can be of paramount importance in certain fields of applications, such as in credit risk modeling, where the incorrect choice of the

[1] Richard Blackwell: “Do oil and loonie share the same flight?”, The Globe and Mail, June 03, 2009. A copy of the complete article is available at http://www.stockhouse.com/blogs/ViewDetailedPost.aspx?p=92442

dependence model can have disastrous consequences. The importance of choosing the right model was emphasized in a number of papers prior to recent financial crisis, e.g. Berrada et al. (2006). However the topic of goodness-of-fit for copulas is only at its early stages of development and there is almost no rigorous method of goodness-of-fit available using residuals. Even the recent literature of goodness-of-fit for copulas based on serially independent observations contains many errors. See, e.g., Genest et al. (2009) for a review.

Perhaps even more important is the fact that in most articles, the (possible) serial dependence is ignored or treated separately from interdependence, when modeling dependence between several time series. And when serial dependence is considered, it is done so by modeling the individual series, while interdependence is modeled through a copula associated with the (serially independent) innovations. See, e.g., van den Goorbergh et al. (2005) or Chen and Fan (2006). See also Yi and Liao (2010) for a similar structure of dependence. However, the use of residuals of parametric models without any modification to the inference procedure leads, in general, to incorrect results. There is always a price to pay when parameters are estimated; see e.g., Ghoudi and Rémillard (2004) for examples of applications involving time series. Furthermore, there is no formal tests of goodness-of-fit yet for these models. It is easy to see that the parametric bootstrap approach proposed by Genest and Rémillard (2008) do not extend to innovations. For more details, see Rémillard (2010).

The introduction of dynamic copulas represents an important step in attempting to fill the gap of modeling intra/interdependence. That idea generalizes the introduction of copulas for modeling serial dependence in a univariate time series, as proposed in Darsow et al. (1992). However the authors never discussed inference procedures. In addition to serial dependence of each time series, dynamic copulas models interdependence, i.e., dependence between time series. To illustrate the concept, consider two Markovian (stationary) time series X and Y. Usually, one models the interdependence, that is, the dependence between X_t and Y_t. Since the time series are stationary, that dependence is not time-varying. The idea behind dynamic copulas is to model the dependence between the four variables X_{t-1}, Y_{t-1}, X_t and Y_t, taking into account both interdependence and serial dependence.

Modeling using dynamic copulas is quite different from the concept of time-varying copulas proposed in van den Goorbergh et al. (2005), and extended in Chen and Fan (2006). In these papers, copulas are only used to model the interdependence between serially independent innovations and this dependence may vary over time. First, the individual time series must be modeled, and then the resulting residuals are used to estimate the parameters of the time-varying copulas of the innovations. The approach of Patton (2006) is a little bit different since he defines conditional copulas, but when it comes to implementing them, he basically does the same thing as van den Goorbergh et al. (2005) and Chen and Fan (2006), by letting the parameters of a given copula family depend on lagged values of the series. The approach of Yi and Liao (2010) is closely related to the one of Patton (2006), but instead of dealing with innovations, they use a Rosenblatt's transform and add a copula for interdependence. Although the notion of time-varying dependence is quite appealing, there are many sensitive issues to address. These include that fact that the inference is very delicate, the time-series are not stationary, and the relationship between the parameters and exogenous variables is, to say the least, far from obvious and quite subjective.

In this paper we also use conditional copulas, however they occur naturally when the serially dependent time series are simulated due to the Markovian nature of the variables. Working with dynamic copulas is much less complicated, and the inference is relatively straightforward, even if it is a little more involved than for serially independent time series. The cost, however, is the need to assume stationarity and a Markovian structure. Fortunately, the main advantage is that one does not need to model the individual time series or calculate their residuals, which is the case of the time-varying approach of van den Goorbergh et al. (2005), Chen and Fan (2006) and Patton (2006). Note that Harvey (2010) proposes an interesting approach of detecting changes in dependence between time series without having to model them. Since our methodology is based on stationarity, one should always apply such tests before using dynamic copulas. The approach of Harvey (2010) is different from that of Rémillard and Scaillet (2009) where the time of change must be given. Finally, it is worth mentioning that Guégan and Zhang (2010) also propose a method of detecting changes in a copula using kernel estimates of copulas and residuals. Their setting is similar to the one of van den Goorbergh et al. (2005) and Chen and Fan (2006) since the time-varying copulas are fitted to residuals. The ideas are interesting but no rigorous proof whatsoever is given that their methodology works since they use residuals. Giacomini et al. (2009) is also closely related to van den Goorbergh et al. (2005) and Guégan and Zhang (2010) and discuss the problem of change-point detection.

The remainder of the paper is structured as follows. In Section 2, we recall some results on known families of copulas (meta-elliptic and Archimedean) and introduce dynamic copulas. Section 3 is dedicated to estimation of parameters and tests of goodness-of-fit. Finally, in Section 4 we give an example of application of the methodology with a data set consisting of oil futures and Canada/US exchange rate, answering the question about positive dependence between these two economic variables. Appendices B and A contains results useful for implementations, while the main results are proved in Appendix C.

2 Dynamic copulas

In this section, we start by recalling the definition of a copula. Then we give some examples of families of copulas that exist for any dimension. Finally we define dynamic copulas and provide some examples.

If H is a d-dimensional distribution function with continuous marginal distributions $F_1, \ldots, F_d$, then according to Sklar (1959), there exists a unique distribution function C with uniform margins over $[0,1]$ such that

$$H(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \tag{1}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. It follows that for any random vector $X = (X_1, \ldots, X_d)$ having distribution function H, denoted $X \sim H$, then $U = (U_1, \ldots, U_d)$, with $U_j = F_j(X_j)$, $j = 1, \ldots, d$, has distribution function C. Note that each component of U is uniformly distributed over $[0,1]$. One can easily check that the variables $X_1, \ldots, X_d$ are independent if and only if $C = C_\perp$, the independence copula, defined by

$$C_\perp(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j, \text{ for all } u = (u_1, \ldots, u_d) \in [0,1]^d.$$

An interesting property of copulas is their invariance under increasing transformations of the components. More precisely, if $Y_j = T_j(X_j)$, where each function T_j is increasing, $j = 1, \ldots, d$, then the copula of Y is the same as the copula of X.

With the exception of the Pearson's correlation (which does not exists if one of the two variables has an infinite second moment), almost all known measures of dependence between pairs of random variables depend only on the copula. For instance, Spearman's rho can be expressed as $\rho^S = 12 \int_0^1 \int_0^1 \{C(u_1,u_2) - u_1 u_2\} du_1 du_2$, while Kendall's tau is defined as $\tau = -1 + 4 \int_0^1 \int_0^1 C(u_1,u_2) dC(u_1,u_2)$. In comparison, Pearson's correlation, provided it exists, can be expressed as

$$\rho^P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[C\{F_1(x_1), F_2(x_2)\} - F_1(x_1)F_2(x_2) \right] dx_1 dx_2.$$

It is then obvious that by varying the marginal distribution functions F_1, F_2, leaving the copula unchanged, one changes in general the value of ρ^P, except of course when C is the independence copula $C_\perp$. For more details on copulas and measures of dependence, one can refer to Nelsen (2006), Joe (1997), and Cherubini et al. (2004).

2.1 Archimedean copulas

Archimedean copulas were first defined by Genest and MacKay (1986). A copula C is said to be Archimedean (with generator ϕ) when it can be expressed in the form

$$C(u_1, \ldots, u_d) = \phi^{-1}\left\{\phi(u_1) + \cdots + \phi(u_d)\right\},$$

where $\phi : (0,1] \to [0,\infty)$, is a bijection such that $\phi(1) = 0$ and

$$(-1)^i \frac{d^i}{dx^i} \phi^{-1}(x) > 0, \quad 1 \le i \le d.$$

Note that the generator is unique up to a constant.

If the generator yields a copula for any $d \geq 2$, then ϕ^{-1} is necessarily the Laplace transform of a non-negative random variable ξ (Marshall and Olkin, 1988), i.e., $\phi^{-1}(s) = E\left(e^{-s\xi}\right)$, for all $s \geq 0$.

Moreover, if $\tilde{U}_1, \ldots, \tilde{U}_d$ are independent and uniformly distributed on $[0,1]$, then defining $U_i = \phi^{-1}\left\{\frac{-\log(\tilde{U}_i)}{\xi}\right\}$, with $i = 1, \ldots, d$, and setting $U = (U_1, \ldots, U_d)$, it follows that $U \sim C$. As a consequence, these Archimedean copulas can be seen as special cases of one-factor models.

Table 1 gives the generators for three well-known Archimedean copulas: Clayton, Frank, and Gumbel-Hougaard families. These three classes share the interesting property that the copula exists for any dimension, for the values of parameters listed in the table. See Joe (1997) and Nelsen (2006) for further examples on copulas.

Table 1: Multivariate Archimedean copulas and domain of parameter.

Family	$\phi(t)$	Range of θ	Kendall's tau
Clayton	$(t^{-\theta} - 1)/\theta$	$(0, \infty)$	$\theta/(\theta+2)$
Frank	$-\log\left(\frac{1-\theta^t}{1-\theta}\right)$	$(0,1)$	$\frac{\log(\theta)^2 + 4\log(\theta) + 4\text{dilog}(\theta)}{\log(\theta)^2}$
Gumbel-Hougaard	$\lvert \log t \rvert^{1/\theta}$	$(0,1)$	$1-\theta$

Here, $\text{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt$ stands for the dilog function.

For the Clayton family of parameter $\theta \in (0,\infty)$, the associated ξ has Gamma distribution with parameters $(1/\xi, 1)$ since $E\left(e^{-s\xi}\right) = (1+s)^{-1/\theta}$. For the Frank family with parameter $\theta \in (0,1)$, the associated ξ is discrete and has a logarithmic series distribution given by $P(\xi = k) = \frac{1}{\log(1/\theta)} \frac{(1-\theta)^k}{k}$, $k = 1, 2, \ldots$, since

$$E\left(e^{-s\xi}\right) = \log\left\{1 - (1-\theta)e^{-s}\right\} / \log(\theta) = \frac{1}{\log(1/\theta)} \sum_{k=1}^{\infty} (1-\theta)^k \frac{e^{-ks}}{k}.$$

Finally, for the Gumbel-Hougaard family with parameter $\theta \in (0,1)$, ξ has a positive stable distribution of parameter θ, since $E\left(e^{-s\xi}\right) = e^{-s^\theta}$.

2.2 Meta-elliptic copulas

Meta elliptic-copulas are simply copulas associated with elliptic distributions through relation (1), and they are quite popular in the Finance literature, specially the Student copula and the (now infamous) Gaussian copula.[2] Recall that a vector Y has an elliptic distribution with generator g and parameters μ and (positive definite symmetric matrix) Σ, denoted $Y \sim \mathcal{E}(g, \mu, \Sigma)$, if its density h is given by

$$h(y) = \frac{1}{|\Sigma|^{1/2}} g\left\{(y-\mu)^\top \Sigma^{-1} (y-\mu)\right\}, \qquad y \in \mathbb{R}^d,$$

[2] Felix Salmon: "Recipe for disaster: The formula that killed Wall Street", Wired Magazine, March 2009. Available at http://www.wired.com/techbiz/it/magazine/17-03/wp_quant

where

$$\frac{\pi^{d/2}}{\Gamma(d/2)} r^{(d-2)/2} g(r) \tag{2}$$

is a density on $(0,\infty)$.[3] In fact it is the density of $\xi = (Y-\mu)^\top \Sigma^{-1}(Y-\mu)$.

In order to generate Y, simple set $Y = \mu + \xi^{1/2} A^\top \mathcal{S}$, where $A^\top A = \Sigma$, ξ has density (2) and is independent of $\mathcal{S}$, and $\mathcal{S}$ is uniformly distributed over the d-dimensional sphere $S_d = \{y \in \mathbb{R}^d; \|y\| = 1\}$.

It is easy to check that if $Y \sim \mathcal{E}(g,\mu,\Sigma)$ then $Z = \Delta^{-1}(Y-\mu) \sim \mathcal{E}(g,0,R)$, where Δ is the diagonal matrix such that $\Delta_{ii} = \sqrt{\Sigma_{ii}}$ and R is the correlation matrix associated with Σ. It follows that the underlying copula depends only on g and R, since a copula is invariant by increasing transformations.

Here are some general families of elliptic distributions.

Table 2: Generators of some d-dimensional elliptic distributions.

Family	Generator
Gaussian	$g(r) = \frac{1}{(2\pi)^{d/2}} e^{-r/2}$
Pearson type II	$g(r) = \frac{\Gamma(\alpha + d/2)}{\pi^{d/2}\Gamma(\alpha)}(1-r)^{\alpha-1}$, where $0 < r < 1$ and $\alpha > 0$
Pearson type VII	$g(r) = \frac{\Gamma(\alpha + d/2)}{(\pi\nu)^{d/2}\Gamma(\alpha)}(1+r/\nu)^{-\alpha-d/2}$, where $\alpha, \nu > 0$

Remark 1 The case $\alpha = \nu/2$ for the Pearson type VII corresponds to the multivariate Student, while if $\alpha = 1/2$ and $\nu = 1$, it corresponds to the multivariate Cauchy distribution.

Suppose that $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{E}(g,0,\Sigma)$, where $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$.

It is easy to check that $X_1 \sim \mathcal{E}(g_1, 0, R_{11})$, where

$$g_1(r) = \int_{\mathbb{R}^{d_2}} g(\|x_2\|^2 + r) dx_2 = \frac{2\pi^{d_2/2}}{\Gamma(d_2/2)} \int_0^\infty s^{d_2-1} g(s^2 + r) ds. \tag{3}$$

Similarly, $X_2 \sim \mathcal{E}(g_2, 0, R_{22})$, where

$$g_2(r) = \int_{\mathbb{R}^{d_1}} g(\|x_1\|^2 + r) dx_1 = \frac{2\pi^{d_1/2}}{\Gamma(d_1/2)} \int_0^\infty s^{d_1-1} g(s^2 + r) ds. \tag{4}$$

As a consequence, the density of any marginal distribution of a d-dimensional elliptic distribution with generator g and parameters $(0,R)$ is

$$f(x) = \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \int_0^\infty s^{(d-3)/2} g(s + x^2) ds, \qquad x \in \mathbb{R}. \tag{5}$$

For example, if g is the generator of the d-dimensional Pearson type VII with parameters (α,ν), then g_i is the generator of the d_i-dimensional Pearson type VII with parameters (α,ν), $i = 1,2$. One can also show that if g is the generator of the d-dimensional Pearson type II with parameter α, then g_i is the generator of the d_i-dimensional Pearson type II with parameter $\alpha + d_{3-i}$, $i = 1,2$.

[3]For any positive continuous function f, $\int_{\mathbb{R}^d} f(\|x\|^2) dx = \kappa_d \int_0^\infty f(r) r^{d/2-1} dr$, where $\kappa_d = \pi^{d/2}/\Gamma(d/2)$.

In particular, the marginal distributions of a Pearson type VII is a Pearson type VII, with density

$$f(x) = \frac{\Gamma(\alpha+1/2)}{(\pi\nu)^{1/2}\Gamma(\alpha)}(1+x^2/\nu)^{-\alpha-1/2}.$$

For the Pearson type VII, $E(\xi^p) < \infty$ if and only if $p < \alpha$. Moreover, $W = \xi/(\xi+\nu) \sim \text{Beta}(d/2, \alpha)$.

2.3 Dynamic copulas

Our aim here is to present a framework for the modeling of dependence for d-dimensional time series $\{X_t\}_{t=0}^n$ using copulas. That framework is much more general than the one considered in Yi and Liao (2010).

Too often in actuarial and financial applications, serial dependence is not taken into account when modeling dependence between several time series using copulas. To try to fill that gap, we propose to use dynamic copulas.

Here we do not assume any structure for the time series nor do we need to introduce innovations. All we assume is that the process X is Markovian, stationary, has continuous marginal distributions $F_1, \ldots, F_d$ (not time-dependent due to stationarity) and that $C(u, v)$ is the (dynamic) copula associated with the $2d$-dimensional vector (X_{t-1}, X_t). It follows that the copula $D(u) = C(u, \mathbf{1})$ of X_{t-1} is the same as the copula of X_t, i.e., $D(v) = C(\mathbf{1}, v)$. Writing F the transformation $x = (x_1, \ldots, x_d) \mapsto F(x) = \{F_1(x_1), \ldots, F_d(x_d)\}$, then one can define $U_t = F(X_t)$ and U is a d-dimensional time series so that $(U_{t-1}, U_t) \sim C$, and $U_t \sim D$. Note that because F is not known, the (natural scale)[4] Markovian stationary time series U is not observable.

To estimate parameters of the dynamic copula or to simulate observations for the process U_t, one needs to compute the conditional distribution of U_t given U_{t-1}. With this conditional distribution is associated what we call the conditional copula. In a univariate time series context, this would correspond to the copula associated to serial dependence. See, e.g. Fermanian and Wegkamp (2004).

In what follows, we study the properties and construction of the conditional copula in a general context. It is then applied to multivariate time series.

2.3.1 The conditional copula

Let $H(x, y)$ be cumulative function of the joint distribution of the d_1-dimensional random vector X and d_2-dimensional random vector Y, both having continuous marginal distributions $F_1, \ldots, F_{d_1}$ and $G_1, \ldots, G_{d_2}$ respectively. Invoking Sklar's theorem (Sklar, 1959), we know that there exists a unique (d_1+d_2)-dimensional copula C so that

$$H(x, y) = C\left\{F_1(x_1), \ldots, F_{d_1}(x_{d_1}), G_1(y_1), \ldots, G_{d_2}(y_{d_2})\right\}. \tag{6}$$

Assuming that the densities f_i of F_i, g_i of G_i and c of C exist, the density of H is

$$h(x, y) = c(F_1(x_1), \ldots, F_{d_1}(x_{d_1}), G_1(y_1), \ldots, G_{d_2}(y_{d_2})) \cdot \prod_{i=1}^{d_1} f_i(x_i) \cdot \prod_{j=1}^{d_2} g_j(y_j), \tag{7}$$

where c is the density of the copula C.

Using (6) and (7), the distribution function H_X of X is

$$H_X(x) = C(F_1(x_1), \ldots, F_{d_1}(x_{d_1}), 1, \ldots, 1),$$

with density

$$f_X(x) = c_X(F_1(x_1), \ldots, F_{d_1}(x_{d_1})) \prod_{j=1}^{d_1} f_j(x_j).$$

[4]The term "natural scale" is used because the marginal distributions of the components of U_t are uniformly distributed over $[0, 1]$.

Hence, setting $u = F(x) = \{F_1(x_1), \ldots, F_{d_1}(x_{d_1})\}$ and $v = G(y) = \{G_1(y_1), \ldots, G_{d_2}(y_{d_2})\}$, one can write the conditional density $f_{Y|X}$ of Y given $X = x$ as

$$f_{Y|X}(y; x) = \frac{f(x, y)}{f_X(x)} = c_{V|U}(v; u) \prod_{j=1}^{d_2} g_j(y_j), \tag{8}$$

where

$$c_{V|U}(v; u) = \frac{c(u, v)}{c_U(u)} \tag{9}$$

is the conditional density of $V = G(Y)$ given $U = F(X) = u$. Note that it is not the density of a copula in general. However the associated (unique) copula is called the conditional copula. This is consistent with the definition given in Patton (2006). In that article, a result similar to the following proposition was called an extension of Sklar's Theorem.

Proposition 2 *The density in (9) is the density of $V = \{G_1(Y_1), \ldots, G_{d_2}(Y_{d_2})\}$ given $U = \{F_1(X_1), \ldots, F_{d_1}(X_{d_1})\} = u$. Therefore the conditional copulas of V given U and Y given X are the same.*

Here are some examples of applications.

2.3.2 Dynamic Gaussian copula

Suppose that

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_{d_1+d_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right),$$

so that $X_1 \sim N_{d_1}(\mu_1, \Sigma_{11})$ and $X_2 \sim N_{d_2}(\mu_2, \Sigma_{22})$ are both Gaussian multivariate random vectors. The associated copula is the so-called Gaussian copula with parameter R, denoted $C^G_{d_1+d_2,R}$, where R is the correlation matrix defined corresponding to Σ.

Further set $\Omega = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, and $B = \Sigma_{21}\Sigma_{11}^{-1}$. Then, it is well-known that the conditional distribution of X_2 given $X_1 = x_1$ is Gaussian, with mean $\mu_2 + B(x_1 - \mu_1)$ and covariance matrix Ω. Note also that $|\Sigma| = |\Sigma_{11}||\Omega|$.

Let $\gamma = \left\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_{d_1})\right\}$ and $\zeta = \left\{\Phi^{-1}(v_1), \ldots, \Phi^{-1}(v_{d_2})\right\}$, where Φ is the distribution function of a standard Gaussian variable. It follows that (9) becomes

$$\begin{aligned} c_{V|U}(v; u) &= \frac{\frac{1}{(2\pi)^{d_2}} \times \frac{1}{\sqrt{|\tilde{\Omega}|}} \exp\left\{-\frac{1}{2}(\zeta - B\gamma)^\top \tilde{\Omega}^{-1}(\zeta - B\gamma)\right\}}{\prod_{i=1}^{d_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\zeta_i^2\right)} \\ &= \frac{1}{\sqrt{|\tilde{\Omega}|}} \exp\left\{-\frac{1}{2}(\zeta - B\gamma)^\top \tilde{\Omega}^{-1}(\zeta - B\gamma) + \frac{1}{2}\zeta^\top\zeta\right\}, \end{aligned} \tag{10}$$

where $(\tilde{\Omega})_{ij} = \frac{\Omega_{ij}}{\sqrt{\Omega_{ii}\Omega_{jj}}}$, $i, j \in \{1, \ldots, d_2\}$, is the correlation matrix associated with Ω.

Since the conditional Gaussian distribution is Gaussian, the conditional copula is Gaussian as well. These results are summarized in the following lemma.

Lemma 3 *Let $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ be a correlation matrix and suppose that $(U, V) \sim C_{d_1+d_2,R}$. Then the conditional copula of V given $U = u$, i.e. the copula associated with the conditional distribution of V given $U = u \in (0, 1)^{d_1}$, is the Gaussian copula with correlation matrix $\tilde{\Omega}$.*

Using the previous calculations, one can propose an algorithm for generating a Markovian time series having a dynamic Gaussian copula $C^G_{2d,R}$.

Algorithm 1 (Dynamic Gaussian copula) To generate a times series $\{U_t\}_{t=0}^n$ with stationary distribution $C_{d,R_{11}}$ and joint distribution of $(U_{t-1}, U_t) \sim C^G_{2d,R}$, with $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{11} \end{bmatrix}$, do the following steps:

- Generate $X_0 = (X_{01}, \ldots, X_{0d}) \sim N_d(0, R_{11})$ and set $U_0 = \{\Phi(X_{01}), \ldots, \Phi(X_{0d})\}$;
- For t = 1 to n,
  - Generate $V_t \sim N_d(0, \Omega)$;
  - Set $X_t = V_t + B \times X_{t-1}$ and $U_t = (\Phi(X_{t1}), \ldots, \Phi(X_{td}))$.

2.3.3 Dynamic meta-elliptic copulas

Suppose now that $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{E}(g, 0, R)$, with correlation matrix $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$.

As before set $\Omega = R_{22} - R_{21} R_{11}^{-1} R_{12}$, and $B = R_{21} R_{11}^{-1}$. Then, $|R| = |R_{11}||\Omega|$, and

$$x^\top R^{-1} x = (x_2 - Bx_1)^\top \Omega^{-1} (x_2 - Bx_1) + x_1^\top R_{11}^{-1} x_1,$$

so $X_1 \sim \mathcal{E}(g_1, 0, R_{11})$, and the conditional distribution of X_2 given $X_1 = x_1$ is $\mathcal{E}(g_2, Bx_1, \Omega)$ where g_1 is given by (3) and

$$g_2(r) = g(r + x_1^\top R_{11}^{-1} x_1)/g_1(x_1^\top R_{11}^{-1} x_1). \tag{11}$$

Lemma 4 *Let* $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ *be a correlation matrix and suppose* $C_{g,R}$ *is the copula associated with the elliptic distribution* $\mathcal{E}(g, 0, R)$*. Then the conditional copula of* V *given* $U = u$ *is* $C_{g_2,\tilde{\Omega}}$*, where* $\tilde{\Omega}$ *is the correlation matrix built from* $\Omega = R_{11} - R_{12} R_{22}^{-1} R_{21}$*, and* g_2 *is defined by* (11).

For example, if g is the generator of the d-dimensional Pearson type VII with parameters (α, ν), then g_1 is the generator of the d_1-dimensional Pearson type VII with parameters (α, ν), and g_2 is the generator of the d_2-dimensional Pearson type VII with parameters (α', ν'), with $\alpha' = \alpha + \frac{d_1}{2}$ and $\nu' = \nu + x_1^\top R_{11}^{-1} x_1$. In particular, if the joint distribution of (X_1, X_2) is Student with parameters (ν, R), then the conditional distribution of X_2, given $X_1 = x_1$, is $\mathcal{E}\left(g, Bx_1, \left(\frac{\nu + x_1^\top R_{11}^{-1} x_1}{\nu + d_1}\right)\Omega\right)$ where g is the generator of a Student with $\nu + d_1$ degrees of freedom. It follows that the conditional copula of a Student with parameters (ν, R) is a Student with parameters $(\nu + d_1, \tilde{\Omega})$.

Using Lemma 4, we propose an algorithm for generating Markovian time series having a dynamic meta-elliptic copula. To that end, suppose that F_1 is the distribution function associated with density (5).

Algorithm 2 (Dynamic meta-elliptic copula) Let g be the generator of a $2d$-dimensional elliptic distribution. To generate a times series $\{U_t\}_{t=0}^n$ with stationary distribution C_{g_1,R_1} and joint distribution of $(U_{t-1}, U_t) \sim C_{g,R}$, with $R = \begin{bmatrix} R_1 & R_{12} \\ R_{21} & R_1 \end{bmatrix}$, do the following steps:

- Generate $X_0 = (X_{01}, \ldots, X_{0d}) \sim \mathcal{E}(g, 0, R_1)$ and set $U_0 = \{F_1(X_{01}), \ldots, F_1(X_{0d})\}$;
- For t = 1 to n,
  - Generate $V_t \sim \mathcal{E}(g_2, 0, \Omega)$;
  - Set $X_t = V_t + B \times X_{t-1}$ and $U_t = (F(X_{t1}), \ldots, F(X_{td}))$.

2.3.4 Dynamic Archimedean copulas

Suppose C is a $(d_1 + d_2)$-dimensional Archimedean copula with generator ϕ. Set

$$h_{d_1}(s) = (-1)^{d_1} \frac{d^{d_1}}{ds^{d_1}} \phi^{-1}(s).$$

Then, by hypothesis, for any $j = 0, \ldots, d_2$, $(-1)^j \frac{d^j}{dt^j} h_{d_1} \geq 0$.

Set $A_u = C(u, \mathbf{1})$. Then the conditional distribution function H_u of V given $U = u$, is

$$
\begin{aligned}
H_u(v) &= P(V \leq v | U = u) = \frac{(-1)^{d_1} \left. \frac{d^{d_1}}{ds^{d_1}} \phi^{-1}(s) \right|_{s=\phi(A)+\sum_{j=1}^{d_2} \phi(v_j)}}{(-1)^{d_1} \left. \frac{d^{d_1}}{ds^{d_1}} \phi^{-1}(s) \right|_{s=\phi(A_u)}} \\
&= \frac{h_{d_1} \left\{ \phi(A_u) + \sum_{j=1}^{d_2} \phi(v_j) \right\}}{h_{d_1} \left\{ \phi(A_u) \right\}}.
\end{aligned}
$$

It follows that the margins of H_u are given, for any $t \in (0,1]$, by

$$
F_{j,u}(t) = P(V_j \leq t | U = u) = \frac{h_{d_1} \left\{ \phi(A_u) + \phi(t) \right\}}{h_{d_1} \left\{ \phi(A_u) \right\}}
$$

and the quantile function is

$$
Q_u(s) = \phi^{-1} \left[h_{d_1}^{-1} \left[s h_{d_1} \{ \phi(A_u) \} \right] - \phi(A_u) \right].
$$

Hence, the copula associated with the conditional distribution of V given $U = u$ is

$$
\begin{aligned}
\mathcal{C}_u(v) &= H_u \left\{ Q_u(v_1), \ldots, Q_u(v_d) \right\} \\
&= \frac{h_d \left[-(d-1)\phi(A_u) + \sum_{i=1}^{d} h_d^{-1} \left[v_j h_d \{ \phi(A_u) \} \right] \right]}{h_d \{ \phi(A_u) \}} \\
&= \psi_u^{-1} \left\{ \sum_{j=1}^{d} \psi_u(v_j) \right\},
\end{aligned}
$$

where

$$
\psi_u^{-1}(s) = \frac{h_{d_1} \{ s + \phi(A_u) \}}{h_{d_1} \{ \phi(A_u) \}} \quad \text{and } \psi_u(t) = h_{d_1}^{-1} \left[t h_{d_1} \{ \phi(A_u) \} \right] - \phi(A_u).
$$

It is easy to check that ψ_u is a generator so $\mathcal{C}_u$ is an Archimedean copula.

Therefore, we have proven that the conditional copula of an Archimedean copula is also Archimedean.

Lemma 5 *If $(U, V) \sim C_{d_1+d_2,\phi}$, then the conditional copula, i.e. the copula associated with the conditional distribution of V given $U = u \in (0,1)^{d_1}$ is Archimedean with generator*

$$
\psi_u(t) = h_{d_1}^{-1} \left[t h_{d_1} \{ \phi(A_u) \} \right] - \phi(A_u), \qquad t \in (0,1], \tag{12}
$$

where $A_u = C_{d_1+d_2,\phi}(u, \mathbf{1}) = C_{d_1,\phi}(u)$ and $h_{d_1}(s) = (-1)^{d_1} \frac{d^{d_1}}{ds^{d_1}} \phi^{-1}(s)$.

Remark 6 For most interesting families, it is quite easy to evaluate h_d. For the Frank and Gumbel-Hougaard families, see Appendices A and B. As for the Clayton family with generator $\phi_\theta(t) = \frac{t^{-\theta}-1}{\theta}$, one gets

$$
h_d(s) = (1 + s\theta)^{-d-1/\theta} \prod_{j=0}^{d-1} (1 + j\theta), \qquad s \geq 0. \tag{13}
$$

Using the previous calculations and Lemma 5, we are able to propose a general algorithm to simulate a Markovian time series with dynamic copula $C_{2d,\phi}$.

Algorithm 3 (Dynamic Archimedean copula) To generate a Markov chain $\{U_t\}_{t=0}^{n}$ with stationary distribution $C_{d,\phi}$ and joint distribution of $(U_{t-1}, U_t) \sim C_{2d,\phi}$, do the following:

- Generate $U_0 \sim C_{d,\phi}$;
- For $t = 1$ to n,
  - Set $A_{U_{t-1}} = C_{2d,\phi}(U_{t-1}, \mathbf{1}) = C_{d,\phi}(U_{t-1})$;
  - Generate $V_t \sim C_{d,\psi_{U_{t-1}}}$, where ψ_u is defined by (12);
  - Set $U_t = (U_{t,1}, \ldots, U_{t,d})$, where $U_{t,j} = Q_{U_{t-1}}(V_{t,j}), \quad j = 1, \ldots, d$.

Example 7 (Dynamic Clayton copula) For the Clayton family with generator $\phi_\theta(t) = \frac{t^{-\theta}-1}{\theta}$, it is easy to check that

$$F_{j,u}(t) = P(V_j \le t | U = u) = A_u^{-(1+d_1\theta)} \left(A_u^{-\theta} + v_j^{-\theta} - 1 \right)^{-1/\theta - d_1},$$

where $A_u = C(u, \mathbf{1}) = \left(\sum_{i=1}^{d_1} u_i^{-\theta} - d_1 + 1 \right)^{-1/\theta}$. Setting $a = \theta/(1 + d_1\theta)$, one obtains, for all $t \in (0, 1)$,

$$Q_u(t) = F_{j,u}^{-1}(t) = \left(1 - A_u^{-\theta} + t^{-a} A_u^{-\theta} \right)^{-1/\theta}.$$

Moreover $\psi_u^{-1}(s) = \left(1 + \frac{s}{1+\phi(A_u)} \right)^{-1/a}$ for any $u \in (0, 1]$ and $s \ge 0$, so the conditional copula is the Clayton copula with parameter $a = \theta/(1 + d_1\theta)$. To our knowledge, this is the only Archimedean family having that property.

Remark 8 (Special Archimedean families) Suppose that ϕ^{-1} is completely monotone, i.e., ϕ^{-1} is the Laplace transform of a non-negative random variable ξ with law μ. These include the Clayton, Frank, and Gumbel-Hougaard families. In that case, based on Lemma 5,

$$h_d(s) = E\left(\xi^d e^{-s\xi} \right), \quad s > 0,$$

so since $\phi(A_u) > 0$ for any $0 < A_u < 1$, and it turns out that

$$\psi_u^{-1}(s) = \frac{h_s\{s + \phi(A_u)\}}{h_s\{\phi(A_u)\}} = \frac{E\left[\xi^d e^{-\{s+\phi(A_u)\}\xi} \right]}{E\left\{ \xi^d e^{-\phi(A_u)\xi} \right\}}$$

is the Laplace transform of the law ν_u with density with respect to μ proportional to the bounded function $x^d e^{-\phi(A_u)x}$. Hence variables having law ν_u can be easily simulated by the rejection method if one can generate $\xi \sim \mu$.

2.4 p-Dynamic copulas

In many applications, a process X_t can depend not just on the previous observation X_{t-1}, but also of a given finite number, leading to p-Markov processes. Recall that X_t is a p-Markov process if $Y_t = (X_{t-p+1}, \ldots, X_t)$ is a Markov process. Of course, a 1-Markov process is a Markov process.

The notion of dynamic copula can be easily extended to cover the case of p-Markov processes. One says that C is a p-dynamic copula if C is the copula associated with $(Y_{t-1}, X_t) = (X_{t-p}, \ldots, X_t)$, whenever X is a p-Markov process. It follows that for any $v \in [0, 1]^{pd}$, one has $D(v) = C(v, \mathbf{1}) = C(\mathbf{1}, v)$.

Using the results in Section 2.3, it is then easy to check that if C is meta-elliptic, then the conditional copula of U_p given $V_{p-1} = (U_1, \ldots, U_{p-1})$ is also meta-elliptic. Similarly, if C is Archimedean, then the conditional copula of U_p given V_{p-1} is Archimedean. Furthermore, the results of the next section, stated for dynamic copulas, extends to p-dynamic copulas.

3 Estimation and goodness-of-fit

Start with a time series of d-dimensional vectors $X_t = (X_{t,1}, \ldots, X_{t,d})$, $t = 1, \ldots, n$, where C_θ is the copula associated with (X_{t-1}, X_t). The goal is to estimate θ belonging to a subset $\mathcal{O}$ of $\mathbb{R}^s$, without any prior knowledge of the margins.

First, since the margins are unknown, replace $X_{t,j}$ by its rank $R_{t,j}$ among $X_{1,j}, \ldots, X_{n,j}$. Next, define the sequence $\hat{U}_t = \frac{R_t}{n+1}$ of normalized ranks (pseudo-observations). In doing so, these pseudo-observations are close to be uniformly distributed over $[0,1]$, when n is large enough.

Set $D_\theta(u) = C(u, \mathbf{1})$ and recall that

$$C_\theta(\mathbf{1}, v) = D_\theta(v), \qquad v \in [0,1]^d. \tag{14}$$

3.1 Estimation by the pseudo maximum likelihood method

An obvious extension of the pseudo maximum likelihood method (Genest et al., 1995) to the Markovian case consists is maximizing

$$\sum_{t=2}^{n} \log \left\{ \frac{c_\theta\left(\hat{U}_{t-1}, \hat{U}_t\right)}{d_\theta\left(\hat{U}_{t-1}\right)} \right\} \tag{15}$$

with respect to θ, where c_θ is the density of C_θ, assumed to be non vanishing on $(0,1)^{2d}$, and d_θ is the density of D_θ. Note that (15) is the logarithm of the conditional density of $U_2, \ldots, U_n$, given U_1, evaluated at $\hat{U}_1, \ldots, \hat{U}_d$.[5]

If c_θ is smooth enough as a function of θ (thrice continuously differentiable) and the sequence U_t is ergodic, then the maximum likelihood estimator obtained by maximizing

$$\sum_{t=2}^{n} \log \left\{ \frac{c_\theta\left(U_{t-1}, U_t\right)}{d_\theta\left(U_{t-1}\right)} \right\} \tag{16}$$

with respect to θ behaves nicely. In fact, $n^{1/2}\left(\theta_n - \theta\right) \rightsquigarrow \Theta \sim N_p\left(0, \mathcal{I}^{-1}\right)$, where

$$\mathcal{I} = \int_{(0,1)^{2d}} \frac{\dot{c}_\theta(u,v)\dot{c}_\theta(u,v)^\top}{c_\theta(u,v)} du dv - \int_{(0,1)^d} \frac{\dot{d}_\theta(u)\dot{d}_\theta(u)^\top}{d_\theta(u)} du,$$

where $\dot{f}$ denotes the gradient with respect to θ.

That follows from the fact that

$$\Delta M_t = G_\theta(U_{t-1}, U_t) = \frac{\dot{c}_\theta(U_{t-1}, U_t)}{c_\theta(U_{t-1}, U_t)} - \frac{\dot{d}_\theta(U_{t-1})}{d_\theta(U_{t-1})}$$

form a martingale difference sequence, i.e., $E(\Delta M_t | U_{t-1}, \ldots, U_1) = 0$, so the central limit theorem for martingales (Durrett, 1996) applies to yield $n^{-1/2}\sum_{t=2}^{n} \Delta M_t \rightsquigarrow N_p(0, \mathcal{I})$, if the chain is ergodic, which is true because $c_\theta(u,v) > 0$ for all $u, v \in (0,1)^d$. See Robert and Casella (2004) for more details.

Mimicking the proof in Genest et al. (1995) or using the pseudo-observations methodology developed in Ghoudi and Rémillard (2004), one can prove that

$$n^{1/2}\left(\hat{\theta}_n - \theta\right) \rightsquigarrow \Theta + \tilde{\Theta} \sim N_p\left(0, J\right),$$

[5] In the case of a p-dynamic copula, one needs to maximize $\sum_{t=p+1}^{n} \log \left\{ \frac{c_\theta(\hat{V}_{t-1}, \hat{U}_t)}{d_\theta(\hat{V}_{t-1})} \right\}$, where $\hat{V}_t = \left(\hat{U}_{t-p+1}, \ldots, \hat{U}_t\right)$, and $d_\theta(v)$ is the density of the copula $C_\theta(v, \mathbf{1})$, $v \in (0,1)^{pd}$.

for some covariance matric J, if $G_\theta(u,v)$ is continuously differentiable with respect to (u,v) and if $(\mathbb{F}_{1,n},\ldots,\mathbb{F}_{d,n}) \rightsquigarrow (\mathbb{F}_1,\ldots,\mathbb{F}_d)$ in the Skorohod space $\mathcal{D}([0,1])^{\otimes d}$, where

$$\mathbb{F}_{j,n}(u_j) = n^{1/2}\left(F_{j,n}(u_j) - u_j\right), \qquad F_{j,n}(u_j) = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}(U_{j,t} \le u_j),$$

for all $j = 1,\ldots,d$, and if $\tilde{\Theta}$ can be expressed as a linear function of $\mathbb{F}_1,\ldots,\mathbb{F}_d$. In fact,

$$\begin{aligned}\tilde{\Theta} &= \mathcal{I}^{-1}\int \nabla_u G_\theta(u,v)\{\mathbb{F}_1(u_1),\ldots,\mathbb{F}_d(u_d)\}^\top dC_\theta(u,v)\\ &\quad + \mathcal{I}^{-1}\int \nabla_v G_\theta(u,v)\{\mathbb{F}_1(v_1),\ldots,\mathbb{F}_d(v_d)\}^\top dC_\theta(u,v).\end{aligned}$$

See Ghoudi and Rémillard (2004) for details.

The proof of the convergence of $(\mathbb{F}_{1,n},\ldots,\mathbb{F}_{d,n}) \rightsquigarrow (\mathbb{F}_1,\ldots,\mathbb{F}_d)$ in $\mathcal{D}([0,1])^{\otimes d}$ is done in Appendix C.

Example 9 (Dynamic Gaussian Copula) The pseudo maximum likelihood estimator of the density in (10) is written as follows: Let $\hat{\zeta}_{j,t} = \Phi^{-1}\left(\hat{U}_{j,t}\right)$, $j = 1,\ldots,d$. Then

$$L(\theta) = \prod_{t=2}^{n} \frac{1}{\sqrt{|\Omega|}} \exp\left\{-\frac{1}{2}\left(\hat{\zeta}_t - B\hat{\zeta}_{t-1}\right)^\top \Omega^{-1}\left(\hat{\zeta}_t - B\hat{\zeta}_{t-1}\right) + \frac{1}{2}\hat{\zeta}_t^\top\hat{\zeta}_t\right\}, \tag{17}$$

where $\theta = (B,\Omega)$; recall that $B = R_{21}R_{11}^{-1}$ and $\Omega = R_{11} - R_{21}R_{11}^{-1}R_{12} = R_{11} - BR_{11}B^\top$.
We need to find $\hat{B}$ and $\hat{\Omega}$ that maximize equation (17), which in turn is equivalent to minimize

$$-2l(\theta) = -2\log L(\theta) = n\ln|\Omega| + \sum_{t=2}^{n}\left((\zeta_t - B\zeta_{t-1})^\top \Omega^{-1}(\zeta_t - B\zeta_{t-1})\right).$$

For the theory of multivariate regression, the solution is known to be

$$\hat{B} = \left(\sum_{t=2}^{n}\hat{\zeta}_t\hat{\zeta}_{t-1}^\top\right)\left(\sum_{t=2}^{n}\hat{\zeta}_{t-1}\hat{\zeta}_{t-1}^\top\right)^{-1}, \quad \hat{\Omega} = \frac{1}{n-1}\sum_{t=2}^{n}\left(\hat{\zeta}_t - \hat{B}\hat{\zeta}_{t-1}\right)\left(\hat{\zeta}_t - \hat{B}\hat{\zeta}_{t-1}\right)^\top.$$

It is easy to show that these estimators are consistent. However, since one needs to estimate R_{11}, which is a correlation matrix, one can set $\xi_t = \hat{\Delta}\hat{\zeta}_t$, where $\hat{\Delta}$ is the diagonal matrix so that $\hat{R}_{11} = \frac{1}{n-1}\sum_{t=1}^{n-1}\xi_t\xi_t^\top$ is a correlation matrix. In fact it is the so-called van der Waerden estimator of R_{11}. Then simply set

$$\hat{R}_{21} = \frac{1}{n-1}\sum_{t=2}^{n}\xi_t\xi_{t-1}^\top, \quad \hat{B} = \hat{R}_{21}\left(\hat{R}_{11}\right)^{-1}, \qquad \hat{\Omega} = \hat{R}_{11} - \hat{B}\hat{R}_{11}\hat{B}^\top.$$

Note that these estimates can be obtained by using the van der Waerden estimator of R through the pseudo-observations $\begin{pmatrix}\hat{U}_{t-1}\\ \hat{U}_t\end{pmatrix}$, for $t = 2,\ldots,n$.

3.1.1 Dynamic meta-elliptic copulas

Due to the large number of constraints in the parameters (R_{11},B,Ω) when writing the likelihood function corresponding to a dynamic meta-elliptic copula, e.g., for the Pearson type VII associated copula, it is preferable to use a mix of moment matching and pseudo maximum likelihood. For meta-elliptic copulas, the obvious choice is Kendall's tau because of its relation with the correlation matrix R. More precisely, if $\tau = \begin{bmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{11}\end{bmatrix}$ is the matrix of Kendall's tau associated with the random vector $\begin{pmatrix}U_{t-1}\\ U_t\end{pmatrix}$, then $R = sin(\pi\tau/2)$, where the transformation is applied pointwise.

Let $\hat{\tau} = \begin{bmatrix} \hat{\tau}_{11} & \hat{\tau}_{12} \\ \hat{\tau}_{21} & \hat{\tau}_{11} \end{bmatrix}$ be the empirical Kendall's tau matrix calculated from the pseudo-sample $\begin{pmatrix} \hat{U}_{t-1} \\ \hat{U}_t \end{pmatrix}$, for $t = 2, \ldots, n$. Then $\hat{\tau}$ converges in probability to the theoretical $\tau = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{11} \end{bmatrix}$ associated with the random vector $\begin{pmatrix} U_{t-1} \\ U_t \end{pmatrix}$. In fact, using the methodology developed in Ghoudi and Rémillard (2004) together with the convergence of the process $\mathbb{C}_n$ defined in Appendix C, one can show that $n^{1/2}(\hat{\tau} - \tau)$ converges in law to a centered Gaussian random variable.

It then follows that $\sin(\pi\hat{\tau}/2)$ is a consistent estimator of R. However the latter is not necessarily positive definite, so maybe it has to be transformed into non-degenerate correlation matrix $\hat{R}$. Then, simply set $\hat{B} = \hat{R}_{21}\left(\hat{R}_{11}\right)^{-1}$ and $\hat{\Omega} = \hat{R}_{11} - \hat{B}\hat{R}_{11}\hat{B}^\top$. The remaining parameters can then be estimated using pseudo maximum likelihood. This approach is particularly well-suited for the dynamic Student copula.

3.1.2 Dynamic Archimedean copulas

Suppose $C(u, v) = C_{2d,\phi}(u, v)$ and recall that $h_k(s) = (-1)^k \frac{d^k}{ds^k}\phi^{-1}(s)$. It follows that

$$c_{V|U}(v; u) = \frac{h_{2d}\left\{\phi \circ C_{2d,\phi}(u, v)\right\}}{h_d\left\{\phi \circ C_{d,\phi}(u)\right\}} \left|\prod_{j=1}^d \phi'(v_j)\right|.$$

Example 10 (Dynamic Clayton copula) In the Clayton case, it is easy to check that one simply needs to maximize

$$\begin{aligned} l(\theta) &= \sum_{t=2}^n \left\{\sum_{j=d}^{2d-1} \ln(1 + j\theta) - \sum_{k=1}^d (\theta + 1)\ln(u_{k,t})\right\} \\ &\quad - \left(\frac{1}{\theta} + 2d\sum_{t=2}^n\right) \ln\left\{\sum_{k=1}^d \left(u_{k,t-1}^{-\theta} + u_{k,t}^{-\theta}\right) - 2d + 1\right\} \\ &\quad + \left(\frac{1}{\theta} + d\right)\sum_{t=2}^n \ln\left(\sum_{k=1}^d u_{k,t-1}^{-\theta} - d + 1\right). \end{aligned}$$

3.2 Goodness-of-fit

As mentioned previously, there exists almost no formal test of goodness-of-fit for copulas in a serially dependent context. Even for serially independent observations, the literature is quite recent, one of the first formal test being Genest et al. (2006), using the parametric bootstrap methodology to compute approximate p-values. It was followed by several other articles, including Kole et al. (2007) in the Finance literature. Although they also proposed goodness-of-fit tests using the parametric bootstrap technique, they provide no evidence as to the validity of their methodology, which is far from obvious (see Genest and Rémillard (2008) for more detail). Furthermore, the tests proposed in Kole et al. (2007) have been shown to be either incorrect (Anderson-Darling type tests) or not powerful (Kolmogorov-Smirnov type tests). For an exhaustive review of tests of goodness-of-fit for copula models in the context of serially independent time series, see Genest et al. (2009).

Based on recent results in Genest et al. (2009), we propose to use goodness-of-fit tests constructed from the Rosenblatt's transform (Rosenblatt, 1952), which were almost always the more powerful. Recall that the Rosenblatt's mapping of a d-dimensional copula C is the mapping $\mathcal{R}$ from $(0,1)^d \to (0,1)^d$ so that for $u = (u_1, \ldots, u_d) \mapsto \mathcal{R}(u) = (e_1, \ldots, e_d)$ with $e_1 = u_1$ and

$$e_i = \frac{\partial^{i-1} C(u_1, \ldots, u_i, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1}, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}}, \tag{18}$$

$i = 2, \ldots, d$.

The main property of Rosenblatt's transform is that $U \sim C$ if and only if $E = \mathcal{R}(U) \sim C_\perp$, i.e., E is uniformly distributed on $[0,1]^d$. It also follows that by inverting the mapping, one can generate $U \sim C$ viz. $U = \mathcal{R}^{-1}(E)$, by generating E according to the d-dimensional independence copula.

In a univariate time series context, the use of the Rosenblatt's transform was suggested by Diebold et al. (1998). However the authors never took into account the fact that the parameters were estimated, leading to a defective test of goodness-of-fit. A corrected version based on parametric bootstrap was proposed in Rémillard and Papageorgiou (2008) for a multivariate regime-switching Gaussian model.

In the present context, recall that (U_t) is a stationary Markov process so that $(U_{t-1}, U_t) \sim C$.

The goal here is to test the null hypothesis $\mathcal{H}_0$ that C belongs to a given parametric family, more precisely that $C = C_\theta$ for some $\theta \in \mathcal{O}$. Denote by $\mathcal{R}_\theta(u,v) = \left\{\mathcal{R}_\theta^{(1)}(u), \mathcal{R}_\theta^{(2)}(u,v)\right\}$ the Rosenblatt's transform associated with the $2d$-dimensional copula C_θ, where $\mathcal{R}_\theta^{(1)}$ is the Rosenblatt's transform associated with the d-dimensional copula D_θ, with $D_\theta(u) = C_\theta(u, \mathbf{1})$ for all $u \in [0,1]^d$.

It then follows that under the null hypothesis $\mathcal{H}_0$, the d-dimensional time observations $E_1 = \mathcal{R}_\theta^{(1)}(U_1)$ and $E_t = \mathcal{R}_\theta^{(2)}(U_{t-1}, U_t)$, $t \geq 2$ are independent and uniformly distributed over $[0,1]^d$.

Because θ is unknown, and also because the U_t are not observable, θ must be estimated and U_t have to be replaced by the pseudo-observation $\hat{U}_t$. Suppose that $\hat{\theta}$ is a "regular" estimator of θ based on the pseudo sample $\hat{U}_1, \ldots, \hat{U}_n$, and set $\hat{E}_1 = \mathcal{R}_{\hat{\theta}}^{(1)}\left(\hat{U}_1\right)$ and $\hat{E}_t = \mathcal{R}_{\hat{\theta}}^{(2)}\left(\hat{U}_{t-1}, \hat{U}_t\right)$, $t \geq 2$.

Under the null hypothesis $\mathcal{H}_0$, the empirical distribution function

$$G_n(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left(\hat{E}_i \leq u\right), \quad u \in [0,1]^d,$$

should be "close" to $C_\perp$, the d-dimensional independence copula. Based on the results in Genest et al. (2009), to test $\mathcal{H}_0$, one proposes to use the Cramér-von Mises type statistic

$$\begin{aligned} S_n &= T(\mathbb{G}_n) = \int_{[0,1]^d} \mathbb{G}_n^2(u) du = \int_{[0,1]^d} \{G_n(u) - C_\perp(u)\}^2 du \qquad (19) \\ &= \frac{n}{3^d} - \frac{1}{2^{d-1}} \sum_{i=1}^{n} \prod_{k=1}^{d} \left(1 - \hat{E}_{ik}^2\right) + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{d} \left\{1 - max\left(\hat{E}_{ik}, \hat{E}_{jk}\right)\right\}, \end{aligned}$$

where $\mathbb{G}_n = n^{1/2}(G_n - C_\perp)$.

Using the tools described in Ghoudi and Rémillard (2004) together with the convergence results of the empirical processes described in the previous section, one can determine that $\mathbb{G}_n$ converges to a (complicated) continuous centered Gaussian processes $\mathbb{G}$. That leads to the weak convergence of $S_n = T(\mathbb{G}_n)$ to $T(\mathbb{G})$, T being a continuous functional on $\mathcal{D}\left([0,1]^d\right)$.

Regarding goodness-of-fit, the results of Genest and Rémillard (2008) can be adapted to a Markovian setting, showing that P-values for tests of goodness-of-fit based on the empirical copula or the Rosenblatt's transform can be estimated by Monte Carlo methods. The proof of the validity of that approach is given in the companion paper Rémillard (2010).

3.2.1 Dynamic meta-elliptic copulas

Because the conditional distributions of an elliptic vector are elliptic, to compute the Rosenblatt's transform $\mathcal{R}_C$ of the associated copula C, it if preferable to compute the Rosenblatt's transform $\mathcal{R}_H$ of the underlying joint distribution function H and then transform it by inverting the marginal distributions. More precisely, if C and H related through

$$H(x) = C\left\{F_1(x_1, \ldots, F_d(x_d)\right\},$$

where $F_1, \ldots, F_d$ are the marginal distributions, then

$$\mathcal{R}_H(x) = \mathcal{R}_C\left\{F_1(x_1, \ldots, F_d(x_d)\right\}. \tag{20}$$

This method of computing the Rosenblatt's transform is particularly well-suited for the dynamic Gaussian and Student copulas.

3.2.2 Dynamic Archimedean copulas

From (18), it follows easily that if $C = C_{2d,\phi}$, then for all $j = 1, \ldots, d$, and all $u, v \in (0,1)^d$,

$$\mathcal{R}_j^{(1)}(u) = \frac{h_{j-1}\left\{\sum_{k=1}^{j}\phi(u_k)\right\}}{h_{j-1}\left\{\sum_{k=1}^{j-1}\phi(u_k)\right\}}, \quad \mathcal{R}_j^{(2)}(u,v) = \frac{h_{d+j-1}\left\{\phi(A_u) + \sum_{k=1}^{j}\phi(v_k)\right\}}{h_{d+j-1}\left\{\phi(A_u) + \sum_{k=1}^{j-1}\phi(v_k)\right\}},$$

with $\phi(A_u) = \phi \circ D(u) = \phi \circ C(u, \mathbf{1}) = \sum_{k=1}^{d}\phi(u_k)$.

Example 11 (Rosenblatt's transform for the dynamic Clayton copula) In that case, using the formula (13), one obtains, for $j = 1, \ldots, d$,

$$\mathcal{R}_j^{(1)}(u) = \frac{\left(1 - j + \sum_{k=1}^{j} u_k^{-1/\theta}\right)^{-1/\theta-(j-1)}}{\left(2 - j + \sum_{k=1}^{j-1} u_k^{-1/\theta}\right)^{-1/\theta-(j-1)}}$$

and

$$\mathcal{R}_j^{(2)}(u,v) = \frac{\left(1 - d - j + \sum_{k=1}^{d} u_k^{-1/\theta} + \sum_{k=1}^{j} v_k^{-1/\theta}\right)^{-1/\theta-(d+j-1)}}{\left(2 - d - j + \sum_{k=1}^{d} u_k^{-1/\theta} + \sum_{k=1}^{j-1} v_k^{-1/\theta}\right)^{-1/\theta-(d+j-1)}}.$$

3.3 Ignoring serial dependence

What would then be the consequences of ignoring serial dependance? Although most of the resulting estimators would still converge, they might not be regular in the sense of Genest and Rémillard (2008). As a result, tests of goodness-of-fit would not be applicable.

An important first step in the inference procedure would be to test for serial dependence, using e.g., Genest and Rémillard (2004). That methodology, together with tests of goodness-of-fit proposed by Genest and Rémillard (2008), has been recently generalized and implemented by Yan and Kojadinovic (2009) for the free statistical package R.[6]

4 Example of application

We now attempt to address the reader's question concerning the dependence between the returns of the Can/US exchange rate and oil prices (NYMEX Oil Futures) by examining the daily returns data of the two variables over the last 10 years. We investigate three overlapping periods of 2, 5 and 10 years respective. These periods correspond to data from 2008 and 2009 (493 returns), 2005–2009 (1225 returns), and 2000–2009 (2440 returns).

[6] See http://www.r-project.org

The returns for both series over the entire 10-year period are plotted in Figure 1.

The first step, as previously suggested, is to test for the presence of serial dependence in the univariate time series (for the three periods), using the statistics I_n and $I_n^\star$ defined in Genest et al. (2007). For lags up to $p = 6$, the tests based on I_n almost never reject the null hypothesis of independence (at the 5% level), while all tests based on $I_n^\star$ reject the same hypothesis.

According to Genest et al. (2007), it seems that both series might exhibit time-dependent conditional variance, e.g.,like in GARCH models. Note that usual tests of independence based on the Ljung-Box statistics did not reject the null hypothesis of independence for the exchange rate returns for any of the three periods, while rejecting the null hypothesis each time for the oil futures returns.

Having identified serial dependence in the time-series of both variables, the next step is to attempt to fit a dynamic copula model. We choose to test the adequacy to 4 families: Clayton, Frank, Gaussian and Student.

The small p-values (calculated with N=100 iterations) for the dynamic Clayton and Frank indicate that both copula are rejected for every time period, t. The corresponding results for the dynamic Gaussian and Student copulas appear in Table 3. First, for each period, the dynamic Student copula systematically exhibits the largest p-value, much larger than those of the dynamic Gaussian copula, which is rejected at the 5% level for the 10-year period.

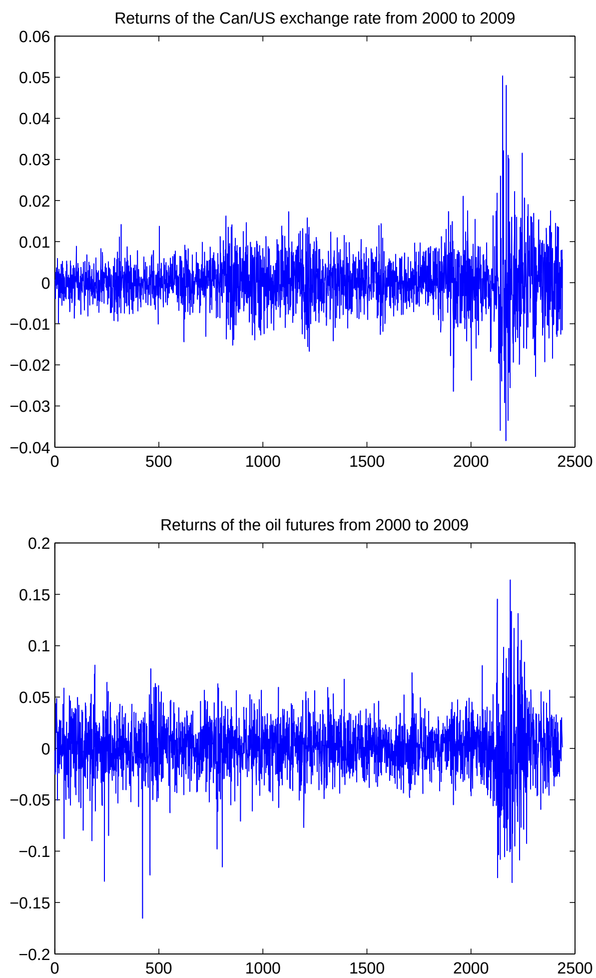

Figure 1: Plot of the returns for both series from 2000 to 2009.

Table 3: Results of the estimation and goodness-of-fit for the dynamic Gaussian and Student copulas, using $N = 100$ iterations.

Period	Gaussian	Student
2008-2009	$\hat{\rho} = .435$, p-value = 12%	$\hat{\rho} = .444$, $\hat{\nu} = 3.51$, p-value = 71%
2005-2009	$\hat{\rho} = .350$, p-value = 53%	$\hat{\rho} = .345$, $\hat{\nu} = 5.60$, p-value = 78%
2000-2009	$\hat{\rho} = .236$, p-value = 1%	$\hat{\rho} = .220$, $\hat{\nu} = 16.7$, p-value = 58%

Table 4: Results of the estimation and goodness-of-fit for the Student copulas, for the non-overlapping periods, using $N = 100$ iterations.

Period	Student
2005-2007	$\hat{\rho} = .228$, $\hat{\nu} = 39.60$, p-value = 31%
2000-2004	$\hat{\rho} = .086$, $\hat{\nu} = \infty$, p-value = 37%

Therefore the best model, for each period, is the dynamic Student model and it allows us to conclude that there is positive dependence between the returns of the two series. However, the strength seems to increase as the length of the period decreases. That may be due to a lack of stationary for these periods, meaning that the dependence changed between 2000 and 2005 and between 2005 and 2007. Figure 1 supports such this hypothesis, at least for the last two years.

To verify the hypothesis of three different regimes corresponding to the periods I: 2000-2004 (1215 returns), II: 2005-2007 (732 returns) and III: 2008-2009 (493 returns), the same analysis is performed using only the dynamic Student copula. The results are given in Table 4. They confirm the impression that the dependence was different during the three non-overlapping periods, the dependence being much stronger over the last two years. However, the surprising result is that for the first period, from 2000 to the end of 2004, the dependence is best modeled by a dynamic Gaussian copula (corresponding to a Student copula with an infinite number of degrees of freedom). The results clearly indicate that the dependence between the Cad/USD exchange rate and the price of oil has become stronger over time. Furthermore, during periods of economic and financial stress, as witnessed over the last two years of the sample, the dependence becomes even more pronounced.

5 Conclusion

In this paper, we introduce an innovative approach, specifically the dynamic copula, to model the interdependence and serial dependence structure between variables. Contrary to the so-called time-varying copulas approach, we do not have to model individual time series. We discuss the issues related to parameter estimation as well as introduced goodness-of-fit tests for model selection. Finally, we implement the approach to investigate the relationship between the Cad/USD exchange rate and the price of oil.

Appendix

A A family of polynomials related to the Frank copula

Based on the formulas in Barbe et al. (1996), it is easy to check that

$$h_{k,\theta}(s) = (-1)^k \frac{d^k}{ds^k} \log\left\{1 - (1-\theta)e^{-s}\right\} / \log(\theta) = p_k\left(\frac{(1-\theta)e^{-s}}{1-(1-\theta)e^{-s}}\right) / \log(1/\theta), \tag{21}$$

where the sequence of polynomials p_k is defined by $p_1(x) = x$ and

$$p_{k+1}(x) = x(1+x)p_k'(x), \qquad k \geq 1. \tag{22}$$

Thus p_k is a polynomial of degree k. For example, $p_2(x) = x(1+x)$, $p_3(x) = x(1+x)(1+2x)$, $p_4(x) = x(1+x)(1+6x+6x^2)$ and $p_5(x) = x(1+x)(1+2x)(1+12x+12x^2)$.

Since $\log\{1-(1-\theta)e^{-s}\}/\log(\theta)$ is completely monotone, and because of representation (21), it follows that for all $x > 0$, $p_k(x) > 0$.

Furthermore,

$$p_k(x) = \sum_{j=1}^{k} a_{k,j} x^j,$$

and it follows from (22) that for any $k \geq 1$,

$$\left\{ \begin{array}{lcl} a_{k+1,1} & = & a_{k,1}, \\ a_{k+1,k+1} & = & k a_{k,k}, \\ a_{k+1,j} & = & j a_{k,j} + (j-1) a_{k,j-1}, \quad j = 2, \ldots, k. \end{array} \right. \tag{23}$$

Note that (23) proves that all coefficients of the polynomials are non-negative and the leading term is positive, reinforcing the observation that $p_k(x) > 0$ for all $x > 0$.

Note that ξ can be generated by

$$\xi = \left\lfloor 1 + \frac{\log(W_2)}{\log\left(1-\theta^{W_1}\right)} \right\rfloor, \tag{24}$$

where $W_1, W_2 \sim \mathrm{Unif}(0,1)$ are independent and $\lfloor x \rfloor$ stands for the integer part of x.

To generate $U \sim C$, first generate S following (24). Then generate $E_1, \ldots, E_d \sim \mathrm{Exp}(1)$, and set $U_i = \phi^{-1}(E_i/\xi) = \log\left\{1-(1-\theta)e^{-E_i/\xi}\right\}/\log(\theta)$, $1 \leq i \leq d$.

B A family of polynomials related to the Gumbel copula

Based on the formulas in Barbe et al. (1996), it is easy to check that

$$h_{k,\theta}(s) = (-1)^k \frac{d^k}{ds^k} e^{-s^\theta} = \frac{p_{k,\theta}(s^\theta)}{s^k} e^{-s^\theta}, k \geq 0, \tag{25}$$

where the sequence of polynomials $p_{k,\theta}$ is defined by $p_{0,\theta} \equiv 1$ and

$$p_{k+1,\theta}(x) = k p_{k,\theta}(x) + \theta x \{p_{k,\theta}(x) - p'_{k,\theta}(x)\}, \qquad k \geq 0. \tag{26}$$

For example, $p_{1,\theta}(x) = \theta x$, $p_{2,\theta}(x) = \theta(1-\theta)x + \theta^2 x^2$, $p_{3,\theta}(x) = \theta(1-\theta)(2-\theta)x + 3\theta^2(1-\theta)x^2 + \theta^3 x^3$, etc.

Since e^{-s^θ} is completely monotone, and because of representation (25), it follows that for all $x > 0$, $p_{k,\theta}(x) > 0$.

Furthermore,

$$p_{k,\theta}(x) = \sum_{j=1}^{k} a_{k,j,\theta} x^j,$$

and it follows from (26) that for any $k \geq 1$,

$$\left\{ \begin{array}{lcl} a_{k+1,1,\theta} & = & (k-\theta) a_{k,1,\theta}, \\ a_{k+1,k+1,\theta} & = & \theta a_{k,k,\theta}, \\ a_{k+1,j,\theta} & = & (k-j\theta) a_{k,j,\theta} + \theta a_{k,j-1,\theta}, \quad j = 2, \ldots, k. \end{array} \right. \tag{27}$$

Note that (27) proves that all coefficients of the polynomials are non-negative and the leading term is positive, reinforcing the observation that $p_{k,\theta}(x) > 0$ for all $x \in (0,1]$.

C Convergence of the empirical processes

First note that $(\mathbb{F}_{1,n},\ldots,\mathbb{F}_{d,n}) \rightsquigarrow (\mathbb{F}_1,\ldots,\mathbb{F}_d)$ in $\mathcal{D}([0,1])^{\otimes d}$ holds true if $\mathbb{F}_n \rightsquigarrow \mathbb{F}$ in $\mathcal{D}\left([0,1]^d\right)$, where $\mathbb{F}_n = n^{1/2}\left(F_n - D_\theta\right)$ and

$$\mathcal{F}_n(u) = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}(U_t \le u),$$

since $F_{1,n},\ldots,F_{d,n}$ are the marginal distributions of $\mathcal{F}_n$.

By hypothesis (14), the Markov chain U_t is reversible, so one can apply the results in Kipnis and Varadhan (1986) to obtain that $\mathbb{F}_n \rightsquigarrow \mathbb{F}$ in $\mathcal{D}\left([0,1]^d\right)$, where $\mathbb{F}$ is a continuous centered Gaussian process with covariance function

$$\Gamma(u,v) = C_\theta(u,v) - D_\theta(u)D_\theta(v) + 2\sum_{t=1}^{\infty}\left\{P(U_0 \le u, U_t \le v) - D_\theta(u)D_\theta(v)\right\}.$$

Remark 12 As a by-product one gets the convergence of the empirical copula based on the pseudo-observations $\hat{U}_t$. More precisely, if

$$G_n(u) = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\left(\hat{U}_t \le u\right)$$

then $\mathbb{G}_n = n^{1/2}(G_n - D_\theta) \rightsquigarrow \mathbb{G}$ in $D\left([0,1]^d\right)$, where

$$\mathbb{G}(u) = \mathbb{F}(u) - \sum_{j=1}^{d} \partial_{u_j} D_\theta(u)\mathbb{F}_j(u_j),$$

provided $\partial_{u_j} D_\theta(u)$ is continuous on $[0,1]$ for all $j = 1,\ldots,d$. The proof follows closely the one of Lemma 3 in Genest et al. (2007) or the proof in Doukhan et al. (2009).

Based on the convergence of $n^{1/2}(H_n - C_\theta)$ to $\mathbb{H}$, where

$$H_n(u,v) = \frac{1}{n}\sum_{t=2}^{n} \mathbb{I}\left(U_{t-1} \le u, U_t \le v\right),$$

which follows again from the reversibility condition and the convergence result in Kipnis and Varadhan (1986), it is possible to show the convergence of $\mathbb{C}_n = n^{1/2}(C_n - C_\theta) \rightsquigarrow \mathbb{C}$ in $D\left([0,1]^d\right)$, where

$$C_n(u,v) = \frac{1}{n}\sum_{t=2}^{n} \mathbb{I}\left(\hat{U}_{t-1} \le u, \hat{U}_t \le v\right).$$

That time, one has to use the results of Ghoudi and Rémillard (2004) to get the representation of the limiting process $\mathbb{C}$.

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