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E.K. Boukas

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El-Kébir Boukas

GERAD and Mechanical Engineering Department
École Polytechnique de Montréal
P.O. Box 6079, Station Centre-ville
Montréal (Québec) Canada, H3C 3A7
el-kebir.boukas@polymtl.ca

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Abstract

This paper deals with the class of linear systems with random abrupt changes. The stochastic stability and the stochastic stabilization problems of this class of systems are revisited and new conditions are developed in the LMI setting to either check the stochastic stability or to design the state feedback controller that stochastically stabilizes the system under consideration. The corresponding robust problems are also considered. It is shown that all the addressed problems can be solved if the corresponding developed linear matrix inequalities (LMIs) are feasible. Two numerical examples are employed to show the usefulness of the proposed results.

Key Words: Markovian jump systems, Stochastic systems, Systems with random abrupt changes, Continuous-time and Discrete-time linear systems, Linear matrix inequality, Stability, Stabilizability, State feedback.

Résumé

Cet article traite de la classe des systèmes des changements aléatoires brusques. Les problèmes de stochastique stabilité et stochastique stabilisation sont révisités et de nouvelles conditions en forme d'inégalités matricielles sont développées que ce soit pour l'analyse de la stabilité ou le design du correcteur par retour d'état stabilisant le système considéré. Les problèmes de robustesse de ces problèmes sont aussi traités dans cet article. Des exemples numériques sont donnés pour montrer la validité des résultats développés.

1 Introduction

In the last decades Markovian jump systems have attracted a lot of researchers from control and operations research communities. This is due to the fact that this class of systems is more appropriate to model some practical systems that we can found in manufacturing systems, power systems, network control systems, etc. More efforts have been done on different subjects related to this class of systems. Almost all the control problems for these systems have been tackled and interesting results have been reported in the literature. For more details on subject we refer the reader to Boukas [2] for the continuous-time case and Costa et al. [5] for the discrete-time case and the references these volumes.

All the results reported in the literature assumed the complete knowledge of the dynamics of the Markov process that describes the switching between the system modes. But practically this is not valid since it is very hard and more expensive to get all the jump rates for the continuous-time case or all the transition probabilities for the discrete-time case, and therefore the results developed earlier can not be applied to practical systems.

More often in the continuous-time case for instance, we have partial knowledge of the jump rates of the transition probabilities with some bounds for few transitions of the system that we can get by doing some experiment on the practical system that we would like to study (stability or stabilization). In this paper we will assume that we have partial knowledge of the transitions and since that all the jump rates or the transition probabilities for a practical system are bounded with finite values which is the case in practice, we will require only the knowledge of a lower and upper bounds for the jump rates and the transition probabilities in each mode. The aim of this paper is to revise the stochastic stability and stochastic stabilization of the class of systems with random abrupt changes and develop new conditions for such problems that require only partial knowledge of the transition rates or the transition probabilities of the Markov process that describes the switching modes of the systems.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is presented. In Section 3, the main results are given and they include results on stochastic stability, stochastic stabilizability and their robustness. A state feedback controller is used in this paper and a design algorithm in terms of the solutions to linear matrix inequalities is proposed to synthesize the controller gains of the state feedback controller we are using.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions.

2 Problem statement

The goal of this paper is to revise the stochastic stability and stochastic stabilizability of the class of linear systems with random abrupt changes. Both the continuous-time and the discrete-time cases are considered.

Let us consider the class of uncertain continuous-time systems under study be described by the following dynamics:

$$\begin{cases} \dot{x}(t) = A(r_t, t)x(t) + B(r_t, t)u(t), \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $A(r_t, t)$ and $A(r_t, t)$ are defined by:

$$\begin{aligned} A(r_t, t) &= D_A(r_t)F_A(r_t)E_A(r_t) \\ B(r_t, t) &= D_B(r_t)F_B(r_t)E_B(r_t) \end{aligned}$$

with $A(i)$, $B(i)$, $D_A(i)$, $E_A(i)$, $D_B(i)$ and $E_B(i)$ are known matrices with appropriate dimensions and $F_A(i)$ and $F_B(i)$ satisfy:

$$\begin{aligned} F_A^\top(i)F_A(i) &\leq \mathbb{I} \\ F_B^\top(i)F_B(i) &\leq \mathbb{I} \end{aligned}$$

The Markov process $\{r_t, t \geq 0\}$ beside taking values in the finite set $\mathscr{S} = \{1, 2, \cdots, N\}$, represents the switching between the different modes and its dynamics is described by the following probability transitions:

$$\begin{aligned} &\mathbb{P}\left[r_{t+h} = j | r_t = i\right] \\ &= \begin{cases} \lambda_{ij} h + o(h) & \text{when } r_t \text{ jumps from } i \text{ to } j \\ 1 + \lambda_{ii} h + o(h) & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

where λ_{ij} is the transition rate from mode i to mode j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$ and $o(h)$ is such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

For the discrete-time case let $\{r_k, k \geq 0\}$ be a Markov chain with state space $\mathscr{S} = \{1, \cdots, N\}$ and state transition matrix $\mathbb{P} = \left[p_{ij}\right]_{i,j\in\mathscr{S}}$, i.e. the transition probabilities of $\{r_k, k \geq 0\}$ are as follows:

$$\mathbb{P}[r_{k+1} = j | r_k = i] = p_{ij}, \forall i, j \in \mathscr{S}. \tag{3}$$

Consider a discrete-time hybrid system with N modes and suppose that the system mode switching is governed by $\{r_t, t \geq 0\}$. The system dynamics is

$$\begin{cases} x_{t+1} = A(r_t,t)x_t + B(r_t,t)u_t, \\ x_{s=0} = x_0 \end{cases} \tag{4}$$

where x_t and u_t, $A(r_t,t)$ and $B(r_t,t)$ are as for the continuous-time case.

In the rest of the paper we will denote by $x(t;x_0,r_0)$, shortened to $x(t)$, the solution of system (1) or (4) when the initial conditions are respectively x_0 and r_0.

For the continuous-time Markovian jump systems for instance, the jump rates are in general hard to measure for practical systems and therefore the results developed on stability or stabilization in the literature are in some sense useless. Some alternates that consider uncertainties on the jump rates have been proposed. Among them we quote the work done by Benjelloun and Boukas [1] where uncertainties on the jump rates are considered and El-Ghaoui and Ait-Rami [6] where they consider polytopic uncertainties on the transition matrix. In this paper we will assume that we have partial knowledge of the transitions and all the jump rates are bounded with finite values which is a practical assumption. For the continuous-time case we have the following assumption.

Assumption 2.1 *The jump rates are assumed to satisfy the following:*

$$0 < \underline{\lambda}_i \le \lambda_{ij} \le \bar{\lambda}_i, \forall i,j \in \mathscr{S}, j \ne i \tag{5}$$

where $\underline{\lambda}_i$ and $\bar{\lambda}_i$ are known parameters for each mode or may represent the lower and upper bounds when all the jump rates are known, i.e.:

$$0 < \underline{\lambda}_i = \min_{j\in\mathscr{S}} \{\lambda_{ij}, i \ne j\} \tag{6}$$

$$0 < \bar{\lambda}_i = \max_{j\in\mathscr{S}} \{\lambda_{ij}, i \ne j\} \tag{7}$$

with $\underline{\lambda}_i \le \bar{\lambda}_i$.

For the discrete-time case we will have the following assumption:

Assumption 2.2 *The transition probabilities are assumed to satisfy the following:*

$$0 < \underline{p}_i \le p_{ij} \le \bar{p}_i, \forall i,j \in \mathscr{S}, j \ne i \tag{8}$$

where $\underline{p}_i$ and $\bar{p}_i$ are known parameters for each mode or may represent the lower and upper bounds when all the jump rates are known, i.e.:

$$0 < \underline{p}_i = \min\left(p_{i1}, \cdots, p_{iN}\right) \tag{9}$$

$$0 < \bar{p}_i = \max\left(p_{i1}, \cdots, p_{iN}\right) \tag{10}$$

with $\underline{p}_i \le \bar{p}_i$.

Remark 2.1 *The results we are planning to develop in this paper do not require the knowledge of:*

- *the transition jump rates of the continuous-time system but only two bounds, $\underline{\lambda}_i$ and $\bar{\lambda}_i$ representing respectively the minimum lower and the maximum upper bounds for all the jump rates in each mode,*
- *the transition probabilities of the discrete-time system but only the bound $\bar{p}_i$ representing respectively the maximum upper bound for all the transition probabilities in each mode.*

The following definitions will be used in the rest of this paper. For more details on the class of systems with random abrupt changes properties, we refer the reader to [2] and the references therein.

Definition 2.1 *Nominal system (1) is said to be stochastically stable (SS) if there exists a constant $M(x_0, r_0) > 0$ such that the following holds for any initial conditions (x_0, r_0):*

$$\mathbb{E}\left[\int_0^\infty x^\top(t)x(t)|x_0, r_0\right] \leq M(x_0, r_0). \tag{11}$$

Definition 2.2 *Nominal system (4) is said to be stochastically stable (SS) if the following holds:*

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \|x_k\|^2 |x_0, r_0\right] \leq \Gamma(x_0, r_0),$$

where $\Gamma(x_0, r_0)$ is a non-negative function of the system initial values.

Definition 2.3 *Nominal system (1) or (4) is said to be stochastically stabilizable if there exists a control*

$$u(t) = K(r_t)x(t), \tag{12}$$

with $K(i) \in \mathbb{R}^{m\times n}$, $i \in \mathscr{S}$, a constant matrix such that the closed-loop system is stochastically stable.

The definitions of robust stochastic stability and robust stochastic stabilizability are given by the following definitions.

Definition 2.4 *System (1) or (4) with $u(t) \equiv 0$ is said to be robustly stochastically stable if it is stochastically stable for all admissible uncertainties.*

Definition 2.5 *System (1) or (4) is said to be robust stochastically stabilizable if there exists a control of the form (12) such that the closed-loop system is stochastically stable for all admissible uncertainties.*

Combining the systems dynamics and the controller expression, we get the following closed-loop dynamics:

$$\begin{cases} \dot{x}(t) = A_{cl}(r_t)x(t) \\ x(t+1) = A_{cl}(r_t)x(t) \end{cases} \tag{13}$$

where $A_{cl}(r_t) = A(t_t) + B(r_t)K(r_t)$ with $K(r_t)$ is the controller gain that we have to compute.

The goal of this paper is to develop new conditions to check the stochastic stability and to design a state feedback controller that makes the closed-loop system stochastically stable. The robustness of these problems is also tackled. In the rest of this paper, we will assume the complete access to the system state and mode for feedback. Our methodology in this paper will be mainly based on the Lyapunov theory and some algebraic results. The conditions we will develop here will be in terms of the solutions to linear matrix inequalities that can be easily obtained using LMI control toolbox.

Before closing this section, let us give some lemmas that we will use in our development. The proofs of these lemmas can be found in the cited references.

Lemma 2.1 *[4] Let H, F and G be real matrices of appropriate dimensions then, for any scalar $\varepsilon > 0$ for all matrices F satisfying $F^\top F \leq \mathbb{I}$, we have:*

$$HFG + G^\top F^\top H^\top \leq \varepsilon HH^\top + \varepsilon^{-1} G^\top G \tag{14}$$

Lemma 2.2 *[4] The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, H - S^\top R^{-1} S > 0$$

where $H = H^\top$, $R = R^\top$ and S is a constant matrix.

3 Main results

In this section, we will firstly develop results that assure that the free system (i.e. $u(t) = 0$ for all $t \geq 0$) is stochastically stable. Then using these results, we will design a state feedback controller of the form (12) that guarantees the same goal. The continuous-time and discrete-time cases will be both treated.

4 Continuous-time case

Based on the known results on stochastic stability of the class of continuous-time linear systems (see [2]), the system will be stochastically stable if and only if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ such that the following coupled set of LMIs holds:

$$A^\top(i)P(i) + P(i)A(i) + \sum_{j=1}^{N} \lambda_{ij} P(j) < 0.$$

This condition can be rewritten as follows:

$$A^\top(i)P(i) + P(i)A(i) + \lambda_{ii}P(i) + \sum_{j=1,j\neq i}^{N} \lambda_{ij}P(j) < 0$$

Based now on the Assumption 2.1, we get:

$$\sum_{j=1,j\neq i}^{N} \lambda_{ij}P(j) \leq \bar{\lambda}_i \sum_{j=1,j\neq i}^{N} P(j)$$

$$\lambda_{ii}P(i) = -\sum_{j=1,j\neq i}^{N} \lambda_{ij}P(i) \leq -(N-1)\underline{\lambda}_i P(i)$$

Using these relations, we obtain the following condition that we should satisfy to guarantee that the system is stochastically stable:

$$A^\top(i)P(i) + P(i)A(i) - (N-1)\underline{\lambda}_i P(i) + \bar{\lambda}_i \sum_{j=1,j\neq i}^{N} P(j) < 0$$

The following theorem summarizes the results of this development.

Theorem 4.1 *Nominal system (1) is stochastically stable if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ such that the following set of coupled LMIs holds:*

$$A^\top(i)P(i) + P(i)A(i) - (N-1)\underline{\lambda}_i P(i) + \bar{\lambda}_i \sum_{j=1,j\neq i}^{N} P(j) < 0, \forall i \in \mathscr{S}. \quad (15)$$

Remark 4.1 *Notice that when the system has only one mode, the results of Theorem 4.1 reduce to the ones of linear time-invariant systems.*

Let us now concentrate on the design of the state feedback controller of the form (12) that makes the closed-loop system stochastically stable. Using the results of Theorem 4.1, the closed-loop dynamics will be stochastically stable if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ such that the following coupled set of LMIs holds:

$$A_{cl}^\top(i)P(i) + P(i)A_{cl}(i) - (N-1)\underline{\lambda}_i P(i) + \bar{\lambda}_i \sum_{j=1,j\neq i}^{N} P(j) < 0.$$

Pre- and post-multiply this inequality by $X(i) = P^{-1}(i)$ and using the expression of $A_{cl}(i)$, we get:

$$J(i) - (N-1)\underline{\lambda}_i X(i) + \bar{\lambda}_i \sum_{j=1, j\neq i}^{N} X(i)X^{-1}(j)X(i) < 0.$$

where $J(i) = X(i)A^\top(i) + A(i)X(i) + B(i)K(i)X(i) + X(i)K^\top(i)B^\top(i)$.

Letting $Y(i)$, $\mathcal{S}_i(X)$ and $\mathcal{X}_i(X)$ be defined as follows:

$$\begin{aligned} Y(i) &= K(i)X(i) \\ \mathcal{S}_i(X) &= \left[\sqrt{\bar{\lambda}_i}X(i), \cdots \sqrt{\bar{\lambda}_i}X(i), \sqrt{\bar{\lambda}_i}X(i), \cdots, \sqrt{\bar{\lambda}_i}X(i)\right] \\ \mathcal{X}_i(X) &= \text{diag}\left[X(1), \cdots, X(i-1), X(i+1), \cdots, X(N)\right] \end{aligned}$$

we get:

$$J(i) - (N-1)\underline{\lambda}_i X(i) + \mathcal{S}_i(X)\mathcal{X}_i^{-1}(X)\mathcal{S}_i^\top(X) < 0.$$

where $J(i) = X(i)A^\top(i) + A(i)X(i) + B(i)Y(i) + Y^\top(i)B^\top(i)$.

The following theorem gives the results that allow the design of the stabilizing controller.

Theorem 4.2 *There exists a state feedback controller of the form (12) such that the closed-loop state equation of the nominal system (1) is stochastically stable if there exist a set of symmetric and positive-definite matrices* $X = (X(1), \cdots, X(N))$ *and a set of matrices* $Y = (Y(1), \cdots, Y(N))$ *such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} J(i) - (N-1)\underline{\lambda}_i X(i) & \mathcal{S}_i(X) \\ \mathcal{S}_i^\top(X) & -\mathcal{X}_i(X) \end{bmatrix} < 0, \tag{16}$$

where $J(i) = X(i)A^\top(i) + A(i)X(i) + B(i)Y(i) + Y^\top(i)B^\top(i)$. *The gain of the controller is given by* $K(i) = Y(i)X^{-1}(i)$.

Remark 4.2 *When the number of modes is reduced to one, our results become those of linear time-invariant systems.*

Remark 4.3 *Notice that we can get more conservative results either for stochastic stability or stochastic stabilizability by letting* $\underline{\lambda}_i$ *and* $\bar{\lambda}_i$ *be mode independent. This is obtained by choosing* $\underline{\lambda}$ *as the smallest* $\underline{\lambda}_i$ *and* $\bar{\lambda}$ *as the largest* $\bar{\lambda}_i$ *respectively.*

For the uncertain system, based on Theorem 4.1, the free uncertain system (1) will be stochastically stable if there exists a set of symmetric and positive-definite matrices such that the following set of coupled matrix inequalities holds for all admissible uncertainties:

$$A^\top(i,t)P(i) + P(i)A(i,t) - (N-1)\underline{\lambda}_i P(i) + \bar{\lambda}_i \sum_{j=1, j\neq i}^{N} P(j) < 0.$$

Using the expression of $A(i,t)$ and Lemma 2.1, the uncertain system will be robust stochastically stable if the following holds:

$$A^\top(i)P(i) + P(i)A(i) - (N-1)\underline{\lambda}_i P(i) + \varepsilon_A(i)P(i)D_A(i)D_A^\top(i)P(i) \\ + \varepsilon_A^{-1}(i)E_A^\top(i)E_A(i) + \bar{\lambda}_i \sum_{j=1, j\neq i}^{N} P(j) < 0.$$

for $\varepsilon_A(i) > 0$.

The following theorem gives the results for the uncertain case.

Theorem 4.3 *System (1) is robust stochastically stable if there exist a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ and a set of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ such that the following set of coupled matrix inequalities holds:*

$$A^\top(i)P(i) + P(i)A(i) - (N-1)\underline{\lambda}_i P(i) + \varepsilon_A(i)P(i)D_A(i)D_A^\top(i)P(i) \\ + \varepsilon_A^{-1}(i)E_A^\top(i)E_A(i) + \bar{\lambda}_i \sum_{j=1, j\neq i}^{N} P(j) < 0. \tag{17}$$

Remark 4.4 *Notice that the conditions of Theorem 4.3 can be put in the LMI setting and solved by the existing tools in the marketplace.*

For the robust stochastic stabilization we can follow the same steps as before and get the following results.

Theorem 4.4 *System (1) is stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} J(i) - (N-1)\underline{\lambda}_i X(i) & X(i)E_A^\top(i) & Y^\top(i)E_B^\top(i) & \mathcal{S}_i(X) \\ E_A(i)X(i) & -\varepsilon_A(i)\mathbb{I} & 0 & 0 \\ E_B(i)Y(i) & 0 & -\varepsilon_B(i)\mathbb{I} & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0, \tag{18}$$

where

$$J(i) = X(i)A^\top(i) + A(i)X(i) + B(i)Y(i) + Y^\top(i)B^\top(i) \\ + \varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i).$$

The gain of the controller is given by $K(i) = Y(i)X^{-1}(i)$.

Remark 4.5 *As we did for the nominal case we can also establish conservative results by considering the mode-independent lowest and the largest bounds of the jump rates.*

5 Discrete-time case

As we did for the continuous-time, let us extend the results to the discrete-time case. Both the stochastic stability and the stochastic stabilizability are considered in this section and corresponding LMI conditions are developed.

Based on the stochastic stability results for this class of systems [3], the free nominal system (4) will be stable if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ such that the following hold:

$$A^\top(i) \sum_{j=1}^{N} p_{ij} P(j) A(i) - P(i) < 0, \forall i \in \mathscr{S}$$

Using now the Assumption 2.2, the previous inequality will hold if the following one does:

$$A^\top(i) \sum_{j=1}^{N} \bar{p} P(j) A(i) - P(i) < 0, \forall i \in \mathscr{S}$$

Let $\bar{P}$ and W_i be defined as follows:

$$\begin{aligned} \bar{P} &= \mathrm{diag}\left[P(1), \cdots, P(N)\right], \\ W_i &= \left[\sqrt{\bar{p}_i}\mathbb{I}, \cdots, \sqrt{\bar{p}_i}\mathbb{I}\right] \end{aligned}$$

we get the following results.

Theorem 5.1 *Nominal system (4) is stochastically stable if there exists a set of symmetric and positive-definite matrices* $P = (P(1), \cdots, P(N))$ *such that the following set of coupled matrix inequalities holds:*

$$A^\top(i) W_i \bar{P} W_i^\top A(i) - P(i) < 0, \forall i \in \mathscr{S}. \tag{19}$$

Let us now design a state feedback controller with the following form:

$$u_t = K(r_t) x_t \tag{20}$$

where $K(i)$ is a gain to be determined.

Combining the system's dynamics (4) with the controller (20) expression and using Theorem 5.1, the closed-loop dynamics will be stochastically stable if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$ such that the following set of coupled LMIs holds:

$$A_{cl}^{\top}(i)W_i\bar{P}W_i^{\top}A_{cl}(i) - P(i) < 0, \forall i \in \mathscr{S}$$

with $A_{cl}(i) = A(i) + B(i)K(i)$.

Using Lemma 2.2, we obtain:

$$\begin{bmatrix} -P(i) & A_{cl}^{\top}(i)W_i \\ W_i^{\top}A_{cl}(i) & -\bar{P}^{-1} \end{bmatrix} < 0, \forall i \in \mathscr{S}.$$

Let $X(i) = P^{-1}(i)$ and define $\mathscr{X}$ and $Y(i)$ as follows:

$$\begin{aligned} \mathscr{X} &= \text{diag}\left[X(1), \cdots, X(N)\right] \\ Y(i) &= K(i)X(i) \end{aligned}$$

Pre- and post-multiply the previous inequality by diag $[X(i), \mathbb{I}]$, we get the results of the following theorem.

Theorem 5.2 *There exists a state feedback controller of the form (20) such that the closed-loop state equation of the nominal system (4) is stochastically stable if there exist a set of symmetric and positive-definite matrices* $X = (X(1), \cdots, X(N))$ *and a set of matrices* $Y = (Y(1), \cdots, Y(N))$ *such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} -X(i) & \left[A(i)X(i) + B(i)Y(i)\right]^{\top} W_i \\ W_i^{\top}\left[A(i)X(i) + B(i)Y(i)\right] & -\mathscr{X} \end{bmatrix} < 0. \tag{21}$$

The gain of the controller is given by $K(i) = Y(i)X^{-1}(i)$.

Based on Theorem 5.2, there exists a state feedback controller of the form (20) such that the closed-loop state equation of the uncertain system (4) is stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ such that the following set of coupled LMIs holds for all admissible uncertainties:

$$\begin{bmatrix} -X(i) & V^{\top}(i)W_i \\ W_i^{\top}V(i) & -\mathscr{X} \end{bmatrix} < 0.$$

with $V(i) = A(i)X(i) + D_A(i)F_A(i)E_A(i)X(i) + B(i)Y(i) + D_B(i)F_B(i)E_B(i)Y(i)$.

Notice that:

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ W^{\top}(i)D_A(i)F_A(i)E_A(i)X(i) & 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ W^{\top}(i)D_A(i) \end{bmatrix} F_A(i) \begin{bmatrix} E_A(i)X(i) & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 \\ W^{\top}(i)D_B(i)F_B(i)E_B(i)Y(i) & 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ W^{\top}(i)D_B(i) \end{bmatrix} F_B(i) \begin{bmatrix} E_B(i)Y(i) & 0 \end{bmatrix} \end{aligned}$$

Using Lemma 2.1, we get:

$$\begin{bmatrix} 0 & 0 \\ W^\top(i)D_A(i)F_A(i)E_A(i)X(i) & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ W^\top(i)D_A(i)F_A(i)E_A(i)X(i) & 0 \end{bmatrix}^\top$$
$$\leq \begin{bmatrix} \varepsilon_A^{-1}(i)X(i)E_A^\top(i)E_A(i)X(i) & 0 \\ 0 & \varepsilon_A(i)W^\top(i)D_A(i)D_A^\top(i)W(i) \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 \\ W^\top(i)D_B(i)F_B(i)E_B(i)Y(i) & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ W^\top(i)D_B(i)F_B(i)E_B(i)Y(i) & 0 \end{bmatrix}^\top$$
$$\leq \begin{bmatrix} \varepsilon_B^{-1}(i)Y^\top(i)E_B^\top(i)E_B(i)Y(i) & 0 \\ 0 & \varepsilon_B(i)W^\top(i)D_B(i)D_B^\top(i)W(i) \end{bmatrix},$$

for $\varepsilon_A > 0$ and $\varepsilon_B > 0$.

Using Lemma 2.2, we get the results of the following theorem.

Theorem 5.3 *There exists a state feedback controller of the form (20) such that the closed-loop state equation of the uncertain system (4) is stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ and sets of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ and $\varepsilon_B = (\varepsilon_B(1), \cdots, \varepsilon_B(N))$ such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} -X(i) & [A(i)X(i)+B(i)Y(i)]^\top W_i & X(i)E_A^\top(i) & Y^\top(i)E_B^\top(i) \\ W_i^\top [A(i)X(i)+B(i)Y(i)] & \mathscr{W}_i & 0 & 0 \\ E_A(i)X(i) & 0 & -\varepsilon_A(i)\mathbb{I} & 0 \\ E_B(i)Y(i) & 0 & 0 & -\varepsilon_B(i)\mathbb{I} \end{bmatrix} < 0. \qquad (22)$$

with $\mathscr{W}_i = -\mathscr{X} + \varepsilon_A(i)W^\top(i)D_A(i)D_A^\top(i)W(i) + \varepsilon_B(i)W^\top(i)D_B(i)D_B^\top(i)W(i)$. The gain of the controller is given by $K(i) = Y(i)X^{-1}(i)$.

Remark 5.1 *As we did for the continuous-time case, we can also here use an upper and lower bounds on the transitions probabilities that are mode-independent and develop similar conservative results on stochastic stability and stochastic stabilization.*

6 Numerical examples

In this section, we will give numerical examples to show that the results we developed either on stochastic stability or stochastic stabilizability are valid. As it was stated on the theory we will assume that we have partial knowledge of the Markov process $\{r_t, t \geq 0\}$ that describes the switching between the different modes of the systems.

Example 6.1 *To show the validness of stability results, let us consider a two modes Markovian system with states in $\mathbb{R}^2$. The data of this system are as follows:*

- *mode 1:*

$$A(1) = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, D_A(1) = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, E_A(1) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$

- *mode 2:*

$$A(2) = \begin{bmatrix} 0.0 & 1.0 \\ -2.0 & -1.0 \end{bmatrix}, D_A(2) = \begin{bmatrix} 0.0 \\ -0.1 \end{bmatrix}, E_A(2) = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}.$$

The switching between the two modes is described by the following:

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1.1 & -1.1 \end{bmatrix},$$

Solving the coupled set of LMIs (15), we get:

$$P(1) = \begin{bmatrix} 2.1673 & 0.8761 \\ 0.8761 & 0.8672 \end{bmatrix},$$
$$P(2) = \begin{bmatrix} 2.2470 & 0.7346 \\ 0.7346 & 1.3377 \end{bmatrix}.$$

The two matrices $P(1)$ and $P(2)$ are both symmetric and positive-definite matrices and based on Theorem 4.1, we conclude that the system is stochastically stable.

Simulation results of this system shows that the system is stochastically stable as illustrated in Figure 1.

Solving the coupled set of LMIs (17), we get:

$$\varepsilon_A(1) = 0.9759, \varepsilon_A(2) = 1.0230,$$
$$P(1) = \begin{bmatrix} 2.1771 & 0.8896 \\ 0.8896 & 0.8816 \end{bmatrix},$$
$$P(2) = \begin{bmatrix} 2.2507 & 0.7365 \\ 0.7365 & 1.3541 \end{bmatrix}.$$

The two matrices $P(1)$ and $P(2)$ are both symmetric and positive-definite matrices and based on Theorem 4.3, we conclude that the system is stochastically stable.

Example 6.2 *To show the validness of the stabilizability results, let us consider a two modes Markovian system with states in $\mathbb{R}^2$. The data of this system are as follows:*

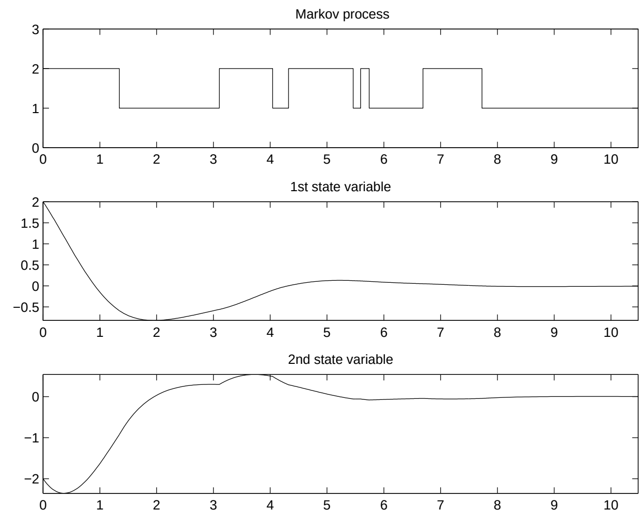

Figure 1: The behaviors of the system states in function of time t

- *mode 1:*

$$A(1) = \begin{bmatrix} 0.0 & 1.0 \\ 1.0 & 2.0 \end{bmatrix}, \qquad D_A(1) = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, \qquad E_A(1) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$
$$B(1) = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}, \qquad D_B(1) = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, \qquad E_B(1) = \begin{bmatrix} 0.1 \end{bmatrix},$$

- *mode 2:*

$$A(2) = \begin{bmatrix} 0.0 & 1.0 \\ 2.0 & 1.0 \end{bmatrix}, \qquad D_A(2) = \begin{bmatrix} 0.0 \\ -0.1 \end{bmatrix}, \qquad E_A(2) = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix},$$
$$B(2) = \begin{bmatrix} 0.0 \\ 2.0 \end{bmatrix}, \qquad D_B(2) = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, \qquad E_B(2) = \begin{bmatrix} -0.1 \end{bmatrix}.$$

The switching between the two modes is described by the following:

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1.1 & -1.1 \end{bmatrix},$$

First of all notice that the system in each mode is not stable since the eigenvalues of $A(1)$ *and* $A(2)$ *are all positive real part. It can be checked even the all system is stochastically unstable.*

Solving the coupled set of LMIs (16), we get:

$$X(1) = \begin{bmatrix} 0.5302 & -0.3693 \\ -0.3693 & 0.8544 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 0.4865 & -0.3819 \\ -0.3819 & 0.8940 \end{bmatrix},$$
$$Y(1) = \begin{bmatrix} -0.4570 & -2.0634 \end{bmatrix},$$
$$Y(2) = \begin{bmatrix} -0.5017 & -0.5989 \end{bmatrix}$$

which gives the following gains for the state-feedback controller:

$$K(1) = \begin{bmatrix} -3.6401 & -3.9883 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -2.3427 & -1.6707 \end{bmatrix}.$$

The two matrices $X(1)$ and $X(2)$ are both symmetric and positive-definite matrices and based on Theorem 4.2, we conclude that the closed-loop system is stochastically stable under the state-feedback controller with the set of gains we computed.

Simulation results of this system shows that the system is stochastically stable as illustrated in Figure 2.

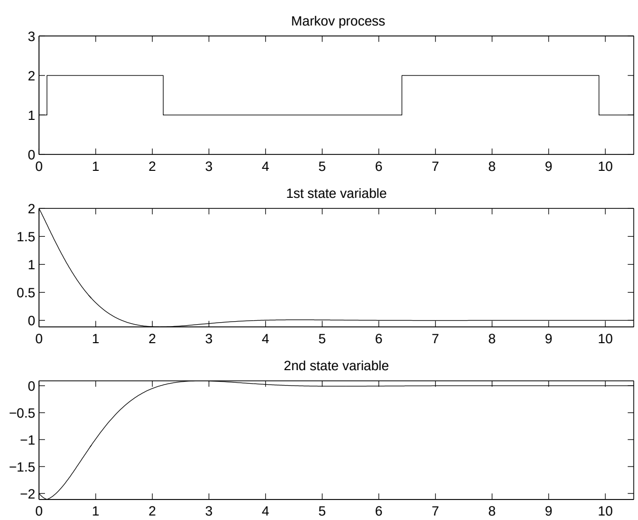

Figure 2: The behaviors of the system states in function of time t

Solving the coupled set of LMIs (16), we get:

$$\begin{aligned}
\varepsilon_A(1) &= 1.0018, \varepsilon_A(2) = 1.0251, \\
\varepsilon_B(1) &= 1.0665, \varepsilon_B(2) = 1.0000, \\
X(1) &= \begin{bmatrix} 0.5246 & -0.3679 \\ -0.3679 & 0.8047 \end{bmatrix}, \\
X(2) &= \begin{bmatrix} 0.4771 & -0.3772 \\ -0.3772 & 0.8594 \end{bmatrix}, \\
Y(1) &= \begin{bmatrix} -0.4136 & -1.9682 \end{bmatrix}, \\
Y(2) &= \begin{bmatrix} -0.4767 & -0.5878 \end{bmatrix}
\end{aligned}$$

which gives the following gains for the state-feedback controller:

$$\begin{aligned}
K(1) &= \begin{bmatrix} -3.6861 & -4.1312 \end{bmatrix}, \\
K(2) &= \begin{bmatrix} -2.3589 & -1.7194 \end{bmatrix}.
\end{aligned}$$

The two matrices $X(1)$ and $X(2)$ are both symmetric and positive-definite matrices and based on Theorem 4.2, we conclude that the closed-loop system is robust stochastically stable under the state-feedback controller with the set of gains we computed.

Example 6.3 *To show the validness of the stabilizability results for the discrete-time case, let us consider the two modes Markovian system with states in $\mathbb{R}^2$ of the previous example. The data of this system are the same as for the continuous-time case and the transition probabilities are as follows:*

$$\Lambda = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix},$$

It can be seen that $\bar{p}_1$ and $\bar{p}_2$ are respectively 0.6 *and* 0.5.

First of all notice that the system in each mode is not stable since at least one of the eigenvalues of $A(1)$ and $A(2)$ are outside the unit circle. It can be checked even the all system is stochastically unstable.

Solving the coupled set of LMIs (22), we get:

$$\begin{aligned}
X(1) &= \begin{bmatrix} 1.3169 & -0.0000 \\ -0.0000 & 0.6831 \end{bmatrix}, \\
X(2) &= \begin{bmatrix} 1.3169 & -0.0000 \\ -0.0000 & 0.6831 \end{bmatrix},
\end{aligned}$$

$$Y(1) = \begin{bmatrix} -1.3169 & -1.3661 \end{bmatrix},$$
$$Y(2) = \begin{bmatrix} -1.3169 & -0.3415 \end{bmatrix}$$

which gives the following gains for the state-feedback controller:

$$K(1) = \begin{bmatrix} -1.0000 & -2.0000 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -1.0000 & -0.5000 \end{bmatrix}.$$

The two matrices $X(1)$ and $X(2)$ are both symmetric and positive-definite matrices and based on Theorem 5.2, we conclude that the closed-loop system is stochastically stable under the state-feedback controller with the set of gains we computed.

Solving the coupled set of LMIs (22), we get:

$$\varepsilon_A(1) = 1.0160, \varepsilon_A(2) = 1.0116,$$
$$\varepsilon_B(1) = 1.1081, \varepsilon_B(2) = 1.0074,$$
$$X(1) = \begin{bmatrix} 1.1822 & -0.0288 \\ -0.0288 & 0.5347 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 1.1901 & 0.0070 \\ 0.0070 & 0.6540 \end{bmatrix},$$
$$Y(1) = \begin{bmatrix} -1.1056 & -1.0301 \end{bmatrix},$$
$$Y(2) = \begin{bmatrix} -1.1911 & -0.3330 \end{bmatrix}$$

which gives the following gains for the state-feedback controller:

$$K(1) = \begin{bmatrix} -0.9834 & -1.9796 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -0.9979 & -0.4985 \end{bmatrix}.$$

The two matrices $X(1)$ and $X(2)$ are both symmetric and positive-definite matrices and based on Theorem 5.3, we conclude that the closed-loop system is robust stochastically stable under the state-feedback controller with the set of gains we computed.

7 Conclusions

This paper dealt with the stochastic stability and stochastic stabilization of the class of linear systems with random abrupt changes and their robustness. Under partial knowledge of the transitions between the system's modes, LMI conditions for stochastic stability and stochastic stabilization and their robustness have been developed. The results we developed can be extended easily for other classes of systems like systems with time-delay and for other type of controllers.

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