**On the Spectral Radius of
Graphs with a Given
Domination Number**

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Abstract

In the set of all connected graphs with a given domination number, we characterize the graphs which achieve the maximum value of the spectral radius of the adjacency matrix.

Résumé

Parmi l'ensemble des graphes connexes de nombre de domination donné, on caractérise les graphes ayant la plus grande valeur du rayon spectral de la matrice d'adjacence.

1   Introduction

All graphs in this note are simple and undirected. For a graph G, let $A(G)$ denote its adjacency matrix and $\rho(G)$ denote the spectral radius of $A(G)$. It is well known that for a connected graph G, there is a unique positive eigenvector corresponding to $\rho(G)$, usually called the *Perron vector*. For other undefined notions, the reader is referred to [4] for general graph theory, and to [7, 8] for spectral graph theory topics.

Brualdi and Solheid [5] proposed the following general problem, which became one of the classic problems of spectral graph theory:

> *Given a set $\mathcal{G}$ of graphs, find* $\min\{\rho(G)\colon G \in \mathcal{G}\}$ *and* $\max\{\rho(G)\colon G \in \mathcal{G}\}$, *and characterize the graphs which achieve the minimum or maximum value.*

The maximization part of this problem has been solved for a number of graph classes so far, although it is interesting that it has been solved only recently for the sets of connected graphs which have a given value of some well-known integer graph invariant: for example, the number of cut vertices [3], the matching number [9], the chromatic number [10], or the clique number [12]. We should also note that the graphs with maximum spectral radius received much more attention in the literature than the graphs with minimum spectral radius (moreover, only the maximization part of the above problem is usually cited).

In this note, we solve the maximum spectral radius problem for the set of connected graphs with a given value of the domination number γ. We will deal with the minimum spectral radius problem in a forthcoming article. To recall, a set S of vertices of a graph G is said to be *dominating* if every vertex of $V(G) \setminus S$ is adjacent to a vertex of S, and the *domination number* $\gamma(G)$ is the minimum number of vertices of a dominating set in G. If G is connected, then $\gamma \le \frac{n}{2}$ [13].

Since the spectral radius of a connected graph strictly increases by adding an edge, we see that the candidates for a graph with the maximum spectral radius are found among *domination-critical* graphs G, which have domination number γ, but for every edge e that does not belong to G, the graph $G+e$ has domination number $\gamma-1$. So far, the domination-critical graphs have been characterized only for $\gamma \le 2$. In the general case, some of their properties are known (for an overview, see [15]) and, in particular, the maximum number of edges in a domination-critical graphs has been determined by Vizing [16] for all graphs and by Sanchis [14] for connected graphs.

We should state here that prior to formulating our main result, the examples of graphs maximizing the spectral radius with a given domination number were found using the computer system *AutoGraphiX* [1, 6]. It is our opinion that, due to its versatility, this system may become an indispensable tool when it comes to looking for examples of extremal graphs.

The *surjective split graph* $SSG(n; a_1, \ldots, a_k)$, defined for positive integers $n, k, a_1, \ldots, a_k$ satisfying $a_1 + \cdots + a_k = n - k$, $a_1 \ge \cdots \ge a_k$, is a split graph on n vertices formed from

a clique K with $n - k$ vertices and an independent set I with k vertices, in such a way that the i-th vertex of I is adjacent to a_i vertices of K, and that no two vertices of I have a common neighbor in K. See Figure 1 for examples of surjective split graphs. Note that $\gamma(SSG(n; a_1, \ldots, a_k)) = k$, and that the surjective split graphs have the maximum number of edges among connected graphs with a given domination number (see [14]).
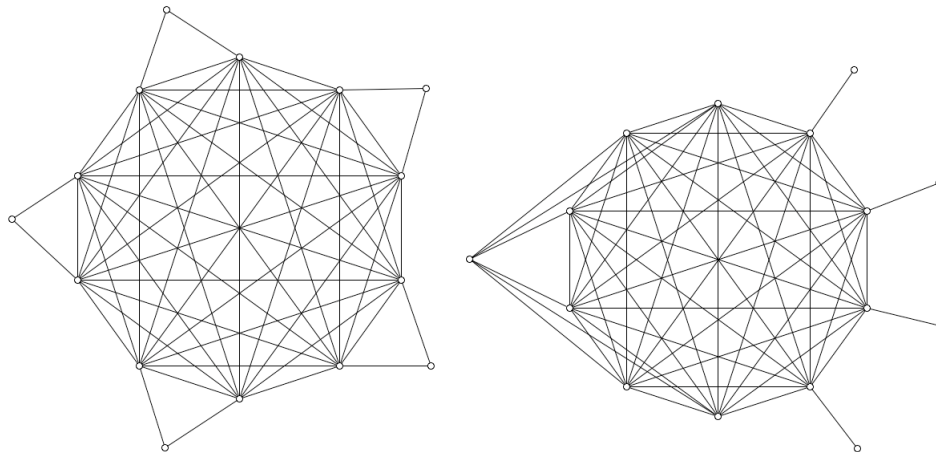

Figure 1: The surjective split graphs $SSG(15; 2, 2, 2, 2, 2)$ and $SSG(15; 6, 1, 1, 1, 1)$.

Our main result is the following

Theorem 1 *If G is a graph on n vertices with domination number γ, $\gamma \leq \frac{n}{2}$, then*

$$\rho(G) \leq \rho(SSG(n; n - 2\gamma + 1, 1, 1, \ldots, 1)).$$

Equality holds if and only if G is isomorphic to $SSG(n; n - 2\gamma + 1, 1, 1, \ldots, 1)$.

2 Proof

For fixed n and γ, $\gamma \leq \frac{n}{2}$, let G^* be a graph with n vertices, domination number γ and the maximum spectral radius $\rho^* = \rho(G^*)$. From the above discussion, G^* must be domination-critical.

If $\gamma = 1$, then $G^* = K_n = SSG(n; n - 1)$ and it is the unique graph with spectral radius $n - 1$. Thus, in the sequel we suppose that $\gamma \geq 2$.

Let S^* be the surjective split graph $SSG(n; n - 2\gamma + 1, 1, \ldots, 1)$. S^* has domination number γ, and thus

$$\rho^* \geq \rho(S^*). \tag{1}$$

Further, S^* contains the complete graph $K_{n-\gamma}$ as a proper induced subgraph, and so

$$\rho(S^*) > n - \gamma - 1. \tag{2}$$

From this and the well-known bound $\Delta(G^*) \geq \rho^*$ (see [7]), where $\Delta(G^*)$ denotes the maximum vertex degree of G^*, we may already conclude that

$$\Delta(G^*) \geq n - \gamma.$$

On the other hand, it must hold that $\Delta(G^*) \leq n - \gamma$: if a vertex u has more than $n - \gamma$ neighbors in G, then u and its nonneighbors form a dominating set with less than γ vertices, a contradiction. Thus,

$$\Delta(G^*) = n - \gamma. \tag{3}$$

The previous equality ensures the existence of one vertex of degree $n-\gamma$ in G^*. However, for our purposes we need to ensure the existence of at least two vertices of degree $n-\gamma$ and sufficiently many edges in G^*, and to do this, we need a much tighter estimate of $\rho(S^*)$ than (2).

Lemma 2 $\rho(S^*) \geq n - \gamma - 1 + \dfrac{1}{n - \gamma} + \dfrac{(n - 2\gamma + 1)(n - 2\gamma)}{(n - \gamma)^2}.$

Proof. Let λ denote the value on the right hand side of the above inequality. It is easy to see that

$$\lambda = n - \gamma - \frac{(2n - 3\gamma)(\gamma - 1)}{(n - \gamma)^2} \leq n - \gamma.$$

Let $I' = \{u_a\}$ be the subset of the independent set I of S^* containing the vertex of degree $n - 2\gamma + 1$, and let K' be the subset of the clique K of S^* containing the $n - 2\gamma + 1$ vertices adjacent to u_a. Set $I'' = I \setminus I'$ and $K'' = K \setminus K'$. From the definition of S^*, each vertex of K'' is adjacent to a unique vertex of I'', and $|K''| = |I''| = \gamma - 1$.

Now, let $y = (y_u)_{u \in V(G)}$ be the vector defined in the following way:

$$y_u = \begin{cases} a = (n - 2\gamma + 1)\left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right), & u \in I', \\ b = \lambda\left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right), & u \in K', \\ c = \lambda, & u \in K'', \\ d = 1, & u \in I''. \end{cases}$$

For $A = A(S^*)$, we have that

$$(Ay)_u = \begin{cases} (n - 2\gamma + 1)b, & u \in I', \\ a + (n - 2\gamma)b + (\gamma - 1)c, & u \in K', \\ (n - 2\gamma + 1)b + (\gamma - 2)c + d, & u \in K'', \\ c, & u \in I''. \end{cases}$$

Let us show that for this particular vector

$$Ay \geq \lambda y,$$

with component-wise inequality. Actually, for $u \in I$ we have the case of equality:

$$
\begin{aligned}
(Ay)_u &= \lambda(n - 2\gamma + 1)\left(1 + \tfrac{n-2\gamma}{(n-\gamma)^2}\right) &&= \lambda y_u, \quad u \in I', \\
(Ay)_u &= \lambda &&= \lambda y_u, \quad u \in I''.
\end{aligned}
$$

Next, for $u \in K'$ we have

$$
\begin{aligned}
(Ay)_u &= n - 2\gamma + 1 + \lambda(n - \gamma - 1) + \frac{n - 2\gamma}{(n - \gamma)^2}\left(n - 2\gamma + 1 + (n - 2\gamma)\lambda\right) \\
&\geq n - 2\gamma + \lambda\left(\frac{1}{n - \gamma} + n - \gamma - 1\right) + \frac{n - 2\gamma}{(n - \gamma)^2}\left(n - 2\gamma + 1 + (n - 2\gamma)\lambda\right) \\
&= \lambda\left(\lambda - \frac{(n - 2\gamma + 1)(n - 2\gamma)}{(n - \gamma)^2}\right) + \frac{n - 2\gamma}{(n - \gamma)^2}\left((n - \gamma)^2 + n - 2\gamma + 1 + (n - 2\gamma)\lambda\right) \\
&= \lambda^2 + \frac{n - 2\gamma}{(n - \gamma)^2}\left((n - \gamma)^2 + n - 2\gamma + 1 - \lambda\right) \\
&> \lambda^2\left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right) = \lambda y_u,
\end{aligned}
$$

where in the first inequality above we used the relation $1 \geq \frac{\lambda}{n-\gamma}$, and the second inequality, based on $(n - \gamma)^2 + n - 2\gamma + 1 - \lambda > \lambda^2$, follows from

$$
(n - \gamma)^2 - \lambda^2 + n - \gamma - \lambda = (n - \gamma - \lambda)(n - \gamma + \lambda + 1) > \frac{(2n - 3\gamma)(\gamma - 1)}{(n - \gamma)^2} \cdot 2(n - \gamma) > \gamma - 1,
$$

thanks to the fact that $\lambda > n - \gamma - 1$ and $2(2n - 3\gamma) > n - \gamma$.

Finally, for $u \in K''$ we have

$$
\begin{aligned}
(Ay)_u &= (n - 2\gamma + 1)\lambda\left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right) + (\gamma - 2)\lambda + 1 \\
&= \lambda\left(n - \gamma - 1 + \frac{(n - 2\gamma + 1)(n - 2\gamma)}{(n - \gamma)^2}\right) + 1 \\
&= \lambda\left(\lambda - \frac{1}{n - \gamma}\right) + 1 = \lambda^2 - \frac{\lambda}{n - \gamma} + 1 \geq \lambda^2 = \lambda y_u.
\end{aligned}
$$

Finally, we can see that

$$
\rho(S^*) = \sup_{x \neq 0} \frac{x^T A x}{x^T x} \geq \frac{y^T A y}{y^T y} \geq \frac{y^T (\lambda y)}{y^T y} = \lambda. \quad \square
$$

We can now get a lower bound on the number m^* of edges of G^* from Lemma 2 and the bound of Yuan Hong [11]

$$
\rho(G^*) \leq \sqrt{2m^* - n + 1}.
$$

Namely, we have

$$
\begin{aligned}
2m^* &\geq \rho(G^*)^2 + n - 1 \\
&= \left(n - \gamma - \frac{(2n - 3\gamma)(\gamma - 1)}{(n - \gamma)^2} \right)^2 + n - 1 \\
&\geq (n - \gamma)^2 - \frac{2(2n - 3\gamma)(\gamma - 1)}{n - \gamma} + n - 1 \\
&> (n - \gamma)^2 - 4(\gamma - 1) + n - 1 = (n - \gamma + 1)(n - \gamma) - 3(\gamma - 1),
\end{aligned}
$$

i.e.,

$$
m^* > \frac{(n - \gamma + 1)(n - \gamma)}{2} - \frac{3}{2}(\gamma - 1). \tag{4}
$$

For comparison, note that any $SSG(n; a_1, \ldots, a_\gamma)$ has $\frac{1}{2}(n - \gamma + 1)(n - \gamma)$ edges (which is also the maximum number of edges in a connected graph with domination number γ, cf. [14]). Thus, G^* is less than $\frac{3}{2}(\gamma - 1)$ edges away from the maximum possible value.

From Lemma 2 we can also deduce that G^* contains at least two vertices of degree $n - \gamma$.

Lemma 3 *If a graph G with maximum degree Δ satisfies*

$$
\rho(G) > \Delta - 1 + \frac{1}{\Delta},
$$

then G contains at least two vertices of degree Δ.

Proof. Suppose the contrary, i.e., that G contains a single vertex u with degree Δ, while the degree of any other vertex is at most $\Delta - 1$. Define a vector $y = (y_v)_{v \in V(G)}$ by

$$
y_v = \begin{cases} 1 + \frac{1}{\Delta}, & v = u, \\ 1, & v \neq u. \end{cases}
$$

Then the adjacency matrix $A = A(G)$ is such that

$$
(Ay)_u = \Delta < \Delta + \frac{1}{\Delta^2} = \left(\Delta - 1 + \frac{1}{\Delta} \right) y_u,
$$

while for $v \neq u$

$$
(Ay)_v \leq d_v + \frac{1}{\Delta} \leq \Delta - 1 + \frac{1}{\Delta} = \left(\Delta - 1 + \frac{1}{\Delta} \right) y_v,
$$

where d_v denotes the degree of v.

Thus, for a positive vector y, the inequality

$$
Ay \leq \left(\Delta - 1 + \frac{1}{\Delta} \right) y
$$

holds component-wise. Let x be the Perron vector of G. Then $x^T y > 0$ and we have

$$\rho(G)x^T y = x^T A y \leq \left(\Delta - 1 + \frac{1}{\Delta} \right) x^T y,$$

from where it follows that $\rho(G) \leq \Delta - 1 + \frac{1}{\Delta}$, which is a contradiction. $\square$

Remark. Only two vertices of degree Δ in a graph may be enough to ensure that $\rho(G) > \Delta - 1 + \frac{1}{\Delta}$. For example, provided that $n(\Delta - 1)$ is even, let G be obtained from an arbitrary $(\Delta - 1)$-regular graph on n vertices by joining two of its nonadjacent vertices by an edge. Then G contains exactly two vertices of degree Δ and its average vertex degree is $\Delta - 1 + \frac{2}{n}$. The spectral radius $\rho(G)$ is bounded from below by the average vertex degree of a graph [7], so for $\Delta > \frac{n}{2}$, we have that $\rho(G) \geq \Delta - 1 + \frac{2}{n} > \Delta - 1 + \frac{1}{\Delta}$. $\square$

From (1), (3) and Lemmas 2 and 3, it now follows that G^* contains at least two vertices of degree $n - \gamma$. Suppose that w' and w'' are vertices of degree $n - \gamma$. Next, we show that G^* is a surjective split graph.

Let us again recall that G^*, as a graph with maximum spectral radius ρ^* among the connected graphs with n vertices and domination number γ, must be domination-critical. Thus, no edge may be added to it without decreasing its domination number.

Consider first the vertex w'. Let $S_{w'}$ be the set of $\gamma - 1$ vertices that are not adjacent to w'. The subgraph induced by $S_{w'}$ contains no edges: otherwise, if uv is an edge between vertices u and v of $S_{w'}$, then $\{w'\} \cup S_{w'} \setminus \{v\}$ would be a dominating set of size $\gamma - 1$, a contradiction.

Similarly, no two vertices from $S_{w'}$ may have a common neighbor: for if t is a vertex of G^* adjacent to vertices u and v of $S_{w'}$, then $\{w', t\} \cup S_{w'} \setminus \{u, v\}$ would be again a dominating set of size $\gamma - 1$.

Let $Y_{w'}$ be the set of vertices that are adjacent both to w' and to a vertex from $S_{w'}$. In particular, for each $u \in S_{w'}$, let $Y_{w',u}$ be the set of vertices that are adjacent to w' and u. The set $Y_{w',u}$ is not empty, as G^* does not contain isolated vertices, and from the previous paragraph it follows that each neighbor of u must also be a neighbor of w'. Moreover, it also follows that the sets $Y_{w',u}$, $u \in S_{w'}$, are mutually distinct.

Finally, let $Z_{w'}$ be the set of remaining vertices of G^*, those which are adjacent to w' and to no vertex of $S_{w'}$. The set $Z_{w'}$ is not empty: otherwise, a set X obtained by choosing an arbitrary vertex from each $Y_{w',u}$, $u \in S_{w'}$, would be a dominating set of size $\gamma - 1$. Actually, for each such X an even stronger statement holds:

There exists a vertex z_X in $Z_{w'}$ that is not adjacent to any vertex in X. (5)

We may now see that for any $u \in S_{w'}$, every dominating set X of G^* must contain either the vertex u or a vertex from $Y_{w',u}$. In particular, if $|X| = \gamma$, then $\gamma - 1$ vertices of X belong to sets $\{u\} \cup Y_{w',u}$, $u \in S_{w'}$, and the remaining vertex belongs to $\{w'\} \cup Z_{w'}$.

Next, the subgraph of G^* induced by $Y_{w'}$ is a clique: otherwise, if uv is not an edge of G^* for $u, v \in Y_{w'}$, then $G^* + uv$ also has domination number γ, but its spectral radius is larger than ρ^*, a contradiction. From a similar reason, the subgraph induced by $Z_{w'}$ is also a clique.

Thus, the only part of G^* that we do not know anything about is the set of edges between vertices of $Y_{w'}$ and $Z_{w'}$. That is where the second vertex w'' of degree $n - \gamma$ will help us. Note that the sets $S_{w''}$, $Y_{w''}$ and $Z_{w''}$ may be defined in the same manner and share similar properties as their counterparts $S_{w'}$, $Y_{w'}$ and $Z_{w'}$. So, let us consider in which of the three sets $S_{w'}$, $Y_{w'}$ and $Z_{w'}$ does w'' appear?

First, w'' may not belong to $S_{w'}$, as the degrees of vertices in $S_{w'}$ are too small. Namely, a vertex $u \in S_{w'}$ is not adjacent to any vertex from

$$\{w'\} \cup Z_{w'} \cup (S_{w'} \setminus \{u\}) \cup (Y_{w'} \setminus Y_{w',u}),$$

and its degree is, thus, at most $n - 1 - (1 + 1 + (\gamma - 2) + (\gamma - 2)) < n - \gamma$.

Next, suppose that $w'' \in Y_{w'}$ and let s be the unique vertex of $S_{w'}$ adjacent to w''. Then w'' is adjacent to all vertices of $Z_{w'}$ but one, which we denote by z. It is easy to see that

$$
\begin{aligned}
S_{w''} &= \{z\} \cup S_{w'} \setminus \{s\}, \\
Y_{w''} &\supseteq \{w'\} \cup (Y_{w'} \setminus Y_{w',s}) \cup (Z_{w'} \setminus \{z\}), \\
Z_{w''} &\subseteq \{s\} \cup Y_{w',s} \setminus \{w''\}.
\end{aligned}
\tag{6}
\tag{7}
$$

We show that equality holds in (6) and (7). Suppose that $t \in Y_{w',s} \cap Y_{w''}$. Since the subgraph induced by $Y_{w''}$ is a clique, t must be adjacent to all vertices of $Z_{w'} \setminus \{z\}$. Further, as an element of $Y_{w''}$, t must be adjacent to a vertex of $S_{w''}$. Since it is not adjacent to any vertex of $S_{w'} \setminus \{s\}$, we conclude that t is adjacent to z as well. But then t has degree $n - \gamma + 1$, a contradiction. Thus, it follows that $Y_{w',s} \cap Y_{w''} = \emptyset$ and then the equality holds in (6) and (7). Moreover, one has

$$Y_{w'',u} = Y_{w',u}, \quad u \in S_{w'} \setminus \{s\}$$

and

$$Y_{w'',z} = \{w'\} \cup Z_{w'} \setminus \{z\}.$$

As a consequence, z is adjacent to vertices of $Y_{w'',z}$ only, and so z is not adjacent to any vertex from $Y_{w'}$. Then G^*, as a domination-critical graph, must already contain all edges between a vertex of $Y_{w'}$ and $Z_{w'} \setminus \{z\}$. In such case, G^* is indeed a surjective split graph:

$$G^* \cong SSG(n; |Z_{w'}|, |Y_{w',s}|_{s \in S_{w'}}).$$

The last option for w'' is that it belongs to $Z_{w'}$. We may freely suppose then that no vertex of $Y_{w'}$ has degree $n - \gamma$ (otherwise, rename any such vertex to w'' and return to

the previous paragraph). Let U be the set of all vertices of G^* having degree $n - \gamma$. Then $U \subseteq \{w'\} \cup Z_{w'}$. The vertices of U imply the same local structure in G^*—for any $w \in U$ one has

$$
\begin{aligned}
S_w &= S_{w'}, \\
Y_w &= Y_{w'}, \\
Z_w &= \{w'\} \cup Z_{w'} \setminus \{w\}.
\end{aligned}
$$

Finally, let $Z' = Z_{w'} \setminus U$. Any vertex $z' \in Z'$ has degree less than $n - \gamma$ and, thus, there exists a vertex $y' \in Y_{w'}$ not adjacent to z'. Since G^* is domination-critical, the graph $G^* + y'z'$ has a dominating set X of cardinality $\gamma - 1$. Note that $y' \in X \subseteq Y_{w'}$ and that X does not dominate z' in G^*. Thus, z' is not adjacent to any vertex of X in G^*. In other words, for any $z' \in Z'$, G^* does not contain at least $\gamma - 1$ edges of the form $z'v$. This can be used to give an upper bound on the number of edges m^* of G^*:

$$
m^* \leq \binom{n - \gamma + 1}{2} - |Z'|(\gamma - 1) + (\gamma - 1).
$$

(The last term above counts the edges between $S_{w'}$ and $Y_{w'}$.) This inequality, together with (4), yields:

$$
|Z'| \leq \frac{5}{2}.
$$

Note that the case $|Z'| = 2$ is impossible. Namely, since each vertex $y' \in Y_{w'}$ has degree less than $n - \gamma$, there are at least two vertices in Z' not adjacent to y'. Thus, neither of two vertices of Z' is adjacent to any vertex of $Y_{w'}$. However, we can then add to G^* all edges between one vertex of Z' and all vertices of $Y_{w'}$ without decreasing its domination number, which is a contradiction.

Thus, $|Z'| = 1$. Then G^* is again a surjective split graph

$$
G^* \cong SSG(n; |U|, |Y_{w',s}|_{s \in S_{w'}}).
$$

Thus, we may suppose that $G^* \cong SSG(n; a_1, \dots, a_\gamma)$ for some $a_1 \geq \cdots \geq a_\gamma$. Our goal is to show that $a_1 = n - 2\gamma + 1$, while $a_2 = \cdots = a_\gamma = 1$.

For this purpose we will use the concept of *edge rotations* from [2]. Let $G = (V, E)$ be a simple graph with a Perron vector x. If, for vertices $r, s, t \in V$, it holds $rs \in E$, $rt \notin E$ and $x_s \leq x_t$, then the rotation of an edge rs into rt, meaning a deletion of an edge rs followed by addition of an edge rt, strictly increases the index of G. We have that

$$
\rho(G - rs + rt) \geq \frac{x^T A(G - rs + rt)x}{x^T x} = \frac{x^T A(G)x + 2x_r(x_t - x_s)}{x^T x} \geq \rho(G).
$$

However, the equality $\rho(G - rs + rt) = \rho(G)$ cannot hold. In such a case, one would have that $x_s = x_t$ and that x is a Perron vector of $G - rs + rt$. The eigenvalue equations at s in graphs G and $G - rs + rt$ would then give

$$\rho(G)x_s = \sum_{\{u:us\in E\}} x_u,$$

$$\rho(G)x_s = \rho(G - rs + rt)x_s = -x_r + \sum_{\{u:us\in E\}} x_u,$$

implying that $x_r = 0$, which is a contradiction. Thus, the strict inequality holds

$$\rho(G - rs + rt) > \rho(G).$$

Back to G^*, let x^* be the Perron vector of G^*. Let $S = \{s_1, \ldots, s_\gamma\}$ be the independent set of G^* such that, for $1 \le i \le \gamma$, the vertex s_i has a_i neighbors in the clique K of G^*. Suppose that there exists vertices $s_i, s_j \in S$ such that $a_i, a_j \ge 2$, and without loss of generality, suppose that $x^*_{s_i} \le x^*_{s_j}$. Let y be an arbitrary vertex adjacent to s_i. By rotating the edge ys_i to ys_j, we get that

$$\rho(G^* - ys_i + ys_j) > \rho(G^*).$$

However, this is contradiction, as the connected graph

$$G^* - ys_i + ys_j \cong SSG(n; a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_\gamma)$$

also has domination number γ.

Thus, at most one number among $a_1, \ldots, a_\gamma$ may be larger than one. This shows that $G^* \cong SSG(n; n - 2\gamma + 1, 1, \ldots, 1)$. $\square$

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