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Abstract

This paper deals with the stability analysis and the stabilization for the class of continuous-time systems with time-varying delay. The time delay is assumed to be differentiable with respect to time with finite bound not necessary less one and appear in the state. Delay-dependent sufficient conditions on stability and stabilizability are developed. These conditions use some weighting matrices to reduce the conservatism. A design algorithm for a state feedback controller which guarantees that the closed-loop dynamics will be stable is proposed in terms of the solutions to linear matrix inequalities.

Key Words: Delayed systems; linear matrix inequality (LMI) stability; stabilizability; state feedback.

Résumé

Cet article traite des problèmes de stabilité et de stabilisation de la classe des systèmes continus avec retards variants dans le temps. Les retards sont sur les variables d'états du système. Des conditions suffisantes de stabilité et de stabilisation sont développées. Ces conditions utilisent des matrices appropriées pour réduire le conservatisme. Un algorithme de design pour un contrôleur par retour d'état qui assure la stabilité de la boucle fermée du système est proposé.

1 Introduction

Time delays may be encountered in many practical systems and it is well known that their existence in the dynamics is one of the causes of instability and poor performance degradation. Therefore, analysis and synthesis of systems with time-delay have been and continue to be a hot subject of research. Systems with time-delay have attracted researchers from mathematics and control communities. In the literature, we can find different results on deterministic and stochastic systems with time-delay. For stochastic systems with time-delay, we refer the reader to Mahmoud et al. [8], Boukas and Liu [1, 3, 2] Boukas et al. [6], Shi and Boukas [10], Cao and Lam [5] and the references therein. For deterministic systems, we refer reader to He et al. [7], Chen and Zheng [4] and the references therein.

More recently, we witnessed the development of a new approach for the study of delay-dependent stability conditions by introducing some free weighting matrices to express the links between the terms in the Leibnitz-Newton formula (see Chen and Zheng [4], He et al. [7] and the references therein). This approach has shown less conservatism compared to the other ones that have been proposed in the past. All the results reported in the literature dealt with the stability problem and the one of stabilization (using free weighting matrices) remains an open problem.

This paper deals with the class of continuous-time systems with time-varying delays and focus mainly on the problems of stability analysis and stabilization for this class of systems. In terms of a set of linear matrix inequalities (LMIs), we present first a delay-dependent sufficient condition, which guarantees stability of such systems. Based on this, a delay-dependent sufficient condition for the existence of a state feedback controller ensuring stability of the closed-loop dynamics is proposed. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed methods. Some appropriate weighting matrices are introduced in this paper to reduce the conservatism as it will be shown by the proposed example. Our results will be compared with the ones developed in the recent literature to show that they are less conservative.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is clarified. In Section 3, the main results are given and they include results on stability and stabilizability. A memoryless state feedback controller is used in this paper and a design algorithm in terms of the solutions to linear matrix inequalities is proposed to synthesize the controller gain we are using.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions. L_2 is the space of integral vector over $[0, \infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|$ denotes the L_2-norm over $[0, \infty)$ defined as $\|f\|^2 = \int_0^\infty f^T(t)f(t)\ dt$. We will use $\star$ as an ellipsis for terms that are introduced by symmetric in the LMIs.

2 Problem statement

Consider a continuous-time system with time-varying delay with the following dynamics:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bu(t), \\ x(s) = \phi(s), -h \leq s \leq 0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input system, A, A_d and B are known real matrices with appropriate dimensions, $d(t) > 0$ represents the system delay satisfying $0 \leq d(t) \leq h$, $\dot{d}(t) \leq \mu < \infty$, $\phi(.)$ is the initial functional such that $x(s) = \phi(s) \in L_2[-h, 0] \stackrel{\Delta}{=} \{f(\cdot) | \int_0^\infty f^\top(t) f(t) dt < \infty\}$.

In this paper we are interested in the design of a stabilizing memoryless controller of the following form:

$$u(t) = Kx(t) \tag{2}$$

where K is a design parameter that has to be determined.

Plugging the controller expression (2) in (1) we get the following closed-loop dynamics:

$$\begin{cases} \dot{x}(t) = A_{cl} x(t) + A_d x(t - d(t)) \\ x(s) = \phi(s), -h \leq s \leq 0 \end{cases} \tag{3}$$

where $A_{cl} = A + BK$.

This paper studies the stability and the stabilizability of the class of systems (1). Our goal is to design a state feedback controller guaranteing that the closed-loop is stable using some appropriate weighting matrices to reduce the conservatism. In the rest of this paper, we will assume that all the required assumptions are satisfied, i.e. the complete access to the system state. The conditions we will develop here are in terms of the solutions to linear matrix inequalities that can be easily obtained using LMI control toolbox. These conditions are delay-dependent, which makes them less conservative. And the fact to use the weighting matrices will reduce more the conservatism as it was shown in many studies (see He et al. [7] and the references therein).

Lemma 1 *For any symmetric and positive-definite matrix P and a time-varying delay $h > d(t) > 0$, if there exists a differentiable vector function $x(t)$ with appropriate dimensions such that the integrals $\int_{t-h}^{t} \dot{x}^\top(s) P \dot{x}(s) ds$ and $\int_{t-d(t)}^{t} \dot{x}(s) ds$ are well defined, then we have:*

$$\left[\int_{t-d(t)}^{t} \dot{x}(s) ds\right]^\top P \left[\int_{t-d(t)}^{t} \dot{x}(s) ds\right] \leq h \int_{t-d(t)}^{t} \dot{x}^\top(s) P \dot{x}(s) ds \leq h \int_{t-h}^{t} \dot{x}^\top(s) P \dot{x}(s) ds$$

3 Main results

In this section, firstly we will develop results that assure that the unforced system (i.e. $u(t) = 0$ for all $t \geq 0$) is stable. Then, we will design a memoryless state feedback controller of the form (2) that guarantees the same goal. The following theorem gives the results on the stability of the unforced system (1).

Theorem 2 *The unforced system (1) is stable if there exist a symmetric and positive-definite matrix P, matrices W_1, W_2, W_3, W_4 and symmetric and positive-definite matrices Q, R and S such that the following LMI holds:*

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ \star & M_{22} & M_{23} & M_{24} \\ \star & \star & M_{33} & M_{34} \\ \star & \star & \star & M_{44} \end{bmatrix} < 0. \tag{4}$$

where

$$\begin{aligned}
M_{11} &= A^\top P + P^\top A + Q + R - W_1 - W_1^\top + hA^\top SA, \\
M_{12} &= -W_2 + W_1^\top + PA_d + hA^\top SA_d, \\
M_{13} &= -W_3, \\
M_{14} &= -W_4 + W_1^\top \\
M_{22} &= -(1-\mu)Q + W_2 + W_2^\top + hA_d^\top SA_d, \\
M_{23} &= W_3, \\
M_{24} &= W_4 + W_2^\top, \\
M_{33} &= -R, \\
M_{34} &= W_3^\top, \\
M_{44} &= W_4 + W_4^\top - \frac{1}{h}S.
\end{aligned}$$

Proof. To prove this theorem let us consider the following Lyapunov functional:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t))$$

where

$$\begin{aligned}
V_1(x(t)) &= x^\top(t)Px(t), \\
V_2(x(t)) &= \int_{t-d(t)}^{t} x^\top(s)Qx(s)ds \\
V_3(x(t)) &= \int_{t-h}^{t} x^\top(s)Rx(s)ds,
\end{aligned}$$

$$V_4(x(t)) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)S\dot{x}(s)dsd\theta$$

with $P > 0$, $Q > 0$, $R > 0$ and $S > 0$.

The derivatives of these Lyapunov functionals with respect to time along the solution of the unforced system (1) are given by:

$$\begin{aligned}
\dot{V}_1(x(t)) &= x^\top(t)\left[A^\top P + AP\right]x(t) + 2x^\top(t)PA_dx(t-d(t)) \\
\dot{V}_2(x(t)) &= x^\top(t)Qx(t) - (1-\dot{d}(t))x^\top(t-d(t))Qx(t-d(t)) \\
&\leq x^\top(t)Qx(t) - (1-\mu)x^\top(t-d(t))Qx(t-d(t)) \\
\dot{V}_3(x(t)) &= x^\top(t)Rx(t) - x^\top(t-h)Rx(t-h) \\
\dot{V}_4(x(t)) &= \int_{-h}^{0}\dot{x}^\top(t)S\dot{x}(t)dt - \int_{-h}^{0}\dot{x}^\top(t+\theta)S\dot{x}(t+\theta)d\theta \\
&= h\dot{x}^\top(t)S\dot{x}(t) - \int_{t-h}^{t}\dot{x}(s)S\dot{x}(s)ds \\
&\leq x^\top(t)hA^\top SAx(t) + x^\top(t)hA^\top SA_dx(t-d(t)) \\
&\quad + x^\top(t-d(t))hA_d^\top SAx(t) + x^\top(t-d(t))hA_d^\top SA_dx(t-d(t)) \\
&\quad - \frac{1}{h}\left(\int_{t-d(t)}^{t}\dot{x}^\top(s)ds\right)^\top S\left(\int_{t-d(t)}^{t}\dot{x}^\top(s)ds\right)
\end{aligned}$$

Notice that from Leibnitz-Newton formula, we have:

$$[\Psi(x(t),\dot{x}(t))]^\top \times \left[\int_{t-d(t)}^{t}\dot{x}(s)ds - x(t) + x(t-d(t))\right] = 0$$

$$\left[\int_{t-d(t)}^{t}\dot{x}(s)ds - x(t) + x(t-d(t))\right]^\top \times \Psi(x(t),\dot{x}(t)) = 0$$

with $\Psi(x(t),\dot{x}(t)) = W_1x(t) + W_2x(t-d(t)) + W_3x(t-h) + W_4\int_{t-d(t)}^{t}\dot{x}(s)ds$.

Using all these relations, we get:

$$\begin{aligned}
&\dot{V}(x(t)) \leq x^\top(t)M_{11}x(t) + x^\top(t)M_{12}x(t-d(t)) \\
&+ x^\top(t)M_{13}x(t-h) + x^\top(t)M_{14}\int_{t-d(t)}^{t}\dot{x}(s)ds \\
&+ x^\top(t-d(t))M_{12}^\top x(t) + x^\top(t-d(t))M_{22}x(t-d(t)) \\
&+ x^\top(t-d(t))M_{23}x(t-h) + x^\top(t-d(t))M_{24}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)
\end{aligned}$$

$$
\begin{aligned}
&+ x^\top(t-h)M_{13}^\top x(t) + x^\top(t-h)M_{23}^\top x(t-d(t)) \\
&+ x^\top(t-h)M_{33}x(t-h) + x^\top(t-h)M_{34}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right) \\
&+ \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{14}^\top x(t) + \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{24}^\top x(t-d(t)) \\
&+ \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{34}^\top x(t-h) + \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{44}\int_{t-d(t)}^{t}\dot{x}(s)ds
\end{aligned}
$$

which can be rewritten as follows:

$$
\dot{V}(x(t)) \leq \eta^\top(t)M\eta(t)
$$

where

$$
\eta(t) = \left[x^\top(t)\ x^\top(t-d(t))\ x^\top(t-h)\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top\right]^\top .
$$

Using (4) and following similar steps as in [3], we can deduce that the unforced system (1) is stable. This completes the proof. □

Let us now concentrate on the design of a state feedback controller of the form (2) which guarantees that the closed-loop system will be stable. For this purpose, using the results of Theorem 2, the dynamics (3) will be stable if there exist a symmetric and positive-definite matrix P, matrices W_1, W_2, W_3, W_4 and symmetric and positive-definite matrices Q, R and S such that the LMI (4) holds with A replaced by A_{cl}.

Firstly, notice that $\widetilde{M}$ can be rewritten as follows:

$$
\widetilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} & \tilde{M}_{14} \\ \star & \tilde{M}_{22} & \tilde{M}_{23} & \tilde{M}_{24} \\ \star & \star & \tilde{M}_{33} & \tilde{M}_{34} \\ \star & \star & \star & \tilde{M}_{44} \end{bmatrix} + \begin{bmatrix} A_{cl}^\top \\ A_d^\top \\ 0 \\ 0 \end{bmatrix} [hS] \begin{bmatrix} A_{cl} & A_d & 0 & 0 \end{bmatrix}
$$

with

$$
\begin{aligned}
\tilde{M}_{11} &= A_{cl}^\top P + P^\top A_{cl} + Q + R - W_1 - W_1^\top, \\
\tilde{M}_{12} &= -W_2 + W_1^\top + PA_d, \\
\tilde{M}_{13} &= -W_3 \\
\tilde{M}_{14} &= -W_4 + W_1^\top, \\
\tilde{M}_{22} &= -(1-\mu)Q + W_2 + W_2^\top
\end{aligned}
$$

$$\begin{aligned}
\tilde{M}_{23} &= W_3, \\
\tilde{M}_{24} &= W_4 + W_2^\top, \\
\tilde{M}_{33} &= -R, \\
\tilde{M}_{34} &= W_3^\top, \\
\tilde{M}_{44} &= W_4 + W_4^\top - \frac{1}{h}S.
\end{aligned}$$

If the following holds:

$$hS < \varepsilon P, \varepsilon > 0, \tag{5}$$

$\widetilde{M}$ can be rewritten as follows:

$$\widetilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} & \tilde{M}_{14} & A_{cl}^\top \\ \star & \tilde{M}_{22} & \tilde{M}_{23} & \tilde{M}_{24} & A_d^\top \\ \star & \star & \tilde{M}_{33} & \tilde{M}_{34} & 0 \\ \star & \star & \star & \tilde{M}_{44} & 0 \\ A_{cl} & A_d & 0 & 0 & -\frac{1}{\varepsilon}P^{-1} \end{bmatrix}$$

Let $X = P^{-1}$. Pre- and post-multiply (5) respectively by X, we get:

$$h\bar{S} < \varepsilon X, \varepsilon > 0,$$

where $\bar{S} = XSX$.

Let $Y = KX$, $\bar{Q} = XQX$, $\bar{R} = XRX$, $\bar{W}_1 = XW_1X$, $\bar{W}_2 = XW_2X$, $\bar{W}_3 = XW_3X$, $\bar{W}_4 = XW_4X$. Pre- and post-multiply $\widetilde{M}$ respectively by $\mathrm{diag}(X, X, X, X, \mathbb{I})$, we get the following sufficient condition for the design of the memoryless state feedback controller of the form (2).

Theorem 3 *Let ε be a given positive scalar. There exists a state feedback controller of the form (2) such that the closed-loop system (1) is stable if there exist a symmetric and positive-definite matrix X, matrices $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$, $\bar{W}_4$ and symmetric and positive-definite matrices $\bar{Q}$, $\bar{R}$ and $\bar{S}$ such that the following set of coupled LMIs holds:*

$$h\bar{S} < \varepsilon X, \varepsilon > 0, \tag{6}$$

$$\begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} & \bar{M}_{13} & \bar{M}_{14} & XA^\top + Y^\top B^\top \\ \star & \bar{M}_{22} & \bar{M}_{23} & \bar{M}_{24} & XA_d^\top \\ \star & \star & \bar{M}_{33} & \bar{M}_{34} & 0 \\ \star & \star & \star & \bar{M}_{44} & 0 \\ \star & \star & \star & \star & -\frac{1}{\varepsilon}X \end{bmatrix} < 0. \tag{7}$$

where

$$\begin{aligned}
\bar{M}_{11} &= XA^\top + AX + BY + Y^\top B^\top \\
&\quad + \bar{Q} + \bar{R} - \bar{W}_1 - \bar{W}_1^\top, \\
\bar{M}_{12} &= -\bar{W}_2 + \bar{W}_1^\top + A_d X, \bar{M}_{13} = -\bar{W}_3, \\
\bar{M}_{14} &= -\bar{W}_4 + \bar{W}_1^\top, \bar{M}_{22} = -(1-\mu)\bar{Q} + \bar{W}_2 + \bar{W}_2^\top \\
\bar{M}_{23} &= \bar{W}_3, \bar{M}_{24} = \bar{W}_4 + \bar{W}_2^\top, \bar{M}_{33} = -\bar{R}, \\
\bar{M}_{34} &= \bar{W}_3^\top, \bar{M}_{44} = \bar{W}_4 + \bar{W}_4^\top - \frac{1}{h}\bar{S}.
\end{aligned}$$

The stabilizing memoryless controller gain is given by $K = YX^{-1}$.

4 Numerical examples

To show the less conservatism of our results let us consider that has been used in Park and Ko [9] to compare their results with previous ones. The data of this system are:

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

Using our results of Theorem 2 for different $\mu \in [0, 1)$ we found out that our results are the same of the ones in Park and Ko [9] despite that our Lyapunov functional are different.

To show the validness of our results on stabilizability which is the main goal of this paper, let us consider a system of the class we treating with state in $\mathbb{R}^2$. The data of this system are as follow:

$$A = \begin{bmatrix} -0.5 & -2.0 \\ 1.0 & -1.0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5 & -1.0 \\ 0.0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}.$$

First of all notice that the matrix A of the system is stable. In He et al. [7], this system has been considered to show the validness of their results on robust stability. For the condition, we developed in this paper for stability we get $h = 1.1$ when 0.5.

For the design of the controller, we modify the matrix A to get a non stable one given by:

$$A = \begin{bmatrix} -0.5 & -2.0 \\ 1.0 & 1.0 \end{bmatrix}.$$

Fixing now $\varepsilon = 0.1$, $h = 7.9$, $\mu = 0.5$ and solving the LMIs (6)-(7), we get:

$$X = \begin{bmatrix} 0.4919 & 0.2920 \\ 0.2920 & 0.4872 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.4254 & -2.8263 \end{bmatrix},$$

which gives $K = [4.0026 - 8.2001]$. The other matrices are not of importance to compute the controller gain and we omit to give them.

For this system there is no limit for h to get a state feedback controller.

Remark 4 *We have to keep in mind that our goal in this paper is to develop delay-dependent conditions to design a stabilizing state feedback controller for systems with time-varying delay using the free weighting matrices techniques and this aim has been reached and the results of the example presented in this paper proves that.*

5 Conclusion

This paper dealt with the class of continuous-time linear systems with time-varying delay in the state vector. Results on stability and stabilizability are developed. The LMI framework is used to establish the different results on stability and stabilizability. The conditions we developed are delay-dependent. The results we developed can easily be solved using any LMI toolbox in the marketplace.

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