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Abstract

This paper deals with the class of linear discrete-time systems with random abrupt changes also known as class of Markovian jump singular systems. The problems of stochastic stability and the stochastic stabilization (using state-feedback control and static output feedback control) are tackled. Conditions in the LMI setting to design the appropriate gains of the controllers are developed. It is shown that all the addressed problems can be solved if the corresponding developed linear matrix inequalities (LMIs) are feasible. Numerical examples are employed to show the usefulness of the proposed results.

Key Words: singular systems, descriptor systems, systems with random abrupt changes, discrete-time linear systems, linear matrix inequality, stochastic stability, stochastic stabilizability, state feedback control, static output feedback control.

Résumé

Cet article traite de la classe des systèmes disctets singuliers et à sauts markoviens. Les problèmes de stabilité et de stabilisation (avec retour d'état et retour de sortie statique) sont considérés. Des conditions en forme de LMI pour le design de ces contrôleurs sont développées dont la solution dépend de la faisabilité des LMIs developpées. Des exemples numériques sont utilisés pour montrer l'utilité des résultats dvéloppés.

1 Introduction

In the last decades Markovian jump systems have attracted a lot of researchers from control and operations research communities. This is due to the fact that this class of systems is more appropriate to model some practical systems that we can found in manufacturing systems, power systems, network control systems, etc. More efforts have been done on different subjects related to this class of systems. Almost all the control problems for these systems have been tackled and interesting results have been reported in the literature. For more details on subject we refer the reader to Boukas (2005) for the continuous-time case and Costa *et al.* (2005) for the discrete-time case and the references in these volumes.

In parallel, the class of descriptor systems has also attracted a lot of researchers from mathematics and control communities and interesting results on different control problems have been reported in the literature. Both continuous-time and discrete-time systems have been considered. For more details on what has been done on the subject, we refer the reader to Xu and Lam (2006) and the references.

Recently, the class of descriptor systems with random abrupt changes has also been tackled and few problems has been considered. For more details on what has been done on the subject, we refer the reader to Boukas (2007), Boukas *et al.* (2005) and the references therein.

All the results reported in the literature on the class of Markovian jump systems or even on singular system with Markovian jumps assumed the complete knowledge of the dynamics of the Markov process that describes the switching between the system modes. But practically this is not valid since it is very hard and more expensive to get all the jump rates for the continuous-time case or all the transition probabilities for the discrete-time case, and therefore the results developed earlier can not be applied to practical systems.

More often in the discrete-time case for instance, we have partial knowledge of the transition probabilities with some bounds for few transitions of the system that we can get by doing some experiment on the practical system that we would like to study the stabilization. In this paper we will assume that we have partial knowledge of the transitions and since that all the transition probabilities for a practical system are bounded with finite values which is the case in practice, we will require only the knowledge of an upper bound for the transition probabilities in each mode. The aim of this paper is to revise the stochastic stabilization of the class of singular systems with random abrupt changes and develop new conditions for such problems that require only partial knowledge of the transition probabilities of the Markov chain that describes the switching modes of the systems.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is presented. In Section 3, the main results are given and they include results on stochastic stability and stochastic stabilizability. A state feedback controller and a static output feedback controller are used in this paper and a design algorithms in terms of the solutions to linear matrix inequalities are proposed to synthesize the controllers gains we are using. Section 4 presents numerical examples to show the usefulness of the proposed results.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions.

2 Problem statement

The goal of this paper is to revise the stochastic stability and stochastic stabilizability of the class of descriptor discrete-time linear systems with random abrupt changes.

Let $\{r_k, k \geq 0\}$ be a Markov chain with state space $\mathscr{S} = \{1, \cdots, N\}$ and state transition matrix $\mathbb{P} = [p_{ij}]_{i,j\in\mathscr{S}}$, i.e. the transition probabilities of $\{r_k, k \geq 0\}$ are as follows:

$$\mathbb{P}[r_{k+1} = j | r_k = i] = p_{ij}, \forall i, j \in \mathscr{S}. \tag{1}$$

with $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{N} p_{ij} = 1, \forall i \in \mathscr{S}$.

Consider a discrete-time descriptor Markovian jump system with N modes and suppose that the system mode switching is governed by $\{r_t, t \geq 0\}$. The system is described by:

$$\begin{cases} E(r_{t+1})x_{t+1} = A(r_t)x_t + B(r_t)u_t, \quad x_{s=0} = x_0 \\ y_t = C(r_t)x_t \end{cases} \tag{2}$$

where x_t is the state and u_t is the control input, $A(i)$, $B(i)$ and $C(i)$, for all $i \in \mathscr{S}$ are known real matrices with appropriate dimensions; and $E(i) \in \mathbb{R}^{n\times n}$ is a known singular matrix with rank $(E(i)) = n_r \leq n$ for all $i \in \mathscr{S}$.

In the rest of the paper we will denote by $x(t; x_0, r_0)$, shortened to x_t, the solution of system (2) when the initial conditions are respectively x_0 and r_0.

For the continuous-time Markovian jump systems for instance, the jump rates are in general hard to measure for practical systems and therefore the results developed on stability or stabilization in the literature are in some sense useless. Some alternates that consider uncertainties on the jump rates have been proposed. Among them we quote the work done by Benjelloun and Boukas (1998) where uncertainties on the jump rates are considered and El-Ghaoui and Ait-Rami (1994) where they consider polytopic uncertainties on the transition matrix. In this paper we will assume that we have partial knowledge of the transitions and all the probability transitions are bounded with finite values which is a practical assumption. The following assumption is made in the rest of the paper.

Assumption 2.1 *The transition probabilities are assumed to satisfy the following:*

$$0 < \underline{p}_i \leq p_{ij} \leq \bar{p}_i \leq 1, \forall i, j \in \mathscr{S}, j \neq i \tag{3}$$

where $\underline{p}_i$ and $\bar{p}_i$ are known parameters for each mode or may represent the lower and upper bounds when all the jump rates are known, i.e.:

$$0 < \underline{p}_i = \min\left(p_{i1}, \cdots, p_{iN}\right)$$
$$0 < \bar{p}_i = \max\left(p_{i1}, \cdots, p_{iN}\right)$$

with $\underline{p}_i \leq \bar{p}_i$.

Remark 2.1 *Notice that the assumption is realistic and it is always true that the transitions probabilities, p_{ij} $\forall i, j \in \mathscr{S}$ satisfy the assumption for any practical system.*

Remark 2.2 *The results we are planning to develop in this paper do not require the knowledge of the transition probabilities of the discrete-time system but only an upper bound, $\bar{p}_i$, representing the maximum upper bound for all the transition probabilities in each mode. The results can be developed for only an unique upper bound that represents the maximum upper bound for all the modes. The corresponding results will be restrictive compared to the one we are planning to develop here.*

Assumption 2.2 *The matrix $C(i)$, $\forall i \in \mathscr{S}$ is assumed to be full row rank.*

Remark 2.3 *The fact that the matrix $C(i)$, $\forall i \in \mathscr{S}$ is full row rank, this means that there exists a matrix $D(i)$, $\forall i \in \mathscr{S}$ such that the following holds:*

$$C(i)D(i) = \begin{bmatrix} \mathbb{I} & 0 \end{bmatrix}$$

where $\mathbb{I}$ is the identity matrix with appropriate dimension.

The following definitions will be used in the rest of this paper. For more details on the class of systems with random abrupt changes properties, we refer the reader to Boukas (2007) and the references therein.

Definition 2.1 *System (2) is regular if for any $i \in \mathscr{S}$, $det[zE(i) - A(i)]$ is not identically zero.*

Definition 2.2 *System (2) is causal if for any $i \in \mathscr{S}$, $deg[det[zE(i) - A(i)]] = rank[E(i)]$.*

Definition 2.3 *System (2) is said to be stochastically stable (SS) if the following holds:*

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \|x_k\|^2 | x_0, r_0\right] \leq \Gamma(x_0, r_0),$$

where $\Gamma(x_0, r_0)$ is a non-negative function of the system initial values.

Definition 2.4 *System (2) is said to be stochastically stabilizable if there exists a control law with one of the following forms:*

$$\begin{cases} u(t) = K(r_t)x(t), \ \textit{state feedback controller}, \\ \textit{or} \\ u(t) = K(r_t)y(t), \ \textit{static output feedback controller} \end{cases} \tag{4}$$

with $K(i) \in \mathbb{R}^{m\times n}$, $i \in \mathscr{S}$ is a constant matrix such that the closed-loop system is stochastically stable.

Combining the systems dynamics and the controller expression, we get the following closed-loop dynamics:

$$E(r_{t+1})x_{t+1} = A_{cl}(r_t)x(t), \tag{5}$$

where $A_{cl}(r_t) = A(t_t) + B(r_t)K(r_t)$ for the state feedback control and $A_{cl}(r_t) = A(t_t) + B(r_t)K(r_t)C(r_t)$ for the static output feedback control with $K(r_t)$ is the controller gain that we have to compute.

The goal of this paper is to develop new conditions to check the stochastic stability and to design a state feedback controller or a static output feedback controller that makes the closed-loop system regular, causal and stochastically stable. In the rest of this paper, we will assume the complete access to the system state and mode for feedback. Our methodology in this paper will be mainly based on the Lyapunov theory and some algebraic results. The conditions we will develop here will be in terms of the solutions to linear matrix inequalities that can be easily obtained using LMI control toolbox.

Before closing this section, let us give some lemmas that we will use in our development. The proofs of these lemmas can be found in the cited references.

Lemma 2.1 *Let H, F and G be real matrices of appropriate dimensions with F symmetric and definite-positive then, for any scalar ε we have:*

$$-H^\top G^\top F^{-1}GH \leq \varepsilon GH + \varepsilon H^\top G^\top + \varepsilon^2 F \tag{6}$$

Proof: Notice that:

$$\left[H^\top G^\top + \varepsilon F\right] F^{-1}\left[GH + \varepsilon F\right] \geq 0$$

After expanding this expression we get:

$$H^\top G^\top F^{-1}GH + \varepsilon GH + \varepsilon H^\top G^\top + \varepsilon^2 F \geq 0$$

which gives the relation of the Lemma 2.1. This ends the proof of the lemma. □

Lemma 2.2 *(Boukas and Liu, 2002) The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, H - S^\top R^{-1}S > 0$$

where $H = H^\top$, $R = R^\top$ and S is a constant matrix.

3 Main results

In this section, we will firstly develop results that assure that the free system (i.e. $u_t = 0$ for all $t \geq 0$) is regular, causal and stochastically stable. Then using these results, we will design a state feedback controller and a static output feedback controller of the form (4) that guarantees the same goal.

3.1 Stochastic stability

Based on the results for this class of systems, the free system (2) will be regular, causal and stochastically stable if there exists a set of symmetric and nonsingular matrices $P = (P(1), \cdots, P(N))$ such that the following hold:

$$A^\top(i) \sum_{j=1}^{N} p_{ij} P(j) A(i) - E^\top(i) P(i) E(i) < 0, \forall i \in \mathscr{S}$$

with the following constraints:

$$E^\top(i) P(i) E(i) \geq 0 \tag{7}$$

To remove the last non strict inequality constraints, let us introduce a new matrices $R(i) \in \mathbb{R}^{n \times n}$ that satisfies the following condition:

$$E^\top(i) R^\top(i) = 0, \text{ or } R(i)E(i) = 0.$$

Using this, the conditions for our system to be regular, causal and stochastic stability become:

$$A^\top(i) \left[\sum_{j=1}^{N} p_{ij} P(j) - R^\top(i) S(i) R(i) \right] A(i) - E^\top(i) P(i) E(i) < 0$$

for any symmetric and nonsingular matrix $S(i)$, $\forall i \in \mathscr{S}$.

From the other side since the matrix $\begin{bmatrix} \mathbb{I} & A^\top(i) \end{bmatrix}$ has full row rank, for any nonsingular matrix $Q(i)$, this inequality can be rewritten as follows:

$$\begin{aligned} & A^\top(i) \left[\sum_{j=1}^{N} p_{ij} P(j) - R^\top(i) S(i) R(i) \right] A(i) - E^\top(i) P(i) E(i) \\ & = \begin{bmatrix} \mathbb{I} & A^\top(i) \end{bmatrix} \begin{bmatrix} J_1(i) & A^\top(i) Q(i) - Q^\top(i) \\ Q^\top(i) A(i) - Q(i) & \sum_{j=1}^{N} p_{ij} P(j) - Q(i) - Q^\top(i) \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ A(i) \end{bmatrix} < 0 \end{aligned}$$

with

$$J_1(i) = A^\top(i) Q(i) + Q^\top(i) A(i) - A^\top(i) R^\top(i) S(i) R(i) A(i) - E^\top(i) P(i) E(i)$$

Using now the Assumption 2.1, the system will be regular, causal and stochastically stable if the following holds:

$$\begin{bmatrix} J_1(i) & A^\top(i)Q(i) - Q^\top(i) \\ Q^\top(i)A(i) - Q(i) & \sum_{j=1}^N \bar{p}_i P(j) - Q(i) - Q^\top(i) \end{bmatrix} < 0$$

Let $\bar{P}$ and $\mathscr{W}_i$ be defined as follows:

$$\begin{aligned} \bar{P} &= \mathrm{diag}\left[P(1), \cdots, P(N)\right], \\ \mathscr{W}_i &= \left[\sqrt{\bar{p}_i}\mathbb{I}, \cdots, \sqrt{\bar{p}_i}\mathbb{I}\right] \end{aligned}$$

we get the following set of coupled matrix inequalities:

$$\begin{bmatrix} J_1(i) & A^\top(i)Q(i) - Q^\top(i) & 0 \\ Q^\top(i)A(i) - Q(i) & -Q(i) - Q^\top(i) & \mathscr{W}_i \\ 0 & \mathscr{W}_i^\top & -\bar{P}^{-1} \end{bmatrix} < 0, \forall i \in \mathscr{S}.$$

Let $Z(i) = Q^{-1}(i)$, $X(i) = P^{-1}(i)$, and $W(i) = S^{-1}(i)$ and pre- and post-multiplying the previous inequality respectively by diag $\left[Z^\top(i), Z^\top(i), \mathbb{I}\right]$ and its transpose we get:

$$\begin{bmatrix} J_2(i) & Z^\top(i)A^\top(i) - Z(i) & 0 \\ A(i)Z(i) - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{bmatrix} < 0, \forall i \in \mathscr{S}.$$

with

$$\begin{aligned} J_2(i) &= Z^\top(i)A^\top(i) + A(i)Z(i) - Z^\top(i)A^\top(i)R^\top(i)W^{-1}(i)R(i)A(i)Z(i) \\ &\quad - Z^\top(i)E^\top(i)X^{-1}(i)E(i)Z(i) \\ \mathscr{W}_i &= \left[\begin{array}{cc} \sqrt{\bar{p}_i}\mathbb{I} \cdots & \sqrt{\bar{p}_i}\mathbb{I} \end{array}\right], \\ \mathscr{X}_i(X) &= \mathrm{diag}\left[X(1), \cdots, X(N)\right] \end{aligned}$$

Using now Lemma 2.1, we get for any $\varepsilon(i)$ and any $\beta(i)$:

$$\begin{aligned} -Z^\top(i)E^\top(i)X^{-1}(i)E(i)Z(i) &\le \varepsilon(i)E(i)Z(i) + \varepsilon(i)Z^\top(i)E^\top(i) + \varepsilon^2(i)X(i), \\ -Z^\top(i)A^\top(i)R^\top(i)W^{-1}(i)R(i)A(i)Z(i)& \\ &\le \beta(i)R(i)A(i)Z(i) + \beta(i)Z^\top(i)A^\top(i)R^\top(i) + \beta^2(i)W(i). \end{aligned}$$

Using these inequalities, we get the results of the following theorem that gives the conditions to guarantee that system (2) is regular, causal and stochastically stable.

Theorem 3.1 *The free System (2) (i.e.: $u_t = 0, \forall t \geq 0$) is regular, causal and stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$ and a set of symmetric and nonsingular matrices $W = (W(1), \cdots, W(N))$, and a set of nonsingular matrices $Z = (Z(1), \cdots, Z(N))$ such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} J_3(i) & Z^\top(i)A^\top(i) - Z(i) & 0 \\ A(i)Z(i) - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{bmatrix} < 0, \forall i \in \mathscr{S}. \tag{8}$$

with

$$\begin{aligned} J_3(i) &= Z^\top(i)A^\top(i) + A(i)Z(i) + \varepsilon(i)E(i)Z(i) + \varepsilon(i)Z^\top(i)E^\top(i) + \varepsilon^2(i)X(i) \\ &\quad + \beta(i)Z^\top(i)A^\top(i)R^\top(i) + \beta(i)R(i)A(i)Z(i) + \beta^2(i)W(i) \\ \mathscr{W}_i &= \begin{bmatrix} \sqrt{p_i}\mathbb{I} \cdots & \sqrt{p_i}\mathbb{I} \end{bmatrix}, \\ \mathscr{X}_i(X) &= diag\left[X(1), \cdots, X(N)\right] \end{aligned}$$

3.2 State feedback stabilization

Let us now design a state feedback controller with the following form:

$$u_t = K(r_t)x_t \tag{9}$$

where $K(i)$ is a gain to be determined.

Combining the system's dynamics (2) with the controller (9) expression and using Theorem 3.1, the closed-loop system is regular, causal and stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$ and a set of symmetric and nonsingular matrices $W = (W(1), \cdots, W(N))$, and a set of nonsingular matrices $Z = (Z(1), \cdots, Z(N))$ such that the following set of coupled matrix inequalities holds:

$$\begin{bmatrix} J_3(i) & Z^\top(i)A_{cl}^\top(i) - Z(i) & 0 \\ A_{cl}(i)Z(i) - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{bmatrix} < 0, \forall i \in \mathscr{S}, \tag{10}$$

with

$$\begin{aligned} A_{cl}(i) &= A(i) + B(i)K(i) \\ J_3(i) &= Z^\top(i)A_{cl}^\top(i) + A_{cl}(i)Z(i) + \varepsilon(i)E(i)Z(i) + \varepsilon(i)Z^\top(i)E^\top(i) + \varepsilon^2(i)X(i) \\ &\quad + \beta(i)Z^\top(i)A_{cl}^\top(i)R^\top(i) + \beta(i)R(i)A_{cl}(i)Z(i) + \beta^2(i)W(i) \\ \mathscr{W}_i &= \begin{bmatrix} \sqrt{p_i}\mathbb{I} \cdots & \sqrt{p_i}\mathbb{I} \end{bmatrix}, \\ \mathscr{X}_i(X) &= \mathrm{diag}\left[X(1), \cdots, X(N)\right] \end{aligned}$$

Notice that:

$$A_{cl}(i)Z(i) = [A(i) + B(i)K(i)]\, Z(i) = A(i)Z(i) + B(i)K(i)Z(i)$$

Letting $Y(i) = K(i)Z(i)$ we get the results of the following theorem that allows us to design the state feedback controller gains.

Theorem 3.2 *There exists a state feedback controller of the form (9) such that the closed-loop system is regular, causal and stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$, a set of symmetric and nonsingular matrices $W = (W(1), \cdots, W(N))$, a set of nonsingular matrices $Z = (Z(1), \cdots, Z(N))$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ such that the following set of coupled matrix inequalities holds:*

$$\begin{bmatrix} J_4(i) & [A(i)Z(i) + B(i)Y(i)]^\top - Z(i) & 0 \\ [A(i)Z(i) + B(i)Y(i)] - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{bmatrix} < 0, \forall i \in \mathscr{S}, \quad (11)$$

with

$$\begin{aligned} J_4(i) &= Z^\top(i)A^\top(i) + A(i)Z(i) + B(i)Y(i) + Y^\top(i)B^\top(i) + \varepsilon(i)E(i)Z(i) \\ &\quad + \varepsilon(i)Z^\top(i)E^\top(i) + \varepsilon^2(i)X(i) + \beta(i)\,[A(i)Z(i) + B(i)Y(i)]^\top R^\top(i) \\ &\quad + \beta(i)R(i)\,[A(i)Z(i) + B(i)Y(i)] + \beta^2(i)W(i) \\ \mathscr{W}_i &= \begin{bmatrix} \sqrt{p_i}\mathbb{I} \cdots & \sqrt{p_i}\mathbb{I} \end{bmatrix}, \\ \mathscr{X}_i(X) &= diag\,[X(1), \cdots, X(N)] \end{aligned}$$

3.3 Static output feedback stabilization

Let us now design a static output feedback controller with the following form:

$$u_t = K(r_t)y_t = K(r_t)C(r_t)x_t \quad (12)$$

where $K(i)$ is a gain to be determined.

Using the fact that the matrix $C(i)$ is full row rank, we have:

$$C(i)D(i) = \begin{bmatrix} \mathbb{I} & 0 \end{bmatrix}$$

which implies that:

$$[A(i) + B(i)K(i)C(i)]\, D(i) = A(i)D(i) + B(i) \begin{bmatrix} K(i) & 0 \end{bmatrix}$$

Combining the system's dynamics (2) with the controller (12) expression the closed-loop system is regular, causal and stochastically stable if the following condition (obtained after pre- and post-multiply respectively by $D^\top(i)$ and $D(i)$ the corresponding inequality) holds:

$$\left[A(i)D(i) + B(i)\left[\begin{array}{cc} K(i) & 0 \end{array}\right]\right]^\top \left[\sum_{j=1}^{N} \bar{p}_i P(j) - R^\top(i)S(i)R(i)\right] \left[A(i)D(i) + B(i)\left[\begin{array}{cc} K(i) & 0 \end{array}\right]\right] < 0$$

Notice that the presence of the term $\left[\begin{array}{cc} K(i) & 0 \end{array}\right]$ in the condition requires a special choice for the form of the matrix $Z(i)$ to allow us to determine uniquely the gain $K(i)$. One of the form that may help in this matter is given by the following expression:

$$Z(i) = \left[\begin{array}{cc} Z1(i) & 0 \\ Z_2(i) & Z_3(i) \end{array}\right] \tag{13}$$

Using this expression for $Z(i)$ and based Theorem 3.1, the closed-loop system is regular, causal and stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$, a set of symmetric and nonsingular matrices $W = (W(1), \cdots, W(N))$, and a set of nonsingular matrices $Z = (Z(1), \cdots, Z(N))$ such that the following set of coupled matrix inequalities holds:

$$\left[\begin{array}{ccc} J_3(i) & V^\top(i) - Z(i) & 0 \\ V(i) - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{array}\right] < 0, \forall i \in \mathscr{S}, \tag{14}$$

with

$$\begin{aligned} V(i) &= A(i)D(i)Z(i) + B(i)K(i)\left[\begin{array}{cc} \mathbb{I} & 0 \end{array}\right] Z(i) \\ J_3(i) &= V(i) + V^\top(i) + \varepsilon(i)E(i)D(i)Z(i) \\ &\quad + \varepsilon(i)Z^\top(i)D^\top(i)E^\top(i) + \varepsilon^2(i)X(i) + \beta(i)V^\top(i)R^\top(i) \\ &\quad + \beta(i)R(i)V(i) + \beta^2(i)W(i) \\ \mathscr{W}_i &= \left[\begin{array}{cc} \sqrt{\bar{p}_i}\mathbb{I}\cdots & \sqrt{\bar{p}_i}\mathbb{I} \end{array}\right], \\ \mathscr{X}_i(X) &= \mathrm{diag}\left[X(1), \cdots, X(N)\right] \end{aligned}$$

Notice that:

$$\begin{aligned} V(i) &= A(i)D(i)Z(i) + B(i)K(i)\left[\begin{array}{cc} \mathbb{I} & 0 \end{array}\right] Z(i) \\ &= A(i)D(i)Z(i) + B(i)\left[\begin{array}{cc} K(i)Z_1(i) & 0 \end{array}\right] \end{aligned}$$

Letting $Y(i) = K(i)Z_1(i)$ we get the results of the following theorem that allows us to design the static output feedback controller gains.

Theorem 3.3 *There exists a static output feedback controller of the form (9) such that the closed-loop system is regular, causal and stochastically stable if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$, a set of symmetric and nonsingular of matrices $W = (W(1), \cdots, W(N))$, a set of nonsingular matrices $Z = (Z(1), \cdots, Z(N))$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ such that the following set of coupled LMIs holds:*

$$\begin{bmatrix} J_4(i) & V^\top(i) - Z(i) & 0 \\ V(i) - Z^\top(i) & -Z(i) - Z^\top(i) & Z^\top(i)\mathscr{W}_i \\ 0 & \mathscr{W}_i^\top Z(i) & -\mathscr{X}_i(X) \end{bmatrix} < 0, \forall i \in \mathscr{S}, \tag{15}$$

with

$$\begin{aligned} V(i) &= A(i)D(i)Z(i) + B(i)\begin{bmatrix} Y(i) & 0 \end{bmatrix} \\ J_4(i) &= V(i) + V^\top(i) + \varepsilon(i)E(i)D(i)Z(i) \\ &\quad + \varepsilon(i)Z^\top(i)D^\top(i)E^\top(i) + \varepsilon^2(i)X(i) + \beta(i)V^\top(i)R^\top(i) \\ &\quad + \beta(i)R(i)V(i) + \beta^2(i)W(i) \\ \mathscr{W}_i &= \begin{bmatrix} \sqrt{\overline{p}_i}\mathbb{I} \cdots & \sqrt{\overline{p}_i}\mathbb{I} \end{bmatrix}, \\ \mathscr{X}_i(X) &= diag\left[X(1), \cdots, X(N)\right] \end{aligned}$$

The stabilizing controller gain is given by $K(i) = Y(i)Z_1^{-1}(i)$.

Remark 3.1 *Previously we have developed results on stochastic stability and stochastic stabilization via state feedback and static output feedback controllers when the transitions probabilities are partially known. The developed results are mainly based on the knowledge of the upper bound probability in each mode. This assumption can be relaxed and new results can be obtained by assuming only the knowledge only of an unique upper bound for the all the transitions probabilities. The results obtained based on this assumption present some conservatism compared to those established in this paper. But they remain an alternate when the transitions probabilities are not available.*

4 Numerical examples

In this section, we will give some numerical examples to show that the results we developed either on stochastic stability or stochastic stabilizability are valid. As it was stated in the theory we will assume that we have partial knowledge of the Markov chain $\{r_t, t \geq 0\}$ that describes the switching between the different modes of the systems.

Example 4.1 *To show the validness of stability results, let us consider a two modes Markovian system with states in $\mathbb{R}^2$. The data of this system are as follows:*

- *mode 1:*

$$E(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0.0 & 1.0 \\ 0.1 & -0.2 \end{bmatrix}$$

- *mode 2:*

$$E(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0.0 & 1.0 \\ -0.2 & 0.1 \end{bmatrix}.$$

The switching between the two modes is described by the following:

$$\Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix},$$

From Λ*, we get* $\bar{p}_1 = 0.7$ *and* $\bar{p}_2 = 0.8$*.*

Based on the expressions of $E(1)$ *and* $E(2)$ *we get the following possible values for* $R(1)$ *and* $R(2)$*:*

$$R(1) = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, R(2) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

such that $E^\top(1)R^\top(1) = 0$ *and* $E^\top(2)R^\top(2) = 0$*. Solving the coupled set of LMIs (8) with* $\varepsilon(1) = 2$*,* $\varepsilon(1) = 2$*,* $\beta(1) = -2$ *and* $\beta(2) = -2$*, we get:*

$$X(1) = \begin{bmatrix} 1.1097 & 0.0108 \\ 0.0108 & 1.0837 \end{bmatrix}, \qquad X(2) = \begin{bmatrix} 1.1097 & 0.0108 \\ 0.0108 & 1.0837 \end{bmatrix},$$

$$Z(1) = \begin{bmatrix} 0.3076 & -0.0334 \\ 0.1181 & 0.2363 \end{bmatrix}, \qquad Z(2) = \begin{bmatrix} 0.2984 & -0.0652 \\ 0.1239 & 0.2522 \end{bmatrix},$$

$$W(1) = \begin{bmatrix} -1.7502 & 0.0405 \\ 0.0405 & -1.2491 \end{bmatrix}, \qquad W(2) = \begin{bmatrix} -1.7391 & 0.0901 \\ 0.0901 & -0.8542 \end{bmatrix}.$$

Based on Theorem 3.1, we conclude that the system is regular, causal and stochastically stable.

Example 4.2 *To show the validness of the stabilizability results via a state feedback controller, let us consider a two modes Markovian system with states in* $\mathbb{R}^2$*. The data of this system are as follows:*

- *mode 1:*

$$A(1) = \begin{bmatrix} 0.0 & 1.0 \\ 1.0 & 2.0 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}, \quad E(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- *mode 2:*

$$A(2) = \begin{bmatrix} 0.0 & 1.0 \\ 2.0 & 1.0 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.0 \\ 2.0 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The switching between the two modes is described by the following:

$$\Lambda = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix},$$

First of all notice that the system in each mode is not stable since the eigenvalues of $A(1)$ *and* $A(2)$ *are all outside the unit circle. It can be checked even that the all system is stochastically unstable. From* Λ, *we get* $\bar{p}_1 = 0.6$ *and* $\bar{p}_2 = 0.5$.

Solving the coupled set of LMIs (11) with $\varepsilon(1) = 2$, $\varepsilon(1) = 2$, $\beta(1) = -2$ *and* $\beta(2) = -2$, *and the same values for* $R(1)$ *and* $R(2)$ *as in the previous example, we get:*

$$\begin{aligned}
X(1) &= \begin{bmatrix} 1.0938 & 0.0215 \\ 0.0215 & 1.0871 \end{bmatrix}, & X(2) &= \begin{bmatrix} 1.0938 & 0.0215 \\ 0.0215 & 1.0871 \end{bmatrix}, \\
Y(1) &= \begin{bmatrix} -0.7983 & -0.2383 \end{bmatrix}, & Y(2) &= \begin{bmatrix} -0.5202 & 0.1031 \end{bmatrix}, \\
Z(1) &= \begin{bmatrix} 0.3410 & -0.0711 \\ 0.1827 & 0.3283 \end{bmatrix}, & Z(2) &= \begin{bmatrix} 0.3599 & -0.0940 \\ 0.2070 & 0.3482 \end{bmatrix}, \\
W(1) &= \begin{bmatrix} -1.7884 & -0.0346 \\ -0.0346 & -0.4977 \end{bmatrix}, & W(2) &= \begin{bmatrix} -1.8172 & 0.1261 \\ 0.1261 & -0.4656 \end{bmatrix},
\end{aligned}$$

which gives the following gains for the state-feedback controller:

$$\begin{aligned}
K(1) &= \begin{bmatrix} -1.7488 & -1.1046 \end{bmatrix}, \\
K(2) &= \begin{bmatrix} -1.3987 & -0.0814 \end{bmatrix}.
\end{aligned}$$

Based on Theorem 3.2, we conclude that the closed-loop system is regular, causal and stochastically stable under the state-feedback controller with the set of computed gains.

Example 4.3 *To show the validness of the stabilizability results for the discrete-time case via static output feedback controller, let us consider the two modes Markovian system with states in* $\mathbb{R}^2$ *of the previous example with the following extra data* $C(1) = \begin{bmatrix} 1 & 0.1 \end{bmatrix}$ *and* $C(2) = \begin{bmatrix} 1 & 0.1 \end{bmatrix}$.

Solving the coupled set of LMIs (15) with $\varepsilon(1) = 2$, $\varepsilon(1) = 2$, $\beta(1) = -2$ *and* $\beta(2) = -2$, *and the same values for* $R(1)$ *and* $R(2)$ *as in the previous example, we get:*

$$\begin{aligned}
X(1) &= \begin{bmatrix} 1.0428 & 0.0127 \\ 0.0127 & 1.0372 \end{bmatrix}, & X(2) &= \begin{bmatrix} 1.0428 & 0.0127 \\ 0.0127 & 1.0372 \end{bmatrix}, \\
Y(1) &= \begin{bmatrix} -0.9801 \end{bmatrix}, & Y(2) &= \begin{bmatrix} -0.4708 \end{bmatrix}, \\
Z(1) &= \begin{bmatrix} 0.4570 & 0.0 \\ 0.2308 & 0.4138 \end{bmatrix}, & Z(2) &= \begin{bmatrix} 0.3620 & 0.0 \\ 0.1788 & 0.3421 \end{bmatrix}, \\
W(1) &= \begin{bmatrix} -1.8781 & -0.0373 \\ -0.0373 & 0.3164 \end{bmatrix}, & W(2) &= \begin{bmatrix} -1.7595 & 0.0843 \\ 0.0843 & -0.4445 \end{bmatrix},
\end{aligned}$$

which gives the following gains for the state-feedback controller:

$$K(1) = \left[\begin{array}{c} -2.1444 \end{array}\right],$$
$$K(2) = \left[\begin{array}{c} -1.3006 \end{array}\right].$$

Based on Theorem 3.3, we conclude that the closed-loop system is regular, causal and stochastically stable under the static output feedback controller with the set of computed gains.

5 Conclusions

This paper dealt with the stochastic stability and stochastic stabilization of the class of linear systems with random abrupt changes. Under partial knowledge of the transitions between the system's modes, LMI conditions for stochastic stability and stochastic stabilization have been developed. It is shown that all the addressed problems can be solved if the corresponding developed linear matrix inequalities (LMIs) are feasible. The results we developed can be extended easily for other classes of systems like systems with time-delay and for other type of controllers.

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