Les Cahiers du GERAD ISSN: 0711–2440

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G–2004–57

July 2004

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July, 2004

Les Cahiers du GERAD

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Abstract

Two methods of analytical approximations for computing the value of a European option on the conditional variance in a GARCH setting are presented. The first is based on the Johnson density functions family while the second uses the generalized Edgeworth expansion of the conditional variance's risk-neutral density function. The analytical approximations based on the lognormal system of Johnson densities is found to be the most accurate one developed in this paper. However, the precision of the approximation is weaker for deep out-of-the-money options when the conditional variance dynamics of the NGARCH process displays a high level of persistence or almost no variability. When the moments of the conditional-variance dynamics display explosive behaviour, the precision of the approximation is generally good, except for the case of long-maturity options. The numerical analysis also suggests that the error term associated with the generalized Edgeworth expansion will have a critical impact when the skewness and kurtosis to be reproduced deviate, even slightly, from those of the approximate distribution used in the expansion. The proposed methodology can easily be adapted to other GARCH processes and generalized to the case of a volatility option.

Résumé

Nous étudions deux méthodes d'approximations analytiques pour le calcul de la valeur d'une option d'achat de type européenne sur la variance, et ce dans un contexte GARCH. La première méthode est basée sur les fonctions de densité des familles de Johnson, tandis que la seconde utilise le développement généralisé de Edgeworth de la densité de la variance conditionnelle sous la mesure martingale. Nous montrons, pour le modèle NGARCH, que les approximations analytiques dérivées du système lognormal de Johnson donnent les meilleurs résultats. La qualité de l'approximation est cependant moindre lorsque l'option d'achat à évaluer est hors-jeu et que la dynamique de la variance conditionnelle du modèle NGARCH démontre un fort niveau de persistance. Lorsque la dynamique des moments de la variance n'est pas stationnaire, la précision de l'approximation est généralement bonne, à l'exception du cas des options de longues échéances. L'analyse des résultats permet également de constater que le terme d'erreur lié au développement généralisé de Edgeworth est non négligeable lorsque le niveau des coefficients d'asymétrie et d'aplatissement à reproduire est éloigné de ceux de la distribution approximative utilisée. La méthodologie proposée peut facilement être adaptée à d'autres modèles GARCH et peut aussi être généralisée au cas d'options européennes portant sur la volatilité.

Acknowledgments: Gauthier and Rémillard acknowledge support from the National Science and Engineering Research Council of Canada (NSERC), from of the Fonds québécois de recherche sur la nature et les technologies (FQRNT), and from the Institut de Finance Mathématique de Montréal (IFM2). Gauthier also acknowledges support from the Social Sciences and Humanities Research Council of Canada (SSHRC), and Turcotte also acknowledges support from the Institut de Finance Mathématique de Montréal (IFM2).

1 Introduction

In the last decade, there has been a significant increase in the trading of financial products involving exposure to the volatility of financial assets. Events leading to the collapse of Long Term Capital Management pinpointed the risks associated to the movements of volatility. Despite these events, until recently[1], the tools available to cover such volatility risks were hardly available. A possible explanation for this state of affairs is the lack of a simple, flexible, and efficient pricing framework to facilitate the introduction of volatility derivatives as instruments of risk management.

To our knowledge, Brenner & Galai (1989) were the first to tackle the theme of volatility derivatives. They present, qualitatively, a variety of examples showing how an investor might use volatility derivatives either for purposes of speculation or risk hedging. Along the same lines, Whaley (1993) shows that strategies for hedging volatility risks which rely on the dynamic management of a portfolio of options could be replaced by the purchase of volatility derivatives. Neuberger (1994) shows that volatility risk is the primary source of residual risk for an option hedged against the price variations in an underlying asset. Using simulations from historical data, he concludes that it is possible to improve option-hedging strategies by including volatility derivatives. These various empirical observations have engendered the development of a theoretical framework for pricing and hedging variance and volatility derivatives. The approaches proposed to date can be grouped in two categories. The first category includes approaches based on static replication of traded securities and the second category contains models keyed to market equilibrium.

In Carr & Madan (1997), the authors propose implementing a strategy allowing the linear reproduction of the variance in returns on an asset over a given space of time. Assuming the existence of options for a continuum of strike prices, it is possible to reproduce any twice continuously differentiable function of the value of a forward contract, by means of a static portfolio composed of the options and the forward contract. Since the variance is a function of the price trajectory, they establish a strategy to create an exposure to the variance of the asset's return. An intuitive explanation and an example of this strategy are presented in Demeterfi, Derman, Kamal & Zou (1999). Approaches based on replication essentially rely on the usual "no arbitrage" argument and do not require any specification of the dynamic behavior of volatility. However, these approaches are limited to the class of linear products based on variance of the return. Moreover, as Carr & Madan (1997) point out, the hypothesis of the existence of options for a continuum of strike prices seems to be very restrictive.

Grünbichler & Longstaff (1996) propose analytical solutions for evaluating forward contracts and options on the volatility by assuming that the process governing its dynamic is the square root process. Given the mean reverting dynamics captured by the square root process, the long-term volatility will concentrate around the average volatility. Thus the

[1]In March 2004, the Chicago Board Of Exchange (CBOE) introduced a future contract based on the VIX index. The VIX Index is an indicator of market expectations of short-term volatility based on S&P 500 stock index option prices.

value of volatility derivatives with long-term maturities are less sensitive to fluctuations in the level of volatility and this makes them less suitable for hedging against short-term variations in volatility. Detemple & Osakwe (2000) generalize the work of Grünbichler & Longstaff (1996) by extending their analytical framework to other stochastic processes and by adapting the results to American options. The general equilibrium model offers additional flexibility for pricing derivative having non-linear payoffs of the variance or the volatility.

Heston & Nandi (2000) propose an analytical solution to the problem of evaluating a variance option, a solution which nests in a specific market model in the GARCH family. The analytical framework is generalized to the diffusion process corresponding to the continuous-time limit of the discrete model used. In both cases, the solution depends on the inversion of the characteristic function of the future conditional variance. The continuous-time scenario uses a replication strategy capable of reproducing the value of the derivative's variations. This replication strategy works through dynamic adjustments of a portfolio composed of traded financial instruments. The pricing framework proposed by Heston & Nandi (2000) produces analytical solutions for a general class of variance derivatives while also providing a replication strategy that does not rely on the restrictive hypothesis of a continuum of exercise prices for options. It is, however, impossible to adapt the pricing framework proposed by Heston & Nandi (2000) to derivatives whose underlying variable depends on the volatility instead of variance. The discrete model used is a hybrid GARCH-family model and it is not possible to adapt the methodology to other GARCH-family models.

As Brockhaus & Long (2000) explain, financial institutions propose linear derivatives based on variance whereas investors are more interested in a general class of derivative products linked to volatility. This situation is partially explained by the absence of a simple, flexible, and efficient pricing framework enabling financial institutions to evaluate and replicate this type of product and then offer it to their clients. This article proposes an approach nested in GARCH-type processes that is designed to price a vast array of derivatives linked to variance and, as an extension, to volatility. Applying the NGARCH model proposed by Engle & Ng (1993), we propose two analytical approximations capable of effectively evaluating this type of derivatives. The results obtained can easily be generalized to other GARCH-type models such as the EGARCH and GJR-GARCH processes. The first approximation uses Johnson density functions in the same fashion Posner & Milevsky (1998) used to price basket and Asian options. The second is based on the generalized Edgeworth expansion similar to the one presented by Jarrow & Rudd (1982).

Characteristics of the NGARCH market model are studied in Section 2. The analytical approximations developed are presented in Section 3. A numerical study allowing comparison of the various analytical approximations proposed is carried out in Section 4. In Section 5, the adjustments needed to adapt the analytical approximations to other GARCH-family models and to generalize the approach to the pricing of securities tied to the volatility of asset returns are explained. Concluding remarks are presented at Section 6.

2 NGARCH market model

The Non-linear Asymmetric GARCH (NGARCH) proposed by Engle and Ng (1993) is presented in this section. The reasons evoked in Duan, Gauthier & Simonato (1999) motivated our choice. Among other things, the NGARCH model permits a adequate reproduction of the dynamics of financial series and, under the risk-neutral measure, it proves to be the most parsimonious of the GARCH-family models capturing asymmetric effects. This characteristic makes the NGARCH model an interesting tool for modelling of returns and for pricing of derivatives.

According to the NGARCH model, the dynamics of the return on a risky asset, under the real probability measure P, is given by

$$\ln \frac{S_{t+1}}{S_t} = r_t + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \quad (1)$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\varepsilon_t - \theta)^2 \quad (2)$$

where r_t is the one-period continuously compound return on a risk-free asset, λ is a constant risk premium, h_{t+1} is the conditional variance of the risky asset's return, and $\{\varepsilon_t : t = 0, 1, 2, \ldots\}$, is a series of independent standard Gaussian variables. Subject to the restrictions $\beta_0 > 0$, $\beta_1 \geq 0$ and $\beta_2 \geq 0$, equation (2) describes the dynamics of the conditional variance of an NGARCH process. Parameter θ is used to capture the relationship between the level of conditional variance and the return on the risky asset; A positive value (respectively negative) of θ implies a negative (respectively positive) relation between the return and the conditional variance. The NGARCH model is thus capable of modelling the asymmetric effects in the distribution of the risky asset's return. The information structure is given by the filtration $\{\mathcal{F}_t : t \in \{0, 1, 2, \ldots\}\}$ where $\mathcal{F}_0$ is the $\sigma-$field generated by (S_0,h_1) and $\mathcal{F}_t$ is the $\sigma-$field generated by $\{S_0, h_1, \varepsilon_s : s \in \{1, 2, \ldots, t\}\}$.

In Duan (1995), pricing derivatives in a GARCH-model setting is made possible by switching from a GARCH-family model under the P measure to what he calls a "locally risk-neutral measure" Q. Under this measure, the dynamics of the return of a risky asset becomes

$$\ln \frac{S_{t+1}}{S_t} = r_t - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1} \quad (3)$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\epsilon_t - \theta - \lambda)^2. \quad (4)$$

Note that under measure Q, $\varepsilon_{t+1} = \epsilon_{t+1} - \lambda$ where $\{\epsilon_t : t = 0, 1, 2, \ldots\}$ is a series of independent standard Gaussian variables.

In order to establish the value of a variance option in an NGARCH setting, it is necessary to characterize the distribution of the future conditional variance h_{t+s} under the locally risk neutral measure, where $t \geq 0$ denotes the present moment and $s > 0$ represents the number of periods characterizing the option's maturity. Under measure Q, h_{t+s} can be expressed as

$$
\begin{aligned}
h_{t+s} &= \beta_0 + h_t \prod_{i=0}^{s-1} Y_{t+i} + \beta_0 \sum_{i=1}^{s-1} \prod_{j=1}^{i} Y_{t+s-j} && (5)\\
Y_{t+i} &= \beta_1 + \beta_2(\epsilon_{t+i} - \theta - \lambda)^2, i \geq 0. && (6)
\end{aligned}
$$

This is easily proven by induction. For more details, see Duan, Gauthier & Simonato (1999). The future conditional variance is expressed as a sum of products of random terms having noncentral chi-square distribution. To our knowledge, it is impossible to identify the exact density function of such an expression and this precludes any analytical solution to the problem of pricing variance derivatives in an NGARCH setting. However, the moments of the future conditional variance can be identified, so it is possible to resolve the pricing problem with an analytical approximation, either based on the generalized Gram-Charlier expansion or using calibration to the Johnson family of distribution. In the case of derivative contracts written on the volatility, we need to determine the first moments of $\sqrt{h_{t+s}}$.

It is worth noting that for all t, $Y_t \geq \beta_1$, so the conditional variance is bounded by :

$$
h_{t+s} \geq \beta_0 + h_t \prod_{i=0}^{s-1} \beta_1 + \beta_0 \sum_{i=1}^{s-1} \prod_{j=1}^{i} \beta_1 = \beta_0 \frac{1-\beta_1^s}{1-\beta_1} + h_t \beta_1^s. \qquad (7)
$$

2.1 Conditional moments of h_{t+s}

Let $E_t^Q[\cdot]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_t$, under Q. Since $h_{t+1} = \beta_0 + h_t Y_t \in \mathcal{F}_t$, it follows that $\mathrm{E}_t^Q\left[h_{t+1}^n\right] = h_{t+1}^n$.

The evaluation of the n-th conditional moment of h_{t+s} is obtained using a recursive application of the following lemma, the proof of which is given in Appendix A.

Lemma 1 *For any positive integer n, and for any integer $s \geq 2$,*

$$
\mathrm{E}_t^Q\left[h_{t+s}^n\right] = \sum_{k=0}^{n} \binom{n}{k} \beta_0^{n-k} \nu_k \mathrm{E}_t^Q\left[h_{t+s-1}^k\right] \qquad (8)
$$

where

$$
\begin{aligned}
\nu_k &= \mathrm{E}^Q\left[Y_t^k\right] = \sum_{j=0}^{k} \binom{k}{j} \beta_1^{k-j} \beta_2^j \eta_j,\\
\eta_j &= \mathrm{E}^Q\left[(\epsilon_t - \theta - \lambda)^{2j}\right] = \sum_{i=0}^{j} \binom{2j}{2i} (\theta+\lambda)^{2(j-i)} \frac{(2i)!}{2^i i!}.
\end{aligned}
$$

The constants $\nu_k,\ k \geq 1$ determine the asymptotic behaviour of the conditional variance of the risky asset's return. In order to make sure that the first n integer moments of the conditional variance under the Q measure will not explode but rather converge towards a stationary finite value when $s \to \infty$, then, as the following lemma shows, the NGARCH-model parameters must test the following conditions: $\nu_k < 1$ for all $k = 1, 2, ..., n$. Failing to respect these conditions for one or more of the moments is to risk lowering the quality of the analytical approximations based on the generalized Edgeworth expansion or on Johnson's density functions.

Lemma 2 *Let $n = \inf\{k \geq 1 : \nu_k \geq 1\}$. Then for all $0 \leq k < n$,*

$$\lim_{s\to\infty} E_t^Q \left[h_{t+s}^k\right] = \mu_k$$

where $\mu_0 = 1$ and

$$\mu_k = \frac{1}{1-\nu_k} \sum_{j=0}^{k-1} \binom{k}{j} \beta_0^{k-1-j} \nu_j \mu_j, \quad 1 \leq k < n. \tag{9}$$

Moreover, if n is finite, then

$$\lim_{s\to\infty} E_t^Q \left[h_{t+s}^n\right] = +\infty.$$

The proof of Lemma 2 is also given in Appendix A.

Next, one can also ask whether or not there exists a stationary distribution. It is easy to check that if $E[\log(Y_1)] < 0$, which holds true if $\nu_1 < 1$ (using Jensen's inequality), then for any fixed t, the strong law of large number implies that

$$\lim_{s\to\infty} \frac{1}{s} \sum_{i=0}^{s-1} \log(Y_{t+i}) = E[\log(Y_1)] < 0 \text{ a.s.}$$

Hence, for any $E[\log(Y_1)] < r < 0$, taking the exponential on both sides of the last expression yields $\prod_{i=0}^{s-1} Y_{t+i} < r^s$, for s large enough (depending on the trajectory), proving that $\prod_{i=0}^{s-1} Y_{t+i}$ converges to zero almost surely, as s tends to infinity.

It follows from (5) that when s is large, the behaviour of h_{t+s} is the same as the behaviour of

$$\check{h}_{t+s} = \beta_0 \left(1 + \sum_{i=1}^{s-1} \prod_{j=1}^{i} Y_{t+s-j}\right),$$

which in turn has the same law as

$$\tilde{h}_{s-1} = \beta_0 \left(1 + \sum_{i=1}^{s-1} \prod_{j=1}^{i} Y_j\right).$$

It follows from Vervaat (1979) that the condition $E[\log(Y_1)] < 0$ is a sufficient condition for the existence and the uniqueness of the limiting distribution of h_{t+s} and the almost sure convergence of $\tilde{h}_s$ to $\tilde{h}$, having representation

$$\tilde{h} = \beta_0 \sum_{i=0}^{\infty} \prod_{j=1}^{i} Y_j, \tag{10}$$

It follows that the law of $\tilde{h}$ is a stationary distribution for h_t, in the sense that if h_t and $\tilde{h}$ have the same law, denoted by $h_t \overset{d}{=} \tilde{h}$, then for any $s \geq 1$, $h_{t+s} \overset{d}{=} h_t$. In fact, because of the almost sure convergence of $\tilde{h}_s$ to $\tilde{h}$, it is the unique stationary law. Note that β_0 appears naturally as a scaling parameter of the unique stationary distribution. It is easy to see that $\tilde{h}$ satisfies the functional equation

$$\tilde{h} \overset{d}{=} \beta_0 + Y_1 \tilde{h}.$$

Unfortunately, the law of $\tilde{h}$ is not known in our case, that is when Y_1 has an affine noncentral chi-square distribution. However, some solutions of the functional equation are known. For more details, see Dufresne (1996) and references therein.

In the case where we are interested by matching the moments of the volatility in the NGARCH setting, on could approximate the fractional moment of h_{t+s} using the method proposed in Duan, Gauthier & Simonato (1999). The proof is based on the Taylor expansion of $h_{s+t}^{\frac{p}{2}}$ around $\mathrm{E}_t^Q[h_{s+t}]$. However, in this particular case, based on intensive simulations, the latter approximation was found to be unsatisfactory even for $p = 1$.

Lemma 3 *For any positive integer p,*

$$\mathrm{E}_t^Q\left[h_{s+t}^{\frac{p}{2}}\right] \cong \left(\frac{p^2}{8} - \frac{p}{4}\right)\left(\mathrm{E}_t^Q[h_{s+t}]\right)^{\frac{p}{2}-2} \mathrm{E}_t^Q\left[h_{s+t}^2\right] + \left(1 + \frac{p}{4} - \frac{p^2}{8}\right)\left(\mathrm{E}_t^Q[h_{s+t}]\right)^{\frac{p}{2}}. \tag{11}$$

3 Analytical approximations

This section presents four analytical approximations allowing the evaluation of a European call option and forward contract on the future conditional variance h_{t+s} and, generally, on any random variable whose first four moments can be identified under a risk neutral measure. The first two approximations depend respectively on systems S_U and S_L of the Johnson distribution family, whereas the third, based on a lognormal distribution, uses the generalized Edgeworth expansion. Combining these two approaches, the last approximation is obtained using a generalized Edgeworth expansion based on the S_L system of the Johnson family.

Equation (7) can be used to identify the minimal value of the future conditional variance. This is important because both the S_L system of the Johnson family and the generalized

Edgeworth expansion based on a lognormal distribution are bounded at zero and, consequently, the random variable whose moments are to be reproduced will, in both cases, be :

$$h^*_{t+s} = h_{t+s} - \left(\beta_0 \frac{1-\beta_1^s}{1-\beta_1} + h_t \beta_1^s\right) \tag{12}$$

This adjustment will make it possible to reproduce the minimal value[2] of the distribution of h_{t+s} as well as the non-centered moments.

3.1 Johnson families of distributions

Johnson (1949) presents families of density functions (the lognormal (S_L) system, the unbounded (S_U) system, and the bounded (S_B) system), which are continuous transformations of a standard normal variable. These families are characterized by four parameters that can be determined via moment matching for example. The Johnson density functions are used to approximate various non-Gaussian distributions. This approach takes care of two problems associated with the generalized Edgeworth expansion, in that the approximation obtained by the method proposed by Johnson (1949) really yields a density function and, with the exception of the system S_L[3], it permits the exact matching of the first four moments of the distribution to be approximated.

Hill, Hill & Holder (1976) propose an algorithm capable of determining which of Johnson's system should be used in setting the parameters of the density function chosen such that their moments will correspond to the values of the targeted unknown distribution.

3.1.1 Approximation based on the S_U system The unbounded system (S_U) is characterized by the non-linear transformation :

$$S_U = \psi_{S_U}(Z) \tag{13}$$

where $\psi_{S_U}(Z) = a + b \sinh\left(\frac{Z-c}{d}\right)$ and Z is a standard normal variable. If the parameters of the vector $\mathbf{\Theta} = [a\ b\ c\ d]'$ are set so that the moments of the random variable S_U are equivalent to those of the future conditional variance h_{t+s}, using formula (8), the value of a forward contract on the future conditional variance h_{t+s} is given by:

$$F_{S_U}(s; \mathbf{\Theta}) = a - b e^{\frac{1}{2d^2}} \sinh\left(\frac{c}{d}\right). \tag{14}$$

Furthermore, the value of a European call option whose exercise price is K and which matures in s periods can be approximated by the expression :

$$C_{S_U}(K, s; \mathbf{\Theta}) = e^{-rs} \left(a - b e^{\frac{1}{2d^2}} \sinh\left(\frac{c}{d}\right) - K\right) \tag{15}$$

[2] The exception being the case of the S_U system of the Johnson distributions.

[3] The S_L system allows the calibration of the three first moments of the targeted distribution.

$$+\frac{be^{-rs}e^{\frac{1}{2d^2}}}{2}\left(e^{\frac{c}{d}}N\left(\kappa+\frac{1}{d}\right)-e^{\frac{-c}{d}}N\left(\kappa-\frac{1}{d}\right)\right)+e^{-rs}(K-a)N(\kappa)$$

where $\kappa = c+d\ln\left(\frac{K-a}{b}+\sqrt{\left(\frac{K-a}{b}\right)^2+1}\right)$ and $N(\cdot)$ is the cumulative distribution function of a standard normal variable.

Note that the initial level of the future conditional variance h_{t+1} does not appear explicitly in the call option's evaluation formula, but nevertheless does influence the value of the moments of the future conditional variance. The effect of h_{t+1} is reflected implicitly in the value of $\mathbf{\Theta}$.

Since the level of a forward contract corresponds to the expected value under the risk-neutral measure of the underlying random variable, the equation (14) giving the value of the forward contract follows directly from the results of moments of S_U, as derived in Section B. Next, the derivation of the analytical approximation (15) is presented in Appendix B.2.

3.1.2 Approximation based on the S_L system The lognormal (S_L) system is characterized by the non-linear transformation :

$$S_L = \psi_{S_L}(Z) \tag{16}$$

where $\psi_{S_L}(Z) = a + b\exp\left(\frac{Z-c}{d}\right) = a + b'\exp\left(\frac{Z}{d}\right)$, where Z is a standard normal random variable, and $b' = b\exp\left(-\frac{c}{d}\right)$.

Since the density really depends on three parameters a, b', d, the S_L system does permit the reproduction of only the first three moments of the future conditional variance. Moreover, the density puts no mass on $(-\infty, a)$. If the parameters of the vector $\mathbf{\Theta} = [a\ b\ c\ d]'$ are chosen so that the first two moments of the random variable $S_L - a$ are equivalent to those of the translated future conditional variable h^*_{t+s}, the level of a forward contract on the future conditional variance h_{t+s} is given by :

$$F_{S_L}(s;\mathbf{\Theta}) = a + b\exp\left(\frac{1}{2d^2}-\frac{c}{d}\right). \tag{17}$$

Furthermore, the value of a European call option whose exercise price is K and which matures in s periods can be approximated by the expression :

$$\begin{aligned}C_{S_L}(K,s;\mathbf{\Theta}) &= e^{-rs}\left(a+b\exp\left(\frac{1}{2d^2}-\frac{c}{d}\right)-K\right) \\ &+e^{-rs}\left((K-a)N\left(\psi^{-1}_{S_L}(K)\right)-b\exp\left(\frac{1}{2d^2}-\frac{c}{d}\right)N\left(\psi^{-1}_{S_L}(K)-\frac{1}{d}\right)\right).\end{aligned} \tag{18}$$

As for the analytical approximation based on the S_U system, the effect of the initial level of the future conditional variance is hidden in $\mathbf{\Theta}$. The proof of the two analytical approximations (16-18) is very similar to those presented for the S_U system and are omitted.

3.2 Generalized Edgeworth expansion

The next method of approximation is called the "Gram-Charlier" approximation in the statistical literature. The Edgeworth expansion can be seen as a special case, and usually refers to the approximation of the density of normalized sums of the form $Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$, where the terms in the expansion of the density are grouped according to the powers of $n^{-1/2}$.

3.2.1 Approximation based on the lognormal distribution In the expression characterizing the value of a European call option, the first analytical approximation based on the generalized Edgeworth expansion replaces the density function of the underlying random variable by an approximation obtained via an Edgeworth type expansion around a lognormal density function characterized by the vector of parameters $\mathbf{\Theta} = [\alpha, \beta]'$. Denoting the lognormal density function used in the expansion by g_L and setting $\mu_{i,t} = \mathrm{E}_t^Q \left[h_{t+s}^i\right]$, we find

$$g_L(y) = \frac{1}{\sqrt{2\pi} y\beta} \exp\left(-\frac{1}{2\beta^2} \left(\log(y) - \alpha\right)^2\right), \tag{19}$$

where

$$\begin{aligned} \alpha &= 2\log(\mu_{1,t}) - \frac{1}{2}\log(\mu_{2,t}), \\ \beta &= \sqrt{\log(\mu_{2,t}) - 2\log(\mu_{1,t})}. \end{aligned}$$

Using a generalized Edgeworth expansion based on g_L, the value of a forward contract on the future conditional variance h_{t+s} is given by

$$F_{E_L}(s; \mathbf{\Theta}_L) = \exp\left(\alpha + \frac{\beta^2}{2}\right). \tag{20}$$

Furthermore, the value of a European call option, whose exercise price is K and which matures in s periods, is given by the expression :

$$\begin{aligned} C_{E_L}(K, s; \mathbf{\Theta}_L) &= e^{-rs}\left(e^{\alpha + \frac{\beta^2}{2}} N(d_1) - K\, N(d_2)\right) - \frac{e^{-rs}(\kappa_3 - \tilde{\kappa}_3)}{3!} \frac{dg_L(K)}{dy} \\ &\quad + \frac{e^{-rs}(\kappa_4 - \tilde{\kappa}_4)}{4!} \frac{d^2 g_L(K)}{dy^2} \end{aligned} \tag{21}$$

where

$$\left.\begin{aligned} d_2 &= \tfrac{\log(1/K) + \alpha}{\beta}, \\ d_1 &= d_2 + \beta, \\ \tilde{\kappa}_3 &= e^{(3\alpha + \frac{3\beta^2}{2})}(e^{\beta^2} - 1)^2 (e^{\beta^2} + 2), \\ \tilde{\kappa}_4 &= e^{(4\alpha + 2\beta^2)}(e^{\beta^2} - 1)^2 (e^{4\beta^2} + 2e^{3\beta^2} + 3e^{2\beta^2} - 6), \\ \kappa_3 &= 2\mu_{1,t}^3 - 3\mu_{1,t}\mu_{2,t} + \mu_{3,t}, \\ \kappa_4 &= -6\mu_{1,t}^4 + 12\mu_{1,t}^2\mu_{2,t} - 3\mu_{2,t}^2 - 4\mu_{1,t}\mu_{3,t} + \mu_{4,t}. \end{aligned}\right\} \tag{22}$$

Calculations leading to these two expressions are performed in Section C.1.

3.2.2 Approximation based on the S_L system The second analytical approximation based on the generalized Edgeworth expansion uses the density function associated with the S_L system of Johnson density functions. This function is characterized by the vector of parameters $\mathbf{\Theta}_{S_L} = [a\ b\ c\ d]^T$. The density function stemming from the S_L system is given by

$$g_{S_L}(y) = \frac{d}{y-a} n\left(\psi_{S_L}^{-1}(y)\right) \tag{23}$$

where $n(\cdot)$ stands for the density function of a standard normal variable and $\psi_{S_L}(Z) = a + b\exp\left(\frac{Z-c}{d}\right)$. Using a generalized Edgeworth expansion around f_{S_L}, the value of a European call option, whose exercise price is K and which matures in s periods, is given by the following expression :

$$C_{E_{S_L}}(K, s; \mathbf{\Theta}_{S_L}) = C_{S_L}(K, s; \mathbf{\Theta}_{S_L}) + \frac{e^{-rs}(\kappa_4 - \tilde{\kappa}_4)}{4!}\frac{d^2 g_{S_L}(K)}{dy^2} \tag{24}$$

with $\tilde{\kappa_4}$ being the fourth cumulant of the variable obtained from the S_L distribution and κ_4, the forth cumulant of the underlying variable.

This analytical approximation makes it possible to adjust the call option to the difference between the fourth moment of the option's underlying random variable and that of the moment reproduced by the S_L system.

4 Numerical study

This section presents a comparative numerical analysis of the performance of the four analytical approximations developed in the last section. We first examine the ability of the approximations in reproducing certain characteristics of the distribution of the future conditional variance. Focusing on the analytical approximation selected, the second part of this numerical analysis will allow us to assess the validity of this approximation under various conditions, specifically as applied to pricing a European call option on the variance of the S&P500 stock index's return.

Two sets of NGARCH-model parameters are considered : $\{\beta_0 = 0.00001,\ \beta_1 = 0.70,\ \beta_2 = 0.10,\ \theta+\lambda = 0.50\}$ and $\{\beta_0 = 0.00001,\ \beta_1 = 0.70,\ \beta_2 = 0.15,\ \theta+\lambda = 0.35\}$, corresponding respectively to the dynamics of low[4] and high[5] persistence. Recall from Lemma 2 that the coefficients ν_k are linked to the convergence of moments towards a finite stationary value. Moreover the conditions $\nu_k < 1,\ k = 1,\ \ldots\ , 4$ insure that the first four conditional moments of h_{t+s} converge to finite values independent of h_t.

[4] $\{\nu_1 = 0.825,\ \nu_2 = 0.711,\ \nu_3 = 0.650,\ \nu_4 = 0.644\}$
[5] $\{\nu_1 = 0.868,\ \nu_2 = 0.810,\ \nu_3 = 0.838,\ \nu_4 = 0.996\}$

4.1 Reproductions of the characteristics of the distribution of h_{t+s}

Figures 1 to 6 make it possible to compare the shape of the density and distribution functions obtained from the four approximations to an estimation obtained by a Monte Carlo simulation. Figures 1 and 4 are associated with the first set of parameters previously described, whereas Figures 5 and 6 are associated with the second set of parameters. These figures reveal that, in a context of weak persistence, systems S_L and S_U do a better job of reproducing the density and distribution functions of h_{t+s}. Though the S_L system fails to reproduce the fourth moment of h_{t+s}, it does allow a better reproduction of the lower boundary of the distribution and, unlike the S_U system, it does not admit any negative values. Moreover, one may also observe that the functions obtained with a generalized Edgeworth expansion are very unstable and not strictly positive.

4.2 Validation in a context of low and high persistence

In this section, we present the pricing results obtained for the two previously defined sets of parameters. Tables 1 and 2 display the values of a European call option on the future conditional variance as computed by the four analytical approximations, as well as the values obtained from a Monte Carlo simulation of 500 000 trajectories, considered as a benchmark. For each set of parameters, we consider call options with 5 different maturities (10,30, 90, and 270 days), whose initial level of conditional variance is set at 0.80 $E^Q[h]$, 1.00 $E^Q[h]$ and 1.20 $E^Q[h]$, and whose exercise price corresponds to 0.75 h_{t+1}, 1.00 h_{t+1} and 1.25 h_{t+1}. In all examples, the annual risk-free interest rate is set at 5 %. As one can guess from the results of the previous section, the S_L analytical approximation achieves greater accuracy in reproducing the value of a European call option on the future conditional variance than do approximations based on the generalized Edgeworth expansion and on the S_U system of Johnson density functions. The poorness of the approximation of both generalized Edgeworth expansions is transferred to the value of the options, even producing negative values.

Since, under the risk-neutral measure, the level of a forward contract is defined by the expected value of the underlying random variable and since each of the approximations proposed does permit reproduction of the first moment of the future conditional variance, the four analytical approximations are capable of giving a precise reproduction of the value of the forward contract.

In view of the results, one must not recommend the use of the analytical approximations based on a generalized Edgeworth expansion. The comparison of the approximations based on the S_U and S_L systems also indicates that the analytical approximation based on the S_L system offers greater precision and remains more reliable when the dynamics of the conditional variance displays a high level of persistence. However, the value obtained for the S_L analytical approximation, as for all the others, often differs significantly from the Monte Carlo estimation. In order to pinpoint the nature of this reproduction error, the following section presents an analysis carried out in a stochastic framework.

4.3 Validation in a stochastic setting

The results proposed in the previous sections are limited to two sets of parameters and a few option specifications. Though these results do make it possible to determine that the S_L approximation offers the greatest precision of all the analytical approximations developed, a more general validation setting is needed to obtain a global picture of the performance of the S_L approximation. To perform this task, we develop a stochastic validation setting similar to the one proposed by Broadie & Detemple (1996), using pre-specified distributions, NGARCH-process parameters, and characteristics of the call option to be evaluated. The maturity T of the call option is expressed in years and obtained by $T \sim U(0.1, 1)$ with probability 0.75 and $T \sim U(1, 3)$ with probability 0.25. Here $U(a, b)$ stands for the uniform distribution on the interval (a, b). The ratio of the option's exercise price to the conditional value's stationary level $K/E^Q[h] \sim U(0.75, 1.25)$ and the risk-free interest rate $r \sim U(0, 0.10)$. The ratio of the initial level of the conditional variance to its stationary level, $h_{t+1}/E^Q[h] \sim U(0.50, 0.90)$ with probability 0.10, $h_{t+1}/E^Q[h] \sim U(0.90, 1.10)$ with probability 0.80 and $h_{t+1}/E^Q[h] \sim U(1.10, 2)$ with probability 0.10. The NGARCH-process parameters are also generated based on the uniform distributions : $\beta_0 \sim U(0, 0.0001)$, $\beta_1 \sim U(0, 1)$, $\beta_2 \sim U(0, 1)$ and $\theta + \lambda \sim U(0, 1)$. The constraints $\nu_k < 1,\ k = 1, \ldots, 4$ are imposed in order to respect stationary conditions of the conditional moments. If these constraints are not respected when a set of parameters is generated at random, the simulation procedure is repeated until the set of parameters satisfy the conditions. The Root Mean Square Error (RMSE) is the criterion used to evaluate the performance of the S_L analytical approximation

$$RMSE_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \epsilon_i^2}$$

where

$$\epsilon_i = \frac{|C_{SL}(i) - C_{MC}(i)|}{C_{MC}(i)}. \tag{25}$$

The number of scenarios considered is $n = 1000$ and the value (C_{SL}) obtained for these scenario using the S_L approximation is compared to the value obtained by a Monte Carlo (C_{MC}) based on 1 000 000 trajectories. The options for which $C_{MC} < 0.005E^Q[h]$ have been withdrawn from the sample in order to avoid relatively high margins of error caused by a very low option value in the denominator of equation (25).

Figure 7 presents an histogram of the relative errors of evaluation. Such errors rarely exceed 0.06 and the global value of the RMSE is 0.03. This type of error is not very sensitive to the level of the risk-free interest rate and the option's maturity date. However, the RMSE of call options whose exercise price is higher than $1.10h_{t+1}$ and for which at least one of the coefficients $\mu_k < 0.10,\ k = 1, \ldots, 4$ is 0.081. In this case, the RMSE's high value derives from the low value of the option, which, stems, on one side, from the out-moneyness and, on the other side, from very weak variability in the future conditional variance. These observations aside, the RMSE is 0.02.

4.4 Validation for the estimated parameters of the S&P500 index

This section analyzes the European call option pricing results for a set of parameters associated with explosive dynamics of conditional variance moments. These parameters are obtained by estimating (under the physical measure) the NGARCH process, based on daily excess returns[6] on the S&P500 stock index from January 1, 1992 to December 31, 2001. These estimations yield that the coefficients ν_2, ν_3 and ν_4 are all higher than 1, which implies that the second, third, and fourth moments of the conditional variance tends to infinity, as the maturity tends to infinity.

Table 3 presents the value of a European call option as computed by means of analytical approximations and Monte Carlo simulations, for various maturities, exercise prices, and initial conditional-variance levels. The results indicate that for short-term maturities, the S_L approximation offers a satisfactory degree of precision, but the quality of the approximation deteriorates as the option's maturity lengthens. For maturities exceeding 90 days, the results obtained via the analytical approximation no longer fit the value estimated with the Monte Carlo simulation. Therefore, when the dynamics of one of the conditional-variance moments is explosive, the quality of the analytical approximation S_L is satisfactory for short-term options, but seriously limited for options with longer maturity horizons.

5 Extension to other GARCH processes

To extend this methodology to other GARCH processes, it suffices to be able to compute

$$\mu_{n,t,s} = \mathrm{E}_t^Q \left[h_{t+s}^n\right], \ n = 1, 2, 3, 4.$$

For the GJR-GARCH and the EGARCH processes, these moments are available in Duan, Gauthier, Sasseville & Simonato (2003). The expressions of $\mu_{n,t,s}$ for non-integer values of n are also presented for the EGARCH process, allowing the pricing of a volatility derivative instrument in the EGARCH setting.

6 Conclusion

This article proposes two classes of analytical approximations for pricing a European variance call option in an NGARCH market setting. Of the analytical approximations proposed, the one based on the lognormal system of Johnson density functions obtains the best

[6] The excess return is defined as $\ln(S_{t+1}/S_t) - r_t$

results. Except for certain specific cases, the approximation error is generally low. The methodology proposed can easily be adapted to the other GARCH models and generalized to European options on volatility.

A Proofs of the properties of conditional moments of h_{t+s}

A.1 Proof of Lemma 1

Using the independence between h_{t+i} and Y_{t+i}, $i \geq 0$, and the projection property of the conditional expectation,

$$\begin{aligned} \mathrm{E}_t^Q\left[h_{t+s}^n\right] &= \mathrm{E}_t^Q\left[(\beta_0 + h_{t+s-1}Y_{t+s-1})^n\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}\beta_0^{n-k}\mathrm{E}_t^Q\left[h_{t+s-1}^k\right]\mathrm{E}^Q\left[Y_{t+s-1}^k\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}\beta_0^{n-k}\nu_k\mathrm{E}_t^Q\left[h_{t+s-1}^k\right]. \end{aligned}$$

Next,

$$\begin{aligned} \nu_k &= \mathrm{E}^Q\left[Y_1^k\right] \\ &= \mathrm{E}^Q\left[\left(\beta_1 + \beta_2\left(\epsilon_1 - \theta - \lambda\right)^2\right)^k\right] \\ &= \sum_{j=0}^{k}\binom{k}{j}\beta_1^{k-j}\beta_2^j\mathrm{E}^Q\left[\left(\epsilon_1 - \theta - \lambda\right)^{2j}\right] \\ &= \sum_{j=0}^{k}\binom{k}{j}\beta_1^{k-j}\beta_2^j\eta_j. \end{aligned}$$

Finally, since $\mathrm{E}^Q\left[(\epsilon_1)^{2i}\right] = \frac{(2i)!}{2^i i!}$ and $\mathrm{E}^Q\left[(\epsilon_1)^{2i+1}\right] = 0$, we get the expression for η_j. ■

A.2 Proof of Lemma 2

The proof is based on induction. It follows from Lemma 1 that for any $s \geq 2$,

$$E_t^Q[h_{t+s}] = \beta_0 + \nu_1 E_t^Q[h_{t+s-1}].$$

Therefore, if $\nu_1 \geq 1$, $E_t^Q[h_{t+s}] \geq \beta_0 + E_t^Q[h_{t+s-1}]$, so

$$E_t^Q[h_{t+s}] \geq (s-1)\beta_0 + h_{t+1} \to \infty, \text{ as } s \to \infty.$$

If $\nu_1 < 1$, it is easy to check that

$$\begin{aligned} E_t^Q[h_{t+s}] &= \nu_1^{s-1} h_{t+1} + \beta_0 \sum_{j=0}^{s-2} \nu_1^j \\ &= \nu_1^{s-1} h_{t+1} + \beta_0 \frac{1-\nu_1^{s-1}}{1-\nu_1} \\ &\to \frac{\beta_0}{1-\nu_1} = \mu_1, \text{ as } s \to \infty. \end{aligned}$$

Hence formula (9) holds true and the lemma proved when $n = 1$.

The rest of the proof is based on induction on index n. So suppose that the Lemma holds true for all indexes $k \leq n$. One may assume that $\nu_k < 1$ for all $k \leq n$, for otherwise there is nothing left to prove. It follows from the induction hypothesis that

$$\lim_{s\to\infty} E_t^Q \left[h_{t+s}^k\right] = \mu_k, \quad 1 \leq k \leq n,$$

where for any $1 \leq k \leq n$,

$$\mu_k = \sum_{j=0}^{k-1} \binom{k}{j} \beta_0^{k-j} \nu_j \mu_j.$$

One has to show that, as s tends to infinity, $E_t^Q \left[h_{t+s}^{n+1}\right]$ tends to a finite limit when $\nu_{n+1} < 1$ and $E_t^Q \left[h_{t+s}^{n+1}\right]$ tends to infinity when $\nu_{n+1} \geq 1$.

For $s \geq 1$, set

$$L_{n,s} = \sum_{j=0}^{n} \binom{n+1}{j} \beta_0^{n+1-j} \nu_j E_t^Q \left[h_{t+s}^j\right].$$

First, note that $L_{n,s} \geq \beta_0^{n+1}$. Second, it follows from the induction hypothesis that

$$\lim_{s\to\infty} L_{n,s} = \sum_{j=0}^{n} \binom{n+1}{j} \beta_0^{n+1-j} \nu_j \mu_j = L_n.$$

Next, note that equation (8) can be written as follows:

$$E_t^Q \left[h_{t+s}^{n+1}\right] = \nu_{n+1} E_t^Q \left[h_{t+s-1}^{n+1}\right] + L_{n,s-1}, \quad s \geq 2.$$

If $\nu_{n+1} \geq 1$, then

$$E_t^Q \left[h_{t+s}^{n+1}\right] \geq E_t^Q \left[h_{t+s-1}^{n+1}\right] + \beta_0^{n+1},$$

so

$$E_t^Q \left[h_{t+s}^{n+1}\right] \geq h_{t+1}^{n+1} + (s-1)\beta_0^{n+1} \to \infty, \text{ as } s \to \infty.$$

It only remains to show that if $\nu_{n+1} \geq 1$, then $E_t^Q\left[h_{t+s}^{n+1}\right] \to \mu_{n+1}$, where $\mu_{n+1} = \frac{L_n}{1-\nu_{n+1}}$. Finally, if $\nu_{n+1} < 1$, then it is easy to see that

$$E_t^Q\left[h_{t+s}^{n+1}\right] = \nu_{n+1}^{s-1} + \sum_{j=1}^{s-1} \nu_{n+1}^{s-1-j} L_{n,j}$$

Since $L_{n,j} \to L_n$ as $j \to \infty$, it follows that

$$\lim_{s\to\infty} E_t^Q\left[h_{t+s}^{n+1}\right] = \lim_{s\to\infty} \sum_{j=1}^{s-1} \nu_{n+1}^{s-1-j} L_n = \frac{L_n}{1-\nu_{n+1}}.$$

Hence the result. ■

B Analytical approximations based on Johnson density functions

The density functions $f_{S_U}(\cdot)$ and $f_{S_L}(\cdot)$ are given respectively by

$$f_{S_U}(y) = n\left(\psi_{S_U}^{-1}(y)\right) \frac{d}{b\sqrt{1+\left(\frac{y-a}{b}\right)^2}}$$

and

$$f_{S_L}(y) = n\left(\psi_{S_L}^{-1}(y)\right) \frac{d}{y-a},$$

where $n(\cdot)$ is the standard normal density function. In order to ensure that f_{S_U} is positive, the parameters b and d must have the same sign, whereas f_{S_L} is positive if parameter d is positive. $\psi(S_U)$ and $\psi(S_L)$ are defined respectively in Sections 3.1.1 and 3.1.2.

B.1 Moments of S_U and S_L

The moments of S_U and S_L are needed to estimate the values of parameters a, b, c and d.

For any $n \geq 1$, one has

$$\begin{aligned}
\mathrm{E}\left[S_U^n\right] &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \mathrm{E}\left[\left(\frac{\exp\left(\frac{Z-c}{d}\right) - \exp\left(-\frac{Z-c}{d}\right)}{2}\right)^k\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{a^{n-k}b^k}{2^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j}\, \mathrm{E}\left[\exp\left(-(k-2j)\frac{Z-c}{d}\right)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{a^{n-k}b^k}{2^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \exp\left(\frac{c}{d}(k-2j) + \frac{(k-2j)^2}{2d^2}\right) \qquad (26)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[S_L^n\right] &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \mathrm{E}\left[\exp\left(k\frac{Z-c}{d}\right)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \exp\left(\frac{k^2-2kdc}{2d^2}\right). \qquad (27)
\end{aligned}$$

B.2 Value of a European call option for the S_U system

Denote by $F_{S_U}(\cdot)$ the distribution function of S_U. Since ψ_{S_U} is increasing,

$$F_{S_U}(x) = P\left[Z \leq \psi_{S_U}^{-1}(x)\right] = N\left(\psi_{S_U}^{-1}(x)\right),$$

with N is the distribution function of the standard Gaussian law. Following Posner & Milevsky (1998),

$$\begin{aligned}
C_{S_U}(K, s, \mathbf{\Theta}) &= e^{-rs}\int_{-\infty}^{\infty} (y-K)^+ f_{S_U}(y)\ dy \\
&= e^{-rs}\left(\mathrm{E}\left[S_U\right]-K\right) - e^{-rs}\left(\left.(y-K)\ F_{S_U}(z)\right|_{-\infty}^{K} - \int_{-\infty}^{K} F_{S_U}(y)dy\right) \\
&\quad \text{(using integration by part)} \\
&= e^{-rs}\left(\mathrm{E}\left[S_U\right]-K\right) + e^{-rs}\int_{-\infty}^{K} N\left(\psi_{S_U}^{-1}(y)\right) dy \\
&= e^{-rs}\left(\mathrm{E}\left[S_U\right]-K\right) + e^{-rs}\frac{b}{d}\int_{-\infty}^{\psi_{S_U}^{-1}(K)} \cosh\left(\frac{x-c}{d}\right) N(x)dx,
\end{aligned}$$

using the change of variable $x = \psi_{S_U}^{-1}(y)$. Standard computations completes the proof. ■

C Analytical approximations of density functions based on Edgeworth expansion

Following Jarrow & Rudd (1982), the density function f of a random variable Y is approximated by an Edgeworth expansion based on the density g of a random variable A.

$$f(y) \cong f_E(y) = g(y) + \sum_{k=1}^{4} \frac{(-1)^k E_k}{k!} g^{(k)}(y) \qquad (28)$$

where $g^{(k)}$ stands for the derivative of order k of g with respect to y, and

$$E_1 = \kappa_1 - \widetilde{\kappa}_1, \qquad E_2 = \kappa_2 - \widetilde{\kappa}_2 + E_1^2,$$

$$E_3 = \kappa_3 - \widetilde{\kappa}_3 + 3E_1E_2 - 2E_1^3, \qquad E_4 = \kappa_4 - \widetilde{\kappa}_4 + 3E_2^2 + 4E_1^3E_3 - 12E_1^2E_2 + 6E_1^4,$$

where κ_i (respectively $\widetilde{\kappa}_i$) denotes the ith cumulant of Y (respectively A). Note that f_E is not a density function in general, taking negative values or having surface under the curve different from one. Recall that the first four cumulants are defined by

$$\kappa_1 = E[Y], \qquad \kappa_2 = E[Y^2] - \kappa_1^2,$$

$$\kappa_3 = E[Y^3] - 3\kappa_1\kappa_2 - \kappa_1^3, \qquad \kappa_4 = E[Y^4] - 3\kappa_2^2 - 4\kappa_1\kappa_3 - 6\kappa_1^2\kappa_2 - \kappa_1^4.$$

C.1 Approximation based on lognormal distribution

The lognormal distribution function g_L is characterized by two parameters which allow us to match the two first moments of the target distribution. Therefore (28) reduces to

$$f_{E_L}(y) = g_L(y) - \frac{(\kappa_3 - \widetilde{\kappa}_3)}{3!} g_L^{(3)}(y) + \frac{(\kappa_4 - \widetilde{\kappa}_4)}{4!} g_L^{(4)}(y),$$

where $\kappa_3, \widetilde{\kappa}_3, \kappa_4$ and $\widetilde{\kappa}_4$ are defined by (22).

C.2 Approximation based on the S_L system

The density function of S_L is based on three parameters and, therefore, we match the three first moments of the target distribution. Equation (28) reduces to :

$$f_{E_{S_L}}(y) = g_{S_L}(y) + \frac{(\kappa_4 - \widetilde{\kappa}_4)}{4!} g_{S_L}^{(4)}(y).$$

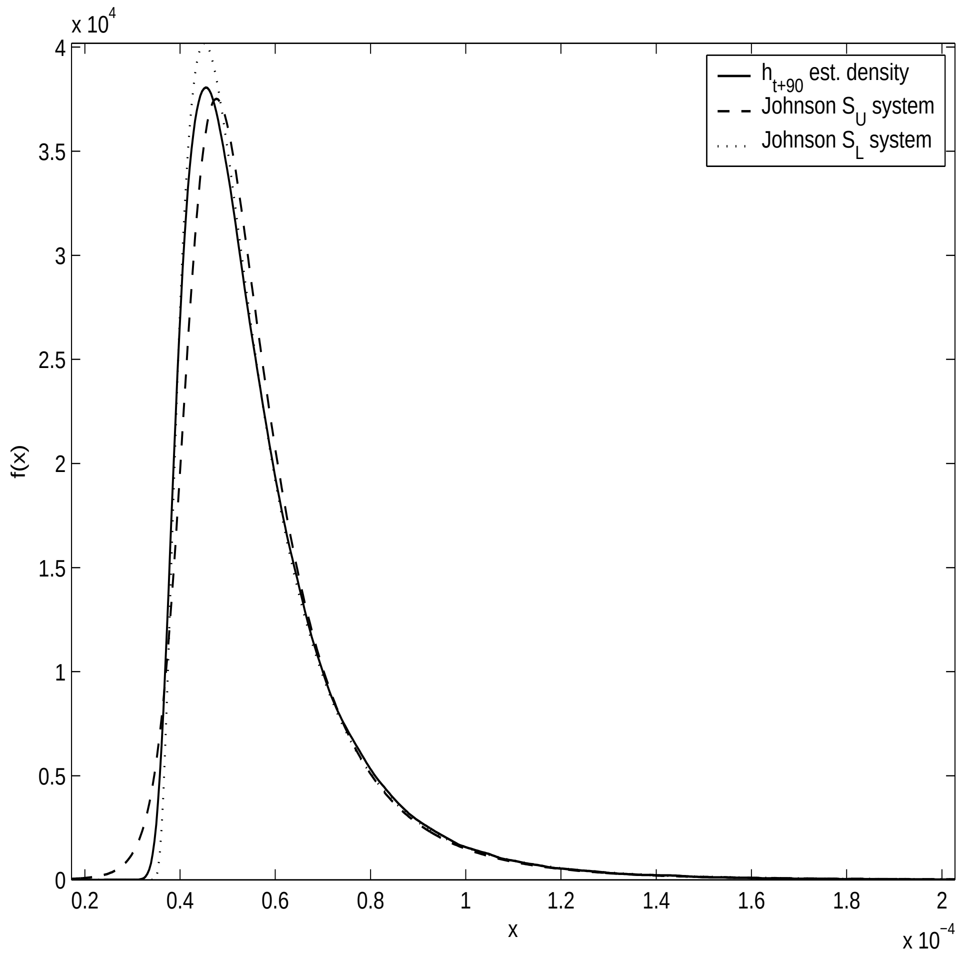
where κ_4 is given by (22) and

$$\widetilde{\kappa}_4 = E[S_L^4] - 4E[S_L]E[S_L^3] - 3E^2[S_L^2] + 12E[S_L^2]E^2[S_L] - 6E^4[S_L],$$

and the moments of S_L are given by (27).

Figure 1: Reproduction of h_{t+s} density function in a low persistence framework

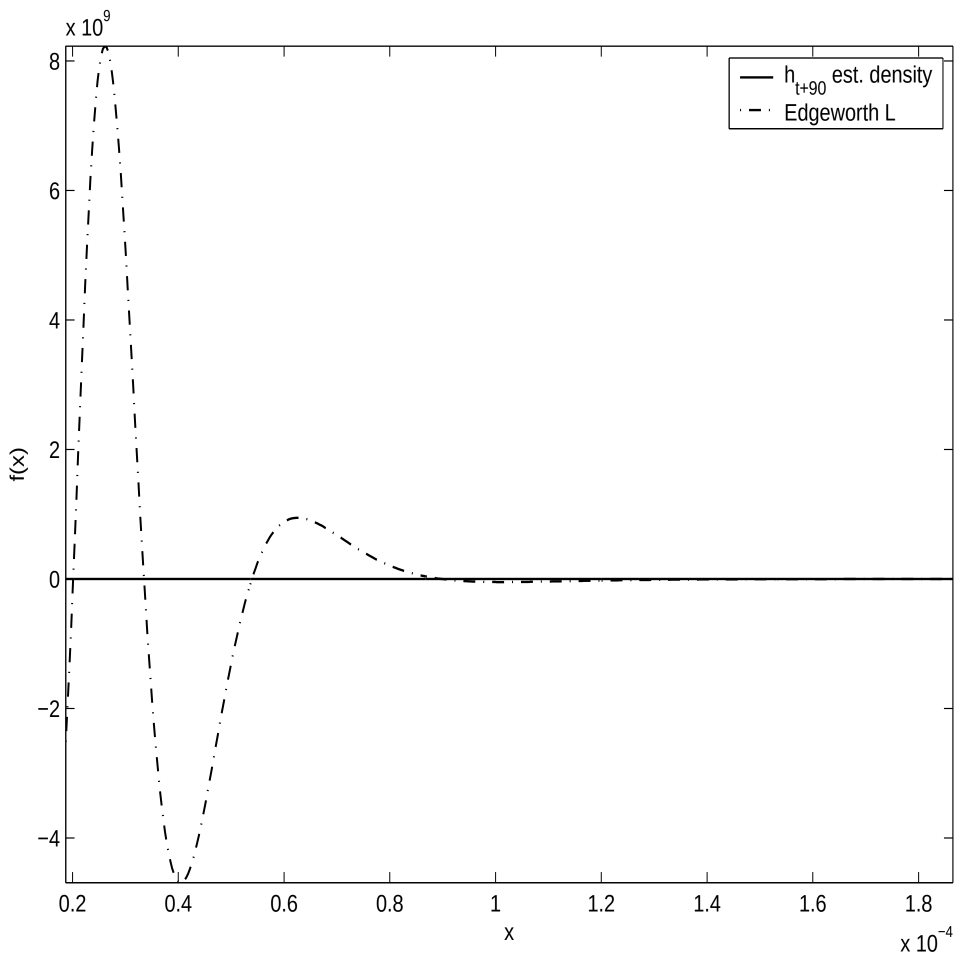
NGARCH parameters : $\beta_0 = 0.00001$, $\beta_1 = 0.70$, $\beta_2 = 0.1000$, $\theta + \lambda = 0.5000$
Critical constants : $\mu_1 = 0.825$, $\nu_2 = 0.711$, $\nu_3 = 0.650$, $\nu_4 = 0.644$
$h_{t+1} = E[h]$



The estimated density function is obtained using a kernel methodology based on a Monte Carlo simulation with 500000 trajectories. The initial value of h_{t+s} is set to its stationary level. The two density function approximations based on Edgeworth expansion are presented separately since they were out of scale.

Figure 2: Reproduction of h_{t+s} density function in a low persistence framework

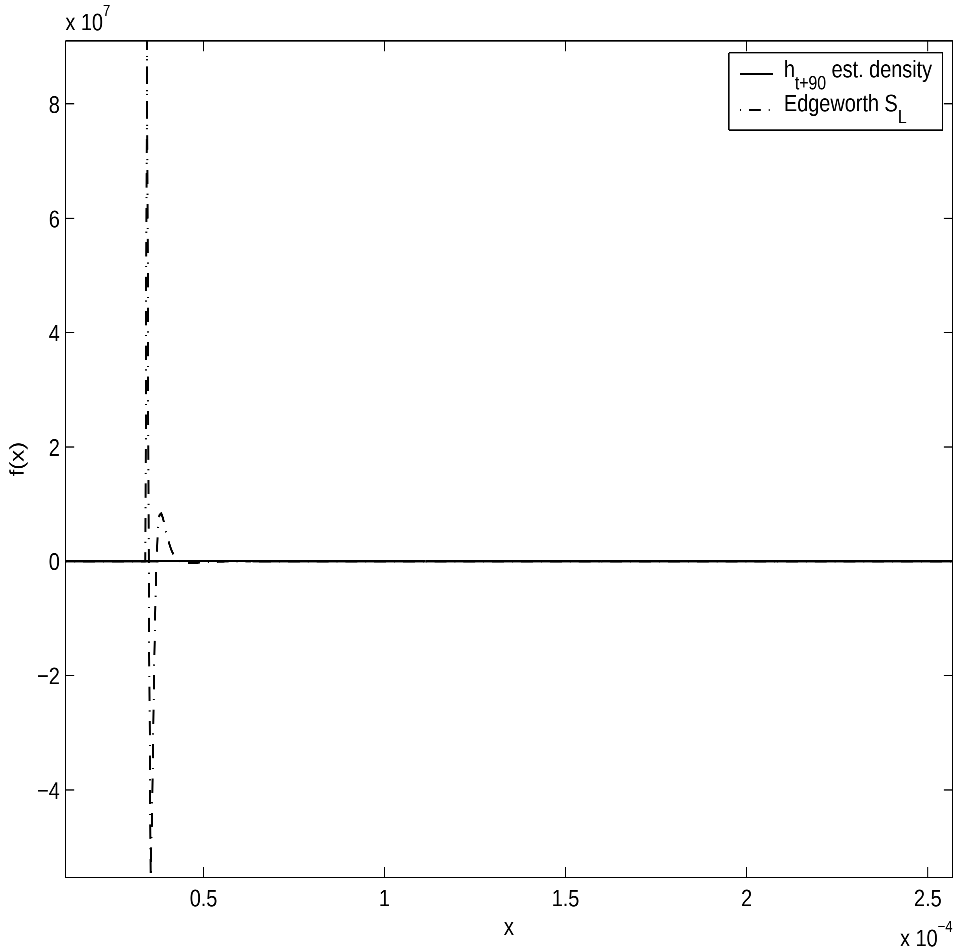
NGARCH parameters : $\beta_0 = 0.00001$, $\beta_1 = 0.70$, $\beta_2 = 0.1000$, $\theta + \lambda = 0.5000$
Critical constants : $\nu_1 = 0.825$, $\nu_2 = 0.711$, $\nu_3 = 0.650$, $\nu_4 = 0.644$
$h_{t+1} = E[h]$



Edgeworth L is the density function approximation based on the Edgeworth expansion using the log normal distribution.

Figure 3: Reproduction of h_{t+s} density function in a low persistence framework

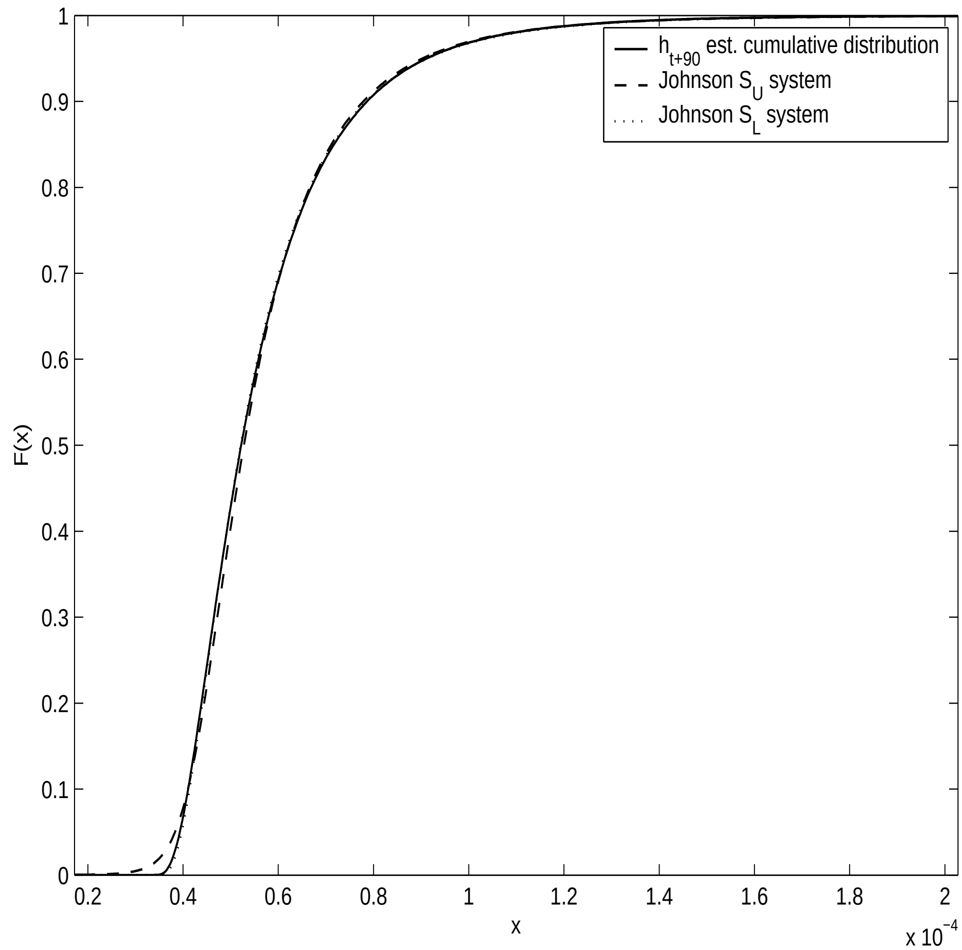
NGARCH parameters : $\beta_0 = 0.00001$, $\beta_1 = 0.70$, $\beta_2 = 0.1000$, $\theta + \lambda = 0.5000$
Critical constants : $\nu_1 = 0.825$, $\nu_2 = 0.711$, $\nu_3 = 0.650$, $\nu_4 = 0.644$
$h_{t+1} = E[h]$



Edgeworth S_L is the density function approximation based on the Edgeworth expansion using the S_L system of Johnson's distribution.

Figure 4: Reproduction of h_{t+s} cumulative distribution function in a low persistence framework

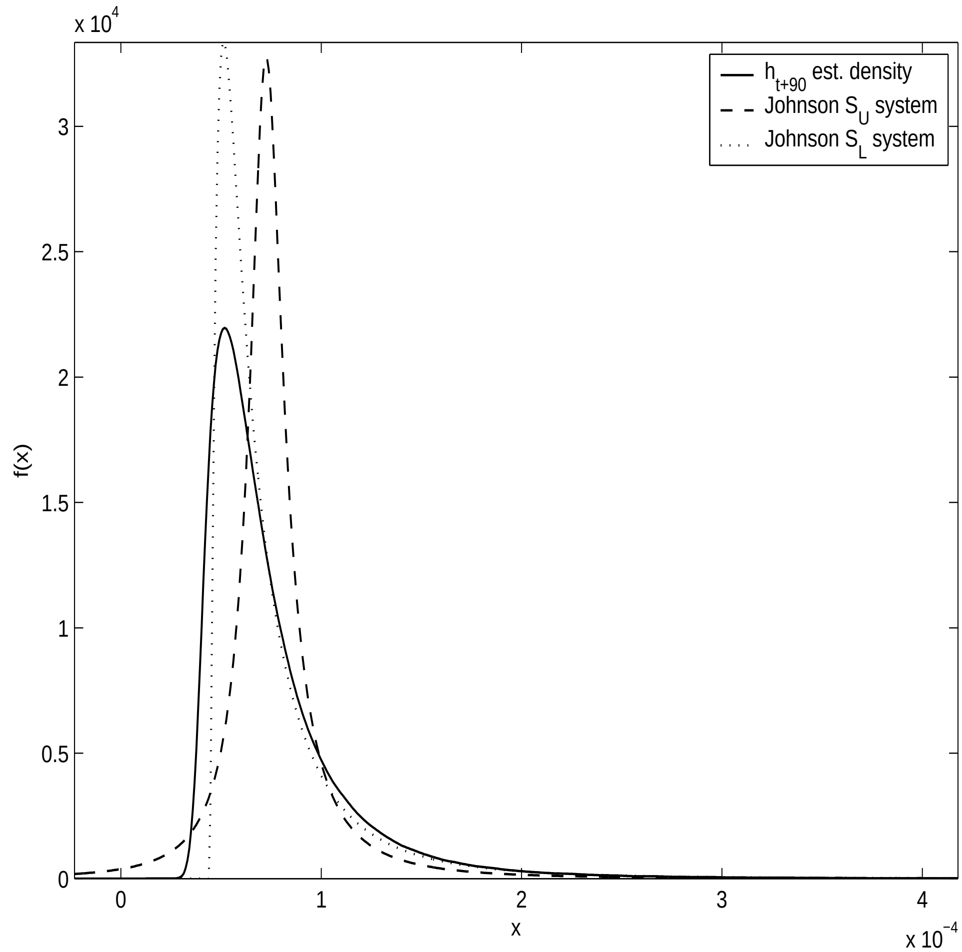
$$\text{NGARCH parameters : } \beta_0 = 0.00001,\ \beta_1 = 0.70,\ \beta_2 = 0.1000,\ \theta+\lambda = 0.5000$$
$$\text{Critical constants : } \nu_1 = 0.825,\ \nu_2 = 0.711,\ \nu_3 = 0.650,\ \nu_4 = 0.644$$
$$h_{t+1} = E[h]$$



The estimated cumulative distribution function is obtained with a Monte Carlo simulation using 500000 trajectories. The initial value of h_{t+s} is set to its stationary level. The cumulative distribution functions from the Edgeworth expansions are omitted here since they were way out of scale.

Figure 5: Reproduction of h_{t+s} density function in a high persistence framework

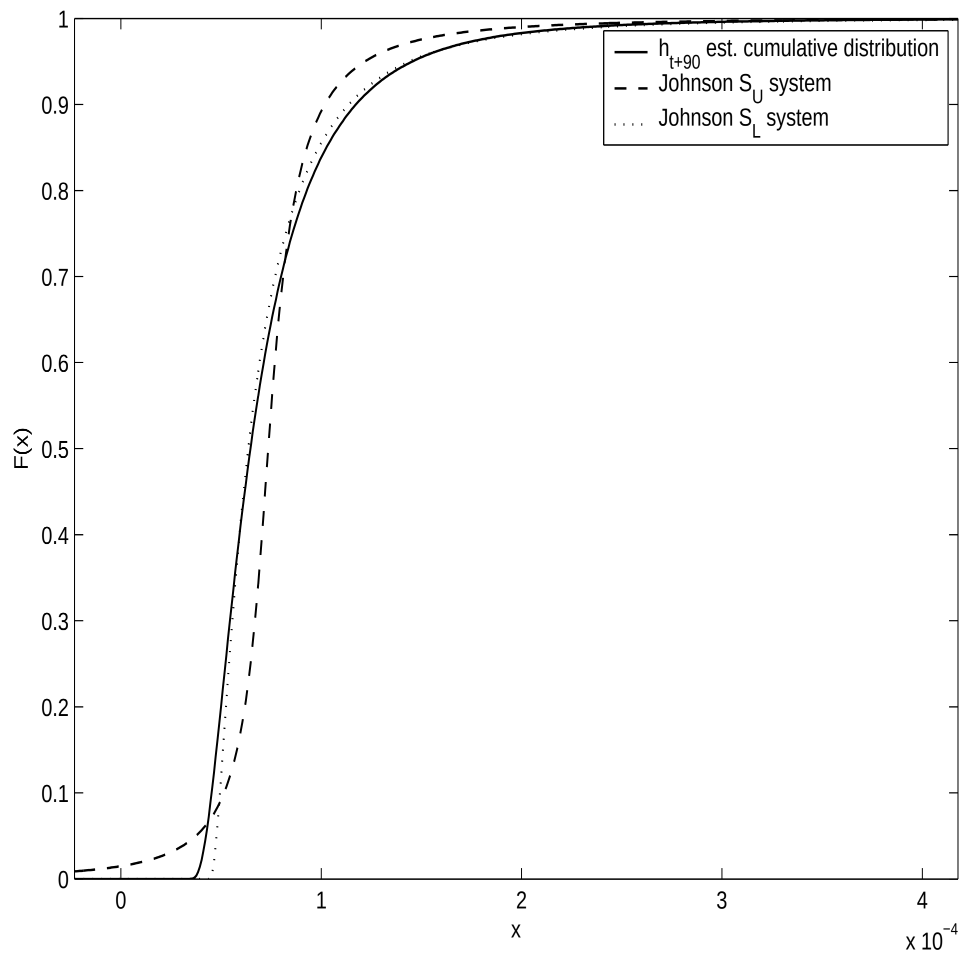
NGARCH parameters : $\beta_0 = 0.00001$, $\beta_1 = 0.70$, $\beta_2 = 0.1500$, $\theta + \lambda = 0.3500$
Critical constants : $\{\nu_1 = 0.868\ ,\ \nu_2 = 0.810,\ \nu_3 = 0.838,\ \ \nu_4 = 0.996\}$
$h_{t+1} = E[h]$



The estimated density function is obtained using a kernel methodology based on a Monte Carlo simulation with 500000 trajectories. The initial value of h_{t+s} is set to its stationary level. The approximated density functions from the Edgeworth expansions are omitted here since they were way out of scale.

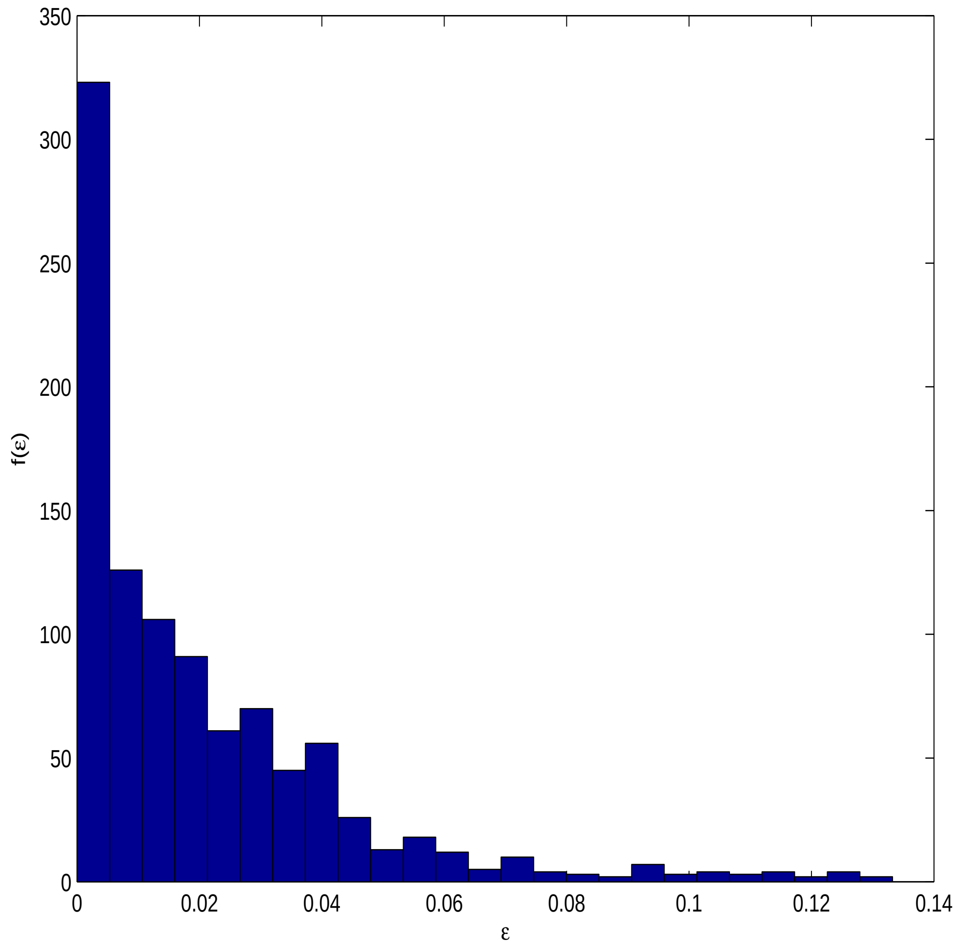
Figure 6: Reproduction of h_{t+s} cumulative distribution function in a high persistence framework

$$\text{NGARCH parameters : } \beta_0 \;=\; 0.00001,\ \beta_1 \;=\; 0.70,\ \beta_2 \;=\; 0.1500,\ \theta+\lambda \;=\; 0.3500$$
$$\text{Critical constants : } \{\nu_1 = 0.868\ ,\ \ \nu_2 = 0.810,\ \ \nu_3 = 0.838,\ \ \ \nu_4 = 0.996\}$$
$$h_{t+1} \;=\; E[h]$$



The estimated cumulative distribution function is obtained with a Monte Carlo simulation using 500000 trajectories. The initial value of h_{t+s} is set to its stationary level. The cumulative distribution functions from the Edgeworth expansions are omitted here since they were way out of scale.

Figure 7: Histogram of the relative pricing errors obtained from the S_L approximation



The relative errors are computed with the ratio of the absolute error by the Monte Carlo estimate using 1 000 000 trajectories. The error term is the difference between the Monte Carlo estimate and the value obtained by the S_L approximation.

Table 1: The performance of the approximations for European Call option prices in a low persistance framework

$\beta_0 = 0.00001$, $\beta_1 = 0.70$, $\beta_2 = 0.10$, $\lambda + \theta = 0.50$

$\nu_1 = 0.825$, $\nu_2 = 0.711$, $\nu_3 = 0.650$, $\nu_4 = 0.644$

	s = 10 days			s = 30 days			s = 90 days			s = 270 days		
K/h_{t+1}	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25
$h_{t+1} = 0.80 \times E[h]$												
Johnson S_U	**2.081e-005**	**1.071e-005**	**5.079e-006**	**2.273e-005**	**1.247e-005**	**6.208e-006**	2.251e-005	**1.236e-005**	**6.160e-006**	**2.172e-005**	**1.193e-005**	**5.944e-006**
Johnson S_L	2.079e-005	1.064e-005	5.118e-006	2.268e-005	**1.229e-005**	6.269e-006	2.245e-005	**1.218e-005**	**6.220e-006**	2.166e-005	**1.175e-005**	6.002e-006
Edgeworth L	**1.887e-005**	**4.235e-006**	**4.748e-006**	**2.008e-005**	**2.534e-006**	**4.466e-006**	**1.983e-005**	**2.314e-006**	**4.364e-006**	**1.904e-005**	**1.843e-006**	**4.118e-006**
Edgeworth S_L	**2.823e-005**	**9.011e-006**	**5.422e-006**	**4.550e-005**	**5.175e-006**	**6.112e-006**	**4.593e-005**	**4.902e-006**	**6.079e-006**	**4.521e-005**	**4.535e-006**	**5.902e-006**
Monte Carlo	2.075e-005	1.063e-005	5.114e-006	2.266e-005	1.235e-005	6.299e-006	2.248e-005	1.228e-005	6.287e-006	2.165e-005	1.181e-005	6.026e-006
Std. Deviation	(2.351e-008)	(2.206e-008)	(1.781e-008)	(2.577e-008)	(2.456e-008)	(2.052e-008)	(2.556e-008)	(2.438e-008)	(2.037e-008)	(2.454e-008)	(2.339e-008)	(1.953e-008)
$h_{t+1} = 1.00 \times E[h]$												
Johnson S_U	**1.476e-005**	**6.181e-006**	**2.730e-006**	**1.480e-005**	**6.233e-006**	**2.791e-006**	**1.462e-005**	**6.160e-006**	**2.758e-006**	**1.411e-005**	**5.944e-006**	**2.661e-006**
Johnson S_L	**1.461e-005**	**6.218e-006**	2.790e-006	**1.457e-005**	6.295e-006	2.890e-006	**1.440e-005**	6.220e-006	2.856e-006	**1.390e-005**	**6.002e-006**	2.756e-006
Edgeworth L	**6.160e-006**	**4.751e-006**	**5.175e-006**	**4.125e-006**	**4.449e-006**	**5.672e-006**	**3.934e-006**	**4.364e-006**	**5.640e-006**	**3.390e-006**	**4.118e-006**	**5.544e-006**
Edgeworth S_L	**8.222e-006**	**6.625e-006**	**3.042e-006**	**1.836e-006**	**7.003e-006**	**3.280e-006**	**1.639e-006**	**6.931e-006**	**3.249e-006**	**1.135e-006**	**6.716e-006**	**3.152e-006**
Monte Carlo	1.468e-005	6.274e-006	2.796e-006	1.462e-005	6.330e-006	2.870e-006	1.446e-005	6.260e-006	2.840e-006	1.396e-005	6.055e-006	2.739e-006
Std. Deviation	(2.493e-008)	(2.022e-008)	(1.510e-008)	(2.536e-008)	(2.068e-008)	(1.564e-008)	(2.502e-008)	(2.039e-008)	(1.538e-008)	(2.414e-008)	(1.966e-008)	(1.483e-008)
$h_{t+1} = 1.20 \times E[h]$												
Johnson S_U	1.028e-005	**3.935e-006**	1.695e-006	8.855e-006	**3.266e-006**	1.389e-006	8.718e-006	**3.211e-006**	1.365e-006	8.412e-006	**3.099e-006**	1.317e-006
Johnson S_L	1.022e-005	4.015e-006	**1.746e-006**	**8.815e-006**	3.372e-006	**1.448e-006**	**8.680e-006**	3.315e-006	**1.422e-006**	**8.375e-006**	3.199e-006	**1.372e-006**
Edgeworth L	**3.258e-006**	**6.140e-006**	**3.867e-006**	**2.626e-006**	**5.920e-006**	**3.375e-006**	**2.550e-006**	**5.855e-006**	**3.332e-006**	**2.205e-006**	**5.738e-006**	**3.297e-006**
Edgeworth S_L	**9.243e-006**	**4.490e-006**	**1.898e-006**	8.930e-006	**4.607e-006**	**1.933e-006**	**8.839e-006**	**4.545e-006**	**1.908e-006**	8.486e-006	**4.407e-006**	**1.848e-006**
Monte Carlo	1.027e-005	4.006e-006	1.714e-006	8.890e-006	3.358e-006	1.409e-006	8.732e-006	3.294e-006	1.382e-006	8.450e-006	3.186e-006	1.338e-006
Std. Deviation	(2.441e-008)	(1.795e-008)	(1.285e-008)	(2.281e-008)	(1.648e-008)	(1.177e-008)	(2.242e-008)	(1.615e-008)	(1.148e-008)	(2.183e-008)	(1.583e-008)	(1.141e-008)

"Johnson S_U" is the analytical approximation based on the Johnson S_U density function, "Johnson S_L" is the analytical approximation based on the Jonhson S_L density function, "Edgeworth L" is the approximation based on the Edgeworth expansion using the log normal distribution , "Edgeworth S_L" is the analytical approximation based on the Edgeworth expansion using the S_L system of Johson's distribution, "Monte Carlo" is the Monte Carlo estimate of the option's value using 500000 trajectories and "Std. Deviation" is the standard deviation of the Monte Carlo estimate. Results presented in bold indicates that the pricing error is statistically different from zero using a confidence level of 5%.

Table 2: The performance of the approximations for European Call option prices in a high persistence framework

$\beta_0 = 0.00001,\ \beta_1 = 0.70,\ \beta_2 = 0.15,\ \lambda + \theta = 0.35$

$\nu_1 = 0.868,\ \nu_2 = 0.810,\ \nu_3 = 0.838,\ \nu_4 = 0.996$

	s = 10 days			s = 30 days			s = 90 days			s = 270 days		
K/h_{t+1}	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25
$h_{t+1} = 0.80 \times E[h]$												
Johnson S_U	**2.709e-005**	1.583e-005	**9.402e-006**	**3.204e-005**	1.933e-005	**1.089e-005**	**3.224e-005**	**1.907e-005**	**9.465e-006**	**3.110e-005**	**1.814e-005**	**8.303e-006**
Johnson S_L	**2.627e-005**	**1.556e-005**	**9.658e-006**	**2.997e-005**	**1.833e-005**	**1.199e-005**	**2.985e-005**	**1.821e-005**	**1.193e-005**	**2.881e-005**	**1.757e-005**	**1.152e-005**
Edgeworth L	**-6.834e-005**	**-1.705e-005**	**1.600e-005**	**-7.246e-004**	**-2.563e-004**	**1.819e-005**	**-3.094e-003**	**-1.148e-003**	**-1.002e-007**	**-7.641e-003**	**-2.863e-003**	**-3.675e-005**
Edgeworth S_L	**-4.072e-004**	**2.607e-005**	**2.324e-005**	**-1.337e-002**	**3.401e-004**	**4.292e-004**	**-7.833e-002**	**1.915e-003**	**2.465e-003**	**-2.046e-001**	**4.976e-003**	**6.422e-003**
Monte Carlo	2.650e-005	1.591e-005	9.806e-006	3.032e-005	1.934e-005	1.264e-005	3.020e-005	1.931e-005	1.266e-005	2.911e-005	1.861e-005	1.219e-005
Std. Deviation	(4.810e-008)	(4.420e-008)	(3.879e-008)	(5.703e-008)	(5.355e-008)	(4.845e-008)	(5.670e-008)	(5.329e-008)	(4.824e-008)	(5.435e-008)	(5.104e-008)	(4.614e-008)
$h_{t+1} = 1.00 \times E[h]$												
Johnson S_U	**2.192e-005**	**1.188e-005**	**6.987e-006**	**2.250e-005**	**1.091e-005**	**6.127e-006**	**2.219e-005**	**9.418e-006**	**4.885e-006**	**2.124e-005**	**8.295e-006**	**4.123e-006**
Johnson S_L	**2.117e-005**	**1.208e-005**	7.399e-006	**2.078e-005**	**1.212e-005**	**7.766e-006**	**2.049e-005**	**1.193e-005**	**7.654e-006**	**1.977e-005**	**1.152e-005**	**7.386e-006**
Edgeworth L	**-5.644e-005**	**1.321e-005**	**2.750e-005**	**-4.460e-004**	**1.691e-005**	**1.203e-004**	**-1.680e-003**	**-1.276e-006**	**3.901e-004**	**-4.018e-003**	**-3.732e-005**	**9.003e-004**
Edgeworth S_L	**-3.264e-005**	**3.543e-005**	**1.733e-005**	**-1.151e-003**	**5.207e-004**	**2.238e-004**	**-5.571e-003**	**2.439e-003**	**1.040e-003**	**-1.399e-002**	**6.093e-003**	**2.594e-003**
Monte Carlo	2.164e-005	1.234e-005	7.423e-006	2.186e-005	1.280e-005	7.962e-006	2.164e-005	1.271e-005	7.924e-006	2.079e-005	1.219e-005	7.578e-006
Std. Deviation	(5.097e-008)	(4.449e-008)	(3.799e-008)	(5.501e-008)	(4.880e-008)	(4.262e-008)	(5.496e-008)	(4.887e-008)	(4.282e-008)	(5.205e-008)	(4.606e-008)	(4.010e-008)
$h_{t+1} = 1.20 \times E[h]$												
Johnson S_U	**1.828e-005**	**9.708e-006**	**5.731e-006**	**1.463e-005**	**6.742e-006**	**3.952e-006**	**1.344e-005**	**5.393e-006**	**3.122e-006**	**1.246e-005**	**4.592e-006**	**2.665e-006**
Johnson S_L	**1.798e-005**	**1.013e-005**	**6.180e-006**	**1.497e-005**	**8.547e-006**	5.396e-006	**1.461e-005**	**8.316e-006**	5.237e-006	**1.410e-005**	**8.024e-006**	**5.053e-006**
Edgeworth L	**-2.333e-005**	**3.151e-005**	**2.970e-005**	**-1.397e-004**	**1.360e-004**	**1.239e-004**	**-4.934e-004**	**3.871e-004**	**3.634e-004**	**-1.153e-003**	**8.535e-004**	**8.097e-004**
Edgeworth S_L	**5.066e-005**	**3.038e-005**	**1.347e-005**	**5.642e-004**	**3.485e-004**	**1.280e-004**	**2.154e-003**	**1.334e-003**	**4.827e-004**	**5.115e-003**	**3.168e-003**	**1.143e-003**
Monte Carlo	1.844e-005	1.023e-005	6.095e-006	1.592e-005	8.856e-006	5.357e-006	1.555e-005	8.606e-006	5.167e-006	1.501e-005	8.300e-006	4.981e-006
Std. Deviation	(5.348e-008)	(4.550e-008)	(3.851e-008)	(5.240e-008)	(4.486e-008)	(3.854e-008)	(5.116e-008)	(4.369e-008)	(3.747e-008)	(4.873e-008)	(4.143e-008)	(3.530e-008)

"Johnson S_U" is the analytical approximation based on the Johnson S_U density function, "Johnson S_L" is the analytical approximation based on the Jonhson S_L density function, "Edgeworth L" is the approximation based on the Edgworth expansion using the log normal distribution, "Edgeworth S_L" is the analytical approximation based on the Edgworth expansion using the S_L system of Johson's distribution, "Monte Carlo" is the Monte Carlo estimate of the option's value using 500000 trajectories and "Std. Deviation" is the standard deviation of the Monte Carlo estimate. Results presented in bold indicates that the pricing error is statistically different from zero using a confidence level of 5%.

Table 3: The performance of the S_L approximation in the case of the S&P500 index

$\beta_0 = 1.3348e-006$, $\beta_1 = 0.87$, $\beta_2 = 0.07$, $\lambda + \theta = -0.85$
$\nu_1 = 0.994$, $\nu_2 = 1.011$, $\nu_3 = 1.061$, $\nu_4 = 1.156$

	s = 10 days			s = 30 days			s = 90 days			s = 180 days		
K/h_{t+1}	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25	0.75	1.00	1.25
$h_{t+1} = 0.80 \times E[h]$												
Johnson S_L	3.411e-005	**1.721e-005**	**1.026e-005**	**5.377e-005**	**3.791e-005**	**2.946e-005**	**6.909e-005**	**5.700e-005**	**5.013e-005**	**5.482e-005**	**4.600e-005**	**4.172e-005**
Monte Carlo	3.415e-005	1.707e-005	1.009e-005	5.507e-005	3.848e-005	2.959e-005	8.090e-005	6.659e-005	5.816e-005	9.248e-005	7.953e-005	7.180e-005
Std. Deviation	(9.539e-008)	(7.487e-008)	(6.102e-008)	(1.935e-007)	(1.754e-007)	(1.619e-007)	(4.623e-007)	(4.500e-007)	(4.406e-007)	(9.063e-007)	(8.990e-007)	(8.933e-007)
$h_{t+1} = 1.00 \times E[h]$												
Johnson S_L	**6.278e-005**	**3.575e-005**	**2.304e-005**	**7.912e-005**	**5.823e-005**	**4.657e-005**	**8.682e-005**	**7.261e-005**	**6.445e-005**	**6.394e-005**	**5.379e-005**	**4.896e-005**
Monte Carlo	6.314e-005	3.582e-005	2.298e-005	8.119e-005	5.949e-005	4.728e-005	9.988e-005	8.367e-005	7.391e-005	1.058e-004	9.214e-005	8.389e-005
Std. Deviation	(1.325e-007)	(1.126e-007)	(9.617e-008)	(2.497e-007)	(2.317e-007)	(2.174e-007)	(5.585e-007)	(5.465e-007)	(5.371e-007)	(1.251e-006)	(1.245e-006)	(1.240e-006)
$h_{t+1} = 1.20 \times E[h]$												
Johnson S_L	**9.723e-005**	**6.098e-005**	**4.185e-005**	**1.071e-004**	**8.149e-005**	**6.664e-005**	**1.054e-004**	**8.909e-005**	**7.965e-005**	**7.364e-005**	**6.200e-005**	**5.659e-005**
Monte Carlo	9.774e-005	6.135e-005	4.196e-005	1.099e-004	8.356e-005	6.807e-005	1.203e-004	1.023e-004	9.124e-005	1.180e-004	1.037e-004	9.502e-005
Std. Deviation	(1.662e-007)	(1.499e-007)	(1.333e-007)	(3.101e-007)	(2.932e-007)	(2.790e-007)	(6.732e-007)	(6.618e-007)	(6.527e-007)	(1.151e-006)	(1.144e-006)	(1.139e-006)

"Johnson S_L" is the analytical approximation based on the S_L system of the Johnson density functions family, "Monte Carlo" is the monte carlo estimate of the option's value based on 500 000 scenarios, "Std.Deviation" is the standard deviation of the monte carlo estimate. Results presented in bold indicates that the pricing error is statistically different from zero with a confidence level of 5%.

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