Les Cahiers du GERAD ISSN: 0711–2440

Non Fragile Controller Design for Linear Markovian Jumping Parameters Systems

E.K. Boukas

G–2003–67

October 2003

Non Fragile Controller Design for Linear Markovian Jumping Parameters Systems

E.K. Boukas

GERAD and Mechanical Engineering Department
École Polytechnique de Montréal
P.O. Box 6079, station "Centre-ville"
Montréal, Québec, Canada H3C 3A7
el-kebir.boukas@polymtl.ca

October, 2003

Les Cahiers du GERAD

G–2003–67

Abstract

This paper deals with the uncertain class of continuous-time linear systems with Markovian jumping parameters. A design method for a non fragile robust controller for this class of systems is proposed when the uncertainties in the system matrices are of norm bounded types. LMI based sufficient condition is developed. The methodology used is mainly based on Lyapunov approach. A numerical example is presented to show the usefulness of the proposed results.

Key Words: Jump linear system, Linear matrix inequality, Stochastic stability, state feedback control, Norm bounded uncertainty.

Résumé

Ce papier traite de la classe des systèmes incertains continus à sauts markoviens. Une méthode de design d'un contrôleur nonfragile et robust pour la classe de systèmes considérée est développée quand les incertitudes sont du type borné norme. Les résultats développés sont en forme de LMI. La méthodologie repose sur la méthode de Lyapunov. Un exemple numérique est présenté pour montrer l'importance des résulats.

1 Introduction

The stabilization problem is a basic control problem that has attracted a lot of researchers and many papers on this subject have been reported in the literature. Its formulation when using state feedback controller can be stated as follow: Given a dynamical linear system how we can design a state feedback controller that stabilizes the closed-loop of the considered system. When the system has uncertainties in its matrices the problem becomes a robust stabilization one.

In practice there exist some systems that can't be modelled by the classical linear model that is widely used in the literature like for instance systems with abrupt changes in their structures that may be caused by many factors like failures, repairs, sudden environmental disturbance, changing subsystem interconnections, abrupt variations of the operating point of a nonlinear system, etc. This class of systems can be modelled by the class of linear system with Markovian jumping parameters which was introduced for the first time by Krasovskii and Lidskii (Ref. [3]). The power of this class of systems to model different practical systems, has been the catalyst of the development of this class of systems. For a recent review on this class of systems and its applications we refer the readers to Boukas and Liu (Ref. [1]) and the references therein on what it has been done on this class of systems. Most of the problems like stability, stabilization, $\mathcal{H}_\infty$ control, filtering and theirs robustness have been tackled and some interesting results already exist in the literature. For example we can refer the readers to Mao (Ref. [4]), Shi and Boukas (Ref. [5]), Shi et al. (Ref. [6]), Wang et al. (Ref. [7]) and to the references therein.

Most of the contributions to the class of systems with Markovian jumping parameters dealt with the design of controllers that cope with the system uncertainties but none of them has addressed the robustness with regard to the controllers uncertainties that may results from different causes like the errors in the electronic components for instance when the controllers are implemented using electronic components. In their study Keel and Bhattacharyya (Ref. [2]) have shown that the controller may be very sensitive or fragile to the errors in the controller parameters even if the design take care of the system uncertainties. To overcome this, the parameters variations should be included in the controller design phase besides the system uncertainties. The goal becomes then how to design a controller that is non fragile in the sense that the closed loop system tolerates a certain changes in the controller parameters and at the same time the system uncertainties that may affect the different matrices.

Our goal in this paper consists of designing a non fragile controller that can cope with norm bounded uncertainties that may affect the class of continuous-time Markovian jumping parameters we are considering in this paper and at the same time tolerate some changes in the controller parameters. To the best of our knowledge this problem has never tackled before for this class of systems. Our choice will be limited to conditions for robust stochastic stabilization in the form of LMI that may be solved easily using the existing convex optimization algorithms. The methodology using in this paper is mainly based on Lyapunov method.

The rest of the paper is organized as follows. In section 2, the stabilization problem is stated. Section 3 gives the main results of the paper. They comprise results on stochastic stability and the design method for a non fragile controller. In section 4, a numerical example is presented to show the usefulness of the developed results.

The notations used in this paper is standard unless it is mentioned otherwise. For symmetric matrices X and Y, $X > Y$ (resp. $X < Y$) means that $X - Y$ is positive-definite (resp. negative-definite). I denotes the identity matrix with the appropriate dimension that may be understood from the context. diag[.] denotes a block diagonal matrix.

2 Problem statement

Consider a continuous-time linear Markovian jumping parameters system defined in a fundamental probability space $(\Omega, \mathcal{F}, P)$ with the following dynamics:

$$\begin{cases} \dot{x}(t) = A(r_t, t)x(t) + B(r_t)u(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time, $u(t) \in \mathbb{R}^m$ is the control at time t, $A(r_t, t)$ is the state matrix that is assumed to contain uncertainties and its expression is given by:

$$A(r_t, t) = A(r_t) + D_A(r_t)F_A(r_t, t)E_A(r_t)$$

with $A(r_t)$, $D_A(r_t)$, $E_A(r_t)$ are known matrices, and $F_A(r_t, t)$ is the uncertainty of the state matrix; and $B(r_t)$ is the control matrix that is supposed to be known; $\{r_t, t \geq 0\}$ is continuous-time homogeneous Markov process with right continuous trajectories taking values in a finite set $\mathcal{S} = \{1, 2, \ldots, N\}$ with the following stationary transition probabilities:

$$P\left[r_{t+\Delta t} = j | r_t = i\right] = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t) & i \neq j \\ 1 + \lambda_{ii}\Delta t + o(\Delta t) & \text{otherwise} \end{cases} \tag{2}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0} \frac{o(\Delta t}{\Delta t} = 0$ and $\lambda_{ij} \geq 0$ is the transition probability from the mode i to the mode j at time t and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$.

The uncertainty in the state matrix is assumed to satisfy the following for every $r_t \in \mathcal{S}$:

$$F_A^\top(r_t, t)F_A(r_t, t) \leq I \tag{3}$$

Let us now define some concepts that will be used in the rest of the paper.

For system (1), when $F_A(r_t, t) \equiv 0$, i.e we drop the system's uncertainties, we have the following definitions.

Definition 2.1 *System (1), with $F_A(r_t, t) = 0$ for all $t \geq 0$, is said to be* *(i) stochastically stable (SS) if there exists a finite positive constant* $T(x_0, r_0)$ *such that the following holds for any initial conditions* (x_0, r_0)*:*

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(r_0, x_0); \tag{4}$$

(ii) mean square stable (MSS) if

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{5}$$

holds for any initial condition (x_0, r_0)*;*

(iii) mean exponentially stable (MES) if there exist positive constants α *and* β *such that the following holds for any initial conditions* (x_0, r_0)*:*

$$\mathbb{E}\left[\|x(t)\|^2 | x_0, r_0\right] \leq \alpha \|x_0\| e^{-\beta t}. \tag{6}$$

Remark 2.1 From the definitions, we can see that the mean exponentially stable (MES) implies the mean square stable (MSS) and the stochastically stable (SS).

When the system's uncertainties are not equal to zero, the concept of stochastic stability becomes robust stochastic stability and is defined for system (1), as follows.

Definition 2.2 *System (1) is said to be*

(i) robustly stochastically stable (RSS) if there exists a finite positive constant $T(x_0, r_0)$ *such that the condition (4) holds for any initial conditions* (x_0, r_0) *and for all admissible uncertainties;*

(ii) robust mean exponentially stable (RMES) if there exist positive constants α *and* β *such that the condition (6) holds for any initial conditions* (x_0, r_0) *and for all admissible uncertainties.*

Remark 2.2 From the definitions, we can see that the robust mean exponentially stable (RMES) implies the stochastically stable (RSS).

Definition 2.3 *System (1) with* $F_A(r_t, t) = 0$ *for all modes and for* $t \geq 0$*, is said to be stabilizable in the SS (MES, MSQS) sense if there exists a controller such that the closed-loop system is SS (MES, MSQS) for every initial conditions* (x_0, r_0)*.*

When the uncertainties are not equal to zero, the previous definition is replaced by the following one:

Definition 2.4 *System (1) is said to be robustly stabilizable in the stochastic sense if there exists a controller such that the closed-loop system is stochastically stable for every initial conditions* (x_0, r_0) *and for all admissible uncertainties.*

The problem we are facing in this paper consists of designing a state feedback controller that robustly stabilizes the closed loop of the system.

In general the state feedback control is given by:

$$u(t) = K(r_t)x(t), \text{ for every } r_t \in \mathcal{S} \tag{7}$$

But in practice the implementation is quite different from this expression and there is always uncertainties in the gain controller which means that the gain is given by:

$$K(r_t, t) = K(r_t) + \Delta K(r_t, t) \tag{8}$$

with $\Delta K(r_t, t)$ is given by:

$$\Delta K(r_t, t) = \rho(r_t) F_K(r_t, t) K(r_t) \tag{9}$$

where $\rho(r_t)$ is an uncertain real parameter indicating the measure of non fragility against controller gain variations and $F_K(r_t, t)$ is the uncertainty that will be supposed to satisfy the following for every $r_t \in \mathcal{S}$:

$$F_K^\top(r_t, t) F_K(r_t, t) \leq I \tag{10}$$

Our goal in this paper is to synthesize the gain for the state feedback controller with the following form for every $r_t \in \mathcal{S}$:

$$K(r_t) = \gamma(r_t) B^\top(r_t) P(r_t) \tag{11}$$

where $\gamma(r_t)$ is a real number and $P(r_t)$ is symmetric and positive-definite matrix for every $r_t \in \mathcal{S}$.

Plugging the controller in the dynamics we get the following closed loop dynamics:

$$\begin{aligned} \dot{x}_t &= \left[A(r_t, t) + B(r_t) K(r_t, t)\right] x_t \\ &= \Big[A(r_t) + D_A(r_t) F_A(r_t, t) E_A(r_t) + B(r_t) \Big[\gamma(r_t) B^\top(r_t) P(r_t) \\ &\quad + \rho(r_t) F_K(r_t, t) \gamma(r_t) B^\top(r_t) P(r_t)\Big]\Big] x_t \end{aligned} \tag{12}$$

In the rest of this we will propose an LMI design approach to compute the controller gain, $P(r_r)$ and $\gamma(r_t)$ for each $r_t \in \mathcal{S}$.

The following lemmas will be used in the rest of the paper. For their proofs, we refer the reader to Boukas and Liu (Ref. [1]).

Lemma 2.1 *Let D, F and E be real constant matrices of compatible dimensions with $F^\top F \leq I$, then, the following:*

$$DFE + E^\top F^\top D^\top \leq \varepsilon DD^\top + \frac{1}{\varepsilon} E^\top E$$

holds for any $\varepsilon > 0$.

Lemma 2.2 *(Schur Complement) Let the symmetric matrix M be partitioned as*

$$M = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix}$$

with X, Z being symmetric matrices. We have
(i) M is nonnegative-definite if and only if either

$$\begin{cases} Z \geq 0 \\ Y = L_1 Z \\ X - L_1 Z L_1^\top \geq 0 \end{cases} \tag{13}$$

or

$$\begin{cases} X \geq 0 \\ Y = X L_2 \\ Z - L_2^\top X L_2 \geq 0 \end{cases} \tag{14}$$

hold, where L_1, L_2 are some (non unique) matrices of compatible dimensions.
(ii) M is positive-definite if and only if either

$$\begin{cases} Z > 0 \\ X - Y Z^{-1} Y^\top > 0 \end{cases} \tag{15}$$

or

$$\begin{cases} X > 0 \\ Z - Y^\top X^{-1} Y > 0 \end{cases} \tag{16}$$

Matrices $X - YZ^{-1}Y^\top$ is called the Schur complement *$X(Z)$ in M.*

3 Main results

In this section we will develop the main results of this paper that are related to the robust stability and the robust stabilization problems for the class of systems we are considering. All the results are LMI based which make them easily solvable using existing convex optimization algorithms.

Let us now study the stability problem. For this purpose let the control $u_t = 0$ for $t \geq 0$. The following theorem give the result on robust stochastic stability.

Theorem 3.1 *If there exist symmetric and positive-definite matrix $P = (P(1), \ldots, P(N))$ and a positive real number ε_A such that the following holds for every $r_t \in \mathcal{S}$ and for all admissible uncertainties:*

$$\begin{bmatrix} J_u(r_t) & P(r_t) D_A(r_t) \\ D_A^\top(r_t) P(r_t) & -\varepsilon_A I \end{bmatrix} < 0 \tag{17}$$

where $J_u(r_t) = A^\top(r_t)P(r_t) + P(r_t)A(r_t) + \sum_{j=1}^N \lambda_{r_tj}P(j) + \varepsilon_A^{-1}E_A^\top(r_t)E_A(r_t)$*, then system (1) is robustly stochastically stable.*

Proof: To prove this theorem, let us consider a Lyapunov function candidate with the following expression:

$$V(x(t), r_t) = x^\top(t)P(r_t)x(t) \tag{18}$$

where $P(r_t)$ is symmetric and positive-definite matrix for every $r_t \in \mathcal{S}$.

Let $\mathbb{L}$ be the infinitesimal generator of the Markov process $(x(t), r_t)$. Then, the expression of the infinitesimal generator is given by:

$$\begin{aligned}
\mathbb{L}V(x(t), r_t) &= \lim_{h\to 0} \frac{\mathbb{E}\left[V(x(t+h), r_{t+h})|x(t) = x_t, r_t = i\right] - V(x_t, i)}{h} \\
&= \quad \dot{x}^\top(t)P(r_t)x(t) + x^\top(t)P(r_t)\dot{x}(t) + \sum_{j=1}^N \lambda_{r_tj}x^\top(t)P(j)x(t) \\
&= x^\top(t)A^\top(r_t, t)P(r_t)x(t) + x^\top(t)P(r_t)A(r_t, t)x(t) \\
&\quad + \sum_{j=1}^N \lambda_{r_tj}x^\top(t)P(j)x(t)
\end{aligned}$$

Using now the structure of the uncertainties we get:

$$\begin{aligned}
\mathbb{L}V(x(t), r_t) &= x^\top(t)\left[A(r_t) + D_A(r_t)F_A(r_t, t)E_A(r_t)\right]^\top P(r_t)x(t) \\
&\quad + x^\top(t)P(r_t)\left[A(r_t) + D_A(r_t)F_A(r_t, t)E_A(r_t)\right]x(t) \\
&\quad + \sum_{j=1}^N \lambda_{r_tj}x^\top(t)P(j)x(t)
\end{aligned}$$

$$\begin{aligned}
&= x^\top(t)\left[A^\top(r_t)P(r_t) + P(r_t)A(r_t)\right]x(t) \\
&\quad + 2x^\top(t)P(r_t)D_A(r_t)F_A(r_t, t)E_A(r_t)x(t) \\
&\quad + x^\top(t)\sum_{j=1}^N \lambda_{r_tj}P(j)x(t)
\end{aligned} \tag{19}$$

Notice that using Lemma 2.1, we have:

$$\begin{aligned}
&2x^\top(t)P(r_t)D_A(r_t)F_A(r_t, t)E_A(r_t)x(t) \leq \varepsilon_A^{-1}x^\top(t)P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)x(t) \\
&\quad +\varepsilon_A x^\top(t)E_A^\top(r_t)E_A(r_t)x(t)
\end{aligned}$$

Using this relation, Eq. (19) becomes:

$$\begin{aligned}\mathbb{L}V(x(t),r_t) &\leq x^\top(t)\left[A^\top(r_t)P(r_t)+P(r_t)A(r_t)+\sum_{j=1}^N \lambda_{r_tj}P(j)\right]x(t)\\ &\quad +\varepsilon_A^{-1}x^\top(t)P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)x(t)\\ &\quad +\varepsilon_A x^\top(t)E_A^\top(r_t)E_A(r_t)x(t)\\ &\leq x^\top(t)\Lambda_u(r_t)x(t)\end{aligned}$$

$$\begin{aligned}\Lambda_u(r_t) &= A^\top(r_t)P(r_t)+P(r_t)A(r_t)+\sum_{j=1}^N \lambda_{r_tj}P(j)\\ &\quad +\varepsilon_A^{-1}x^\top(t)P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)+\varepsilon_A E_A^\top(r_t)E_A(r_t).\end{aligned}$$

Using (17) and Schur complement we conclude that $\Lambda_(r_t) < 0$ and therefore we get:

$$\mathbb{L}V(x(t),r_t) \leq -\min_{i\in\mathbf{S}} x^\top(t)\lambda_{min}\left(-\Lambda_u(i)\right)x(t) \tag{20}$$

with:

Combining this again with Dynkin's formula yields

$$\begin{aligned}\mathbb{E}\left[V(x(t),r_t)\right]-\mathbb{E}\left[V(x(0),r_0)\right] &= \mathbb{E}\left[\int_0^t \mathbb{L}V(x(s),r_s)ds|(x_0,r_0)\right]\\ &\leq -\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Lambda_u(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right],\end{aligned}$$

implying, in turn,

$$\begin{aligned}&\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Lambda_u(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right]\\ &\quad \leq \mathbb{E}\left[V(x(0),r_0)\right]-\mathbb{E}\left[V(x(t),r_t)\right]\\ &\quad \leq \mathbb{E}\left[V(x(0),r_0)\right].\end{aligned}$$

This yields that

$$\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right] \leq \frac{\mathbb{E}\left[V(x(0),r_0)\right]}{\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Lambda_u(i))\}}$$

holds for any $t > 0$. This proves Theorem 3.1. □

Remark 3.1 The condition we give in this theorem is a sufficient one which means that if we are not able to find a set $P = (P(1),\ldots,P(N))$ of symmetric and positive-definite matrices that satisfies the condition (17), this doesn't imply that the dynamical system is not robustly stochastically stable.

Let us now return to the initial problem and see how we can design a robust controller that can handle the uncertainties in the system matrices and the controller gains. The following theorem gives a result in the LMI framework that can be used to design a such controller.

Theorem 3.2 *If there exist a set of symmetric and positive-definite matrix $X = (X(1), \ldots, X(N))$ and positive scalars $\varepsilon_A(r_t)$, $\mu(r_t)$, $\nu(r_t)$ and a scalar $\gamma(r_t)$ satisfying the following LMI for every $r_t \in \mathcal{S}$ and for all admissible uncertainties:*

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^\top(r_t) & \gamma(r_t)B(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A I & 0 & 0 \\ \gamma(r_t)B^\top(r_t) & 0 & -\mu(r_t)I & 0 \\ \mathcal{S}_{r_t}^\top(X) & 0 & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{21}$$

where

$$\begin{aligned} J(r_t) &= X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A D_A(r_t)D_A(r_t) + 2\gamma(r_t)B(r_t)B^\top(r_t) + \lambda_{r_t r_t}X(r_t) \\ &\quad + \nu(r_t)B(r_t)B^\top(r_t) \\ \mu(r_t) &= \frac{\varepsilon_K(r_t)}{\rho(r_t)} \\ \nu(r_t) &= \varepsilon_K(r_t)\rho(r_t) \end{aligned}$$

then closed-loop system is robustly stochastically stable with non fragility $\rho(r_t)$ under the controller (7) with the gain $K(r_t) = \gamma(r_t)B^\top(r_t)X^{-1}(r_t)$.

Proof: Let $P(r_t)$ be a symmetric and positive-definite matrix and consider the following candidate Lyapunov function be given as follows:

$$V(x_t, r_t) = x_t^\top P(r_t)x_t \tag{22}$$

The infinitesimal generator of the process $\{(x_t, r_t), t \geq 0\}$ is given by:

$$\begin{aligned} \mathbb{L}V(x_t, r_t) &= \lim_{h \to 0} \frac{\mathbb{E}\left[[V(x_{t+h}, r_{t+h}) - V(x_t, r_t)|x_t, r_t]\right]}{h} \\ &= \dot{x}_t^\top P(r_t)x_t + x_t^\top P(r_t)\dot{x}_t + \sum_{j=1}^{N} \lambda_{r_t j} x_t^\top P(j)x_t \\ &= \left[[A(r_t, t) + B(r_t)K(r_t, t)]\, x_t\right]^\top P(r_t)x_t + x_t^\top P(r_t)\left[A(r_t, t) + B(r_t)K(r_t, t)\right] x_t \\ &\quad + \sum_{j=1}^{N} \lambda_{r_t j} x_t^\top P(j)x_t \end{aligned}$$

$$= x_t^\top A^\top(r_t)P(r_t)x_t + x_t^\top P(r_t)A(r_t)x_t + 2x_t^\top P(r_t)D_A(r_t)F_A(r_t,t)E_A(r_t)x_t$$
$$2\gamma(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t + \sum_{j=1}^{N} \lambda_{r_tj}x_t^\top P(j)x_t$$
$$+ 2\rho(r_t)\gamma(r_t)x_t^\top P(r_t)B(r_t)F_K(r_t,t)B^\top(r_t)P(r_t)x_t \tag{23}$$

Using now Lemma 2.1, we get:

$$\begin{aligned} 2x_t^\top P(r_t)D_A(r_t)F_A(r_t,t)E_A(r_t)x_t \le\ & \varepsilon_A(r_t)x_t^\top P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)x(t) \\ & + \varepsilon_A^{-1}x^\top(t)(r_t)E_A^\top(r_t)E_A(r_t)x_t \end{aligned}$$

$$\begin{aligned} & 2\rho(r_t)\gamma(r_t)x_t^\top P(r_t)B(r_t)F_K(r_t,t)B^\top(r_t)P(r_t)x_t \\ & \quad \le \varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \\ & \quad\quad + \varepsilon_K(r_t)\rho(r_t)x_t^\top P(r_t)B(r_t)F_K^\top(r_t,t)F_K(r_t,t)B^\top(r_t)P(r_t)x_t \\ & \quad \le \varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \\ & \quad\quad + \varepsilon_K(r_t)\rho(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \end{aligned} \tag{24}$$

Based on this, the previous expression of $\mathbb{L}V(x_t, r_t)$ becomes:

$$\begin{aligned} \mathbb{L}V(x_t,r_t) \le\ & x_t^\top A^\top(r_t)P(r_t)x_t + x_t^\top P(r_t)A(r_t)x_t + \varepsilon_A(r_t)x_t^\top P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)x(t) \\ & +\varepsilon_A^{-1}(r_t)x_t^\top E_A^\top(r_t)E_A(r_t)x_t + \sum_{j=1}^{N}\lambda_{r_tj}x_t^\top P(j)x_t \\ & +\varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \\ & +\varepsilon_K(r_t)\rho(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \\ & +2\gamma(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t \\ \le\ & x_t^\top \Gamma(r_t)x_t \end{aligned} \tag{25}$$

with

$$\begin{aligned} \Gamma(r_t) =\ & A^\top(r_t)P(r_t) + P(r_t)A(r_t) + \varepsilon_A(r_t)P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) \\ & +\varepsilon_A^{-1}(r_t)E_A^\top(r_t)E_A(r_t) + \sum_{j=1}^{N}\lambda_{r_tj}P(j) \\ & +\varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t) + \varepsilon_K(r_t)\rho(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t) \\ & +2\gamma(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t) \end{aligned} \tag{26}$$

The expression $\Gamma(r_t)$ is nonlinear in the design parameters $\gamma(r_t)$ and $P(r_t)$ for every $r_t \in \mathcal{S}$ to cast it into in an LMI, let us put $X(r_t) = P^{-1}(r_t)$ for each $r_t \in \mathcal{S}$. Let us now pre- and post-multiplying (26) by $X(r_t)$ we get:

$$\begin{aligned} X(r_t)\Lambda(r_t)X(r_t) &= X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A(r_t)D_A(r_t)D_A^\top(r_t) \\ &\quad + \varepsilon_A^{-1}(r_t)X(r_t)E_A^\top(r_t)E_A(r_t)X(r_t) + \sum_{j=1}^N \lambda_{r_tj}X(r_t)X^{-1}(j)X(r_t) \\ &\quad + \varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)B(r_t)B^\top(r_t) + \varepsilon_K(r_t)\rho(r_t)B(r_t)B^\top(r_t) \\ &\quad + 2\gamma(r_t)B(r_t)B^\top(r_t) \end{aligned} \tag{27}$$

Letting $S_{r_t}(X)$ and $\mathcal{X}_{r_t}(X)$ be defined as follows:

$$\begin{cases} S_{r_t}(X) = \left[\sqrt{\lambda_{r_t1}}X(r_t), \ldots \sqrt{\lambda_{r_tr_t-1}}X(r_t), \sqrt{\lambda_{r_tr_t+1}}X(r_t), \ldots, \sqrt{\lambda_{r_tN}}X(r_t)\right] \\ \mathcal{X}_{r_t}(X) = \operatorname{diag}\left[X(1), \ldots, X(r_t-1), X(r_t+1), \ldots, X(N)\right] \end{cases} \tag{28}$$

the term $X(r_t)\left[\sum_{j=1}^N \lambda_{r_tj}X^{-1}(j)\right]X(r_t)$ can be rewritten as follows:

$$X(r_t)\left[\sum_{j=1}^N \lambda_{r_tj}X^{-1}(j)\right]X(r_t) = \lambda_{r_tr_t}X(r_t) + S_{r_t}(X)\mathcal{X}_{r_t}^{-1}(X)\mathcal{S}_{r_t}^\top(X)$$

Using now (21) and Schur complement we get:

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^\top(r_t) & \gamma(r_t)B(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A(r_t)I & 0 & 0 \\ \gamma(r_t)B^\top(r_t) & 0 & -\frac{\varepsilon_K(r_t)}{\rho(r_t)}I & 0 \\ \mathcal{S}_{r_t}^\top(X) & 0 & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned} J(r_t) &= X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A(r_t)D_A(r_t)D_A^\top(r_t) \\ &\quad + 2\gamma(r_t)B(r_t)B^\top(r_t) + \lambda_{r_tr_t}X(r_t) \quad + \varepsilon_K(r_t)\rho(r_t)B(r_t)B^\top(r_t) \end{aligned} \tag{30}$$

which is symmetric and negative-definite by hypothesis and therefore we conclude that $\Gamma(r_t) < 0$ which implies:

$$\mathbb{L}V(x_t, r_t) \leq -\min_{i\in\mathbf{S}} x^\top(t)\lambda_{min}\left(-\Gamma(i)\right)x(t) \tag{31}$$

Combining this again with Dynkin's formula yields

$$\begin{aligned} \mathbb{E}\left[V(x_t, r_t)\right] - \mathbb{E}\left[V(x_0, r_0)\right] &= \mathbb{E}\left[\int_0^t \mathbb{L}V(x_s, r_s)ds|(x_0, r_0)\right] \\ &\leq -\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0, r_0)\right], \end{aligned}$$

implying, in turn,

$$\begin{aligned}
&\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right] \\
&\quad\leq \mathbb{E}\left[V(x(0),r_0)\right] - \mathbb{E}\left[V(x(t),r_t)\right] \\
&\quad\leq \mathbb{E}\left[V(x(0),r_0)\right].
\end{aligned}$$

This yields that

$$\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right] \leq \frac{\mathbb{E}\left[V(x(0),r_0)\right]}{\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma(i))\}}$$

holds for any $t > 0$. This proves that the closed-loop is stable under the chosen controller. □

When the controller gains don't have uncertainties the previous result becomes easier and it is summarized in the following theorem:

Theorem 3.3 *If there exist a set of symmetric and positive-definite matrix $X = (X(1), \ldots, X(N))$ and positive scalar $\varepsilon_A(r_t)$, and $\gamma(r_t)$ satisfying the following LMI for every $r_t \in \mathcal{S}$ and all admissible uncertainties:*

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^\top(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A I & 0 \\ \mathcal{S}_{r_t}^\top(X) & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{32}$$

where

$$J(r_t) = X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A D_A(r_t)D_A^\top(r_t) + 2\gamma(r_t)B(r_t)B^\top(r_t) + \lambda_{r_t r_t}X(r_t)$$

then closed loop system is robustly stochastically stable under the controller (7) with the gain $K(r_t) = \gamma(r_t)B^\top(r_t)X^{-1}(r_t)$.

Proof: The details of the proof is similar to the one of the previous theorem and follows the same steps. □

4 Numerical example

In this section we will show the usefulness of the proposed results in this paper. For this purpose let us consider a system with two modes and two components in the state vector. Let the data in each mode be given by:

- mode 1:

$$
\begin{aligned}
A(1) &= \begin{bmatrix} 1.0 & -0.5 \\ 0.1 & 1.0 \end{bmatrix} \\
B(1) &= \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \\
D_A(1) &= \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \\
E_A(1) &= \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}
\end{aligned}
$$

- mode 2:

$$
\begin{aligned}
A(2) &= \begin{bmatrix} -0.2 & 0.5 \\ 0.0 & -0.25 \end{bmatrix} \\
B(2) &= \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \\
D_A(2) &= \begin{bmatrix} 0.13 \\ 0.1 \end{bmatrix} \\
E_A(2) &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}
\end{aligned}
$$

Let the transition probability matrix between these two modes be given by:

$$
\Lambda = \begin{bmatrix} -2.0 & 2.0 \\ 3.0 & -3.0 \end{bmatrix}
$$

Letting $\varepsilon_A(1) = \varepsilon_A(2) = 0.5$, $\varepsilon_K(1) = \varepsilon_K(2) = 0.1$, and $\rho(1) = 0.5$, $\rho(2) = 0.6$ and solving the LMI (21), we get:

$$
\begin{aligned}
X(1) &= \begin{bmatrix} 0.4589 & -0.0270 \\ -0.0270 & 0.3248 \end{bmatrix} \\
X(2) &= \begin{bmatrix} 0.4589 & -0.0270 \\ -0.0270 & 0.3248 \end{bmatrix} \\
\gamma(1) &= -0.1891 \\
\gamma(2) &= -0.1310
\end{aligned}
$$

which gives the following controller gains:

$$
\begin{aligned}
K(1) &= \begin{bmatrix} -3.6899 & 0.4446 \\ 0.4446 & -4.1029 \end{bmatrix} \\
K(2) &= \begin{bmatrix} -0.2868 & -0.0238 \\ -0.0238 & -0.4052 \end{bmatrix}.
\end{aligned}
$$

5 Conclusion

This paper deals with the class of uncertain continuous-time Markovian jump linear systems. The uncertainties we considered in this paper were of norm bounded type. A design LMI method was developed to synthesize a state feedback controller that robustly stochastically stabilizes the class under study. The condition we established is easily solvable using existing convex optimization algorithms.

References

1. BOUKAS, E. K., and LIU, Z. K., *Deterministic and Stochastic Systems with Time-Delay*, Birkhauser, Boston, 2002.

2. KEEL, L. H., and BHATTACHARYYA, *Robust, Fragile, or Optimal*, IEEE, Transactions on Automatic Control, Vol. 42, pp. 1098–1105, 1997.

3. KRASOVSKII, N. N., and LIDSKII, E. A., *Analysis Design of Controller in Systems with Random Attributes-Part 1*, Automation Remote Control, Vol. 35, pp. 1021–1025, 1961.

4. MAO, X., *Stability of Stochastic Differential Equations with Markovian Switching*, Stochastic Processes and their Apllications, Vol. 79, pp. 45–67, 1999.

5. SHI, P., and BOUKAS, E. K., *$\mathcal{H}_\infty$ Control for Markovian Jumping Linear Systems with Parametric Uncertainty*, Journal of Optimization Theory and Apllications, Vol. 95, pp. 75–99, 1997.

6. SHI, P., BOUKAS, E. K., and AGARWAL, R. K., *Control of Markovian Jump Discrete-Time Systems with Norm Bounded Uncertainty and Unknown delay*, IEEE, Transactions on Automatic Control, Vol. 44, pp. 2139–2144, 1999.

7. WANG, Z., QIAO, H., and BURNHAM, K. J., *On Stabilization of Bilinear Uncertain Time-Delay Stochastic Systems with Markovian Jumping Parameters*, IEEE, Transactions on Automatic Control, Vol. 47, No. 4, pp. 460–646, 2002.