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# Dynamic Model of R&D, Spillovers and Efficiency of Bertrand and Cournot Equilibria*

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**Abstract**

Using an inifinite-horizon two-player differential game, we derive and compare Bertrand and Cournot equilibria, for a differentiated duopoly engaging in process R&D competition. The main findings of this study are as follows. First, Bertrand competition is more efficient if either R&D productivity is low, or products are very different. Second, Cournot competition is more efficient provided that R&D productivity is high, products are close substitutes and spillovers are not close to zero. This last result is different from what has been obtained in the literature. This shows hence that considering a dynamic model and more general investment costs do have an impact on efficiency results.

**Keywords:** Differential games, Research and development, Bertrand equilibrium, Cournot equilibrium, Social optimum, Duopoly.

**Résumé**

On considère un jeu différentiel à deux joueurs et à horizon infini où les joueurs investissent en R&D pour réduire leurs coûts de production. On calcule et on compare les équilibres de Bertrand et de Cournot. Les résultats sont comme suit. La concurrence à la Bertrand est plus efficiente si la productivité de la R&D est basse ou si les produits sont très différents. Par contre, si la productivité de la R&D est élevée, les produits sont similaires et les effets de débordement sont différents de zéro, alors la concurrence à la Cournot est plus efficiente. Ce dernier résultat est différent de ceux obtenus dans la littérature avec des modèles statiques et des fonctions de coûts moins générales que celle considérée ici.

**Mots clés :** Jeux différentiels, Recherche et développement, Équilibre de Bertrand, Équilibre de Cournot, Optimum social, Duopole.

# 1 Introduction

We develop in this paper infinite-horizon differential game models for a differentiated duopoly producing substitutable goods. The firms' aims are to maximize their total discounted profit functions by choosing the optimal levels of either their outputs or prices as well as research and development (R&D) investments. The latter investments are of the production process type, i.e., are intended to reduce production costs. Due to the presence of possible R&D spillovers, however, firms may also benefit from each other in decreasing these costs, depending on the degree of substitutability between their products. This implies that each firm may inadvertently make the other firm a tougher competitor.

Our main objectives are to characterize and compare the efficiency of Bertrand and Cournot equilibria and to examine the robustness of the results obtained in the literature in typically static or two-stage games setting.

In a seminal paper, Singh and Vives (Ref. 1) show that in a differentiated duopoly (i) Bertrand (*price*) competition is always more efficient than Cournot (*quantity*) competition, (ii) Bertrand prices (*quantities*) are smaller (*larger*) than Cournot prices (*quantities*) if the goods are substitutes (*complements*), and (iii) it is a dominant strategy for a firm to choose quantity (*price*) as its strategic variable provided that the goods are substitutes (*complements*). These findings attracted economists' attention and two main streams have appeared in the literature. The first stream extends the above model in different ways, e.g., more general class of cost and demand functions, more players, homogeneous product, quality differences (see Refs. 2–8).

A second stream, to which this paper naturally belongs, introduces investments in R&D as additional strategic variables. Such investments may lead to improvement of the quality of product(s) (in the case of product R&D) and/or to reduction in production cost (in the case of process R&D). Hence the demand and/or cost structures of the Cournot and Bertrand markets may change. One objective of this stream is to reexamine the issue of efficiency of the two equilibria (Cournot and Bertrand). For instance, Delbono and Denicolo (Ref. 9) consider a homogenous duopoly with process R&D in the form of patent race and show that although R&D investments are higher in the Cournot competition, the comparison of the efficiency of Bertrand and Cournot outcomes is generally ambiguous. The same conclusions are obtained in Motta (Ref. 10) for a vertically differentiated duopoly with either fixed or variable costs of quality improvements. Qiu (Ref. 11) extends the model in Ref. 1 by introducing a stage of process R&D. He shows that (i) although Cournot competition induces more R&D effort than Bertrand competition, the latter results in lower prices and higher quantities, (ii) Bertrand competition is more efficient if either R&D productivity is low, or spillovers are weak, or products are very different, and (iii) Cournot competition is more efficient if either R&D productivity is high, spillovers are strong, and products are close substitutes. Finally, Symeonidis (Ref. 12) compares the Bertrand and Cournot equilibria in a differentiated duopoly with substitute goods and product R&D.

This paper extends Qiu's model in different respects and checks if the comparative efficiency results still hold when firms are not necessarily symmetric, operate in a dynamic environment rather than in a static one[1], actual spillover is related to product differentiation and face more general production and process R&D costs. Our conclusion is that the results reported in Ref. 11 still hold in general but in the case where R&D productivity and product substitutability are high.

The rest of the paper is organized as follows. In Section 2, the proposed model is outlined. Cournot, Bertrand and first best equilibria are presented in Sections 3, 4 and 5, respectively. In Section 6, Cournot and Bertrand outcomes are compared and the results of numerical experiments are presented and analyzed. Finally, concluding remarks are presented in Section 7.

## 2 The Model

Consider a non-cooperative differential game with two firms producing one variety of differentiated but substitutable goods. Firms independently undertake cost-reducing R&D. They also sell their products in the market.

As in Ref. 1, the representative consumer's preferences are described by the following utility function:

$$U(q_1, q_2) = A(q_1 + q_2) - \frac{\omega}{2}(q_1 + q_2)^2 - \eta q_1 q_2 \tag{1}$$

where $q_i$ is firm $i$'s output, and $0 \leq \eta \leq \omega$. The ratio $\frac{\eta}{\omega}$ represents the degree of product differentiation; the closer this term is to 1 the more substitutes the products are. The resulting market inverse demands are linear and given by

$$P_i = A - \omega q_i - \eta q_j, \qquad i, j = 1, 2, i \neq j. \tag{2}$$

Let $K_i$ be firm $i$'s accumulated capital stock of R&D. The total production cost of firm $i$ is supposed to depend on the quantity produced and, to take into account spillovers in knowledge, on both players' R&D stocks. The following quadratic form is adopted

$$C_i(q_i, K_i, K_j) = (c_i + \sigma q_i - \psi(K_i + \frac{\eta}{\omega}\beta K_j))q_i \tag{3}$$

where $c_i < A, 0 \leq \psi \leq 1, \sigma \geq 0$, and $0 \leq \beta \leq 1$. The degree of spillover of the process R&D is thus given by $\frac{\eta}{\omega}\beta$, where the spillover parameter $\beta$ is assumed to be exogenous. Note that, contrary to previous contributions in this area, our formulation assumes that the degree of spillover depends on the degree of product substitutability; the higher the latter, the higher the degree of spillover can be. In particular, if the products are unrelated, then

[1] Typically in the R&D literature, the game is a two-stage one where in the first stage the firms determine their investments in R&D and in the second stage they engage in market competition. In such framework there is no dynamic or carry-over effects of investments in R&D.

a reasonable assumption would be that firms will not benefit from each other's research effort. Equation (3) is assumed to be strictly positive. Note that $C_i(\cdot)$ is convex increasing in $q_i$ and linear decreasing in capital stocks.

Denote by $I_i$ the investment of firm $i$ in R&D. The resulting investment cost is assumed to be quadratic convex increasing

$$F_i(I_i) = \theta_i I_i + \frac{\phi}{2} I_i^2 \tag{4}$$

where $\theta_i$, $\phi > 0$. This specification is similar to those used in Refs. 13–14.

Firm $i$'s capital stock evolves over time according to the following standard capital accumulation process

$$\dot{K}_i = I_i - \delta K_i, \quad K_i(0) = K_{i0} \tag{5}$$

where $\delta$, $0 \le \delta < 1$, is the depreciation rate.

The following table summarizes the assumptions made on the parameters values in Ref. 11 and in this paper.

| | | | | | | |
|---|---|---|---|---|---|---|
| Qiu (Ref. 11) | $c_i = c_j$ | $\sigma = 0$ | $\psi = 1$ | $\frac{\eta}{\omega} = 1$ | $0 \le \beta \le 1$ | $\theta_i = 0$ |
| This paper | $c_i \ne c_j$ | $\sigma > 0$ | $0 \le \psi \le 1$ | $0 \le \frac{\eta}{\omega} \le 1$ | $0 \le \beta \le 1$ | $\theta_i > 0$ |

As readily seen, the firms' costs of production (4) differ from the one proposed in Ref. 11 in the following aspects. First, for any given degree of spillover, the more substitutes the firms' products are the more each firm benefits from the other's accumulated capital stock, in reducing its costs. This in turn implies that, even if the degree of spillover is very high, if firms' products are unrelated (not substitutes) then these firms will not benefit from each other's capital stocks. Second, firm $i$'s production cost function is assumed to be quadratic in its level of production rather than linear. Note also that this formulation is more general than the one proposed in the seminal paper by d'Aspremont and Jacquemin (Ref. 15), and adopted in subsequent literature of R&D.

Assuming that each player maximizes her total discounted stream of profits over an infinite horizon, then player $i$'s objective functional is given by

$$\pi^i = \int_0^\infty e^{-rt} [P_i(t) q_i(t) - C_i\left(q_i(t), K_i(t), K_j(t)\right) - F_i(I_i(t))] dt \tag{6}$$

where $r$, $0 < r \le 1$, is the discount rate.

In addition to deciding on the level of its R&D, firm $i$ chooses either output (in the Cournot game) or price (in the Bertrand game) levels so as to maximize $\pi^i$ subject to the state dynamics in (5). Using differential game terminology, $K_i, i = 1, 2$, are the state variables, and $I_i$ and $q_i$ (or $P_i$) are the control variables. Note that the differential game at hand is a linear-quadratic one. It is well known (see, e.g., Ref. 16) that value functions of such games are quadratic and strategies are linear in state. Further, we confine our interest to stationary Markovian feedback strategies which is standard in infinite horizon differential games in economics.

We shall compare outputs, prices, investments and R&D stocks obtained at steady state under different modes of play, i.e., Cournot, Bertrand and social planner solution. We then compare consumer surplus ($CS$), profits and total welfare ($TW$) obtained under these different modes of play which are defined as follows (the bar on a variable refers to its steady state value):

$$CS = U(\bar{q}_1, \bar{q}_2) - \bar{P}_1\bar{q}_1 - \bar{P}_2\bar{q}_2 \quad (7)$$

$$\pi_i = \bar{P}_i\bar{q}_i - C_i(\bar{q}_i, \bar{K}_i, \bar{K}_j) - F_i(\bar{I}_i) \quad (8)$$

$$TW = CS + \sum_{i=1}^{2} \pi_i \quad (9)$$

## 3 Cournot Feedback Equilibrium

In the non-cooperative Cournot feedback game, firms select independently their output $q_i$ and R&D investment levels $I_i$. The following proposition characterizes the resulting feedback Cournot equilibrium, where the superscript $C$ refers to Cournot solution.

**Proposition 3.1** *The firms' Cournot feedback equilibrium output and R&D investment strategies are given by*

$$q_1^C = \frac{1}{m_6}\left(m_1K_1 + m_2K_2 + m_3A + m_4c_1 + m_5c_2\right) \quad (10)$$

$$q_2^C = \frac{1}{m_6}\left(m_2K_1 + m_1K_2 + m_3A + m_5c_1 + m_4c_2\right) \quad (11)$$

$$I_i^C = \frac{1}{\phi}[z_{2i}^C + z_5^C K_i + z_4^C K_j - \theta_i], i, j = 1, 2, i \neq j, \quad (12)$$

*where the constants* $m_1, \ldots, m_6$ *are given by*

$$\begin{aligned} m_1 &= 2\psi\omega(\omega+\sigma) - \psi\eta^2\beta \\ m_2 &= \left(2\beta(\omega+\sigma) - \omega\right)\eta\psi \\ m_3 &= \left(2(\omega+\sigma) - \eta\right)\omega \\ m_4 &= -2(\omega+\sigma)\omega \\ m_5 &= \eta\omega \\ m_6 &= \left(4(\omega+\sigma)^2 - \eta^2\right)\omega, \end{aligned}$$

*and the coefficients* $z_{ij}^C$ *solve the system of equations in Appendix 1.*

*The value functions are quadratic and given by*

$$V_i^C(K_i, K_j) = z_{1i}^C + z_{2i}^C K_i + z_{3i}^C K_j + z_4^C K_i K_j + \frac{z_5^C}{2}K_i^2 + \frac{z_6^C}{2}K_j^2, \; i, j = 1, 2, i \neq j. \quad (13)$$

**Proof.** We apply a standard sufficient condition for a stationary feedback equilibrium and wish to find bounded and continuously differentiable functions $V_i(K_i, K_j), i, j = 1, 2, i \neq j$,satisfying the Hamilton-Jacobi-Bellman (HJB) equations

$$\begin{aligned} rV_i(K_i, K_j) \quad = \quad & \max_{I_i, q_i}\{(A - \omega q_i - \eta q_j)\, q_i - (c_i + \sigma q_i - \psi(K_i + \frac{\eta}{\omega}\beta K_j))q_i \\ & - \left(\theta_i I_i + \frac{\phi}{2} I_i^2\right) + \frac{\partial V_i(.)}{\partial K_i}(I_i - \delta K_i) + \frac{\partial V_i(.)}{\partial K_j}(I_j - \delta K_j)\}. \end{aligned} \tag{14}$$

Differentiating the right-hand side w.r.t. $q_i$ and $I_i$ and equating to zero leads to

$$q_i^C = \frac{1}{2(\omega + \sigma)} \left(A + \eta q_j^C - c_i + \psi(K_i + \frac{\eta}{\omega}\beta K_j)\right), \; i, j = 1, 2, i \neq j \tag{15}$$

$$I_i^C = \frac{1}{\phi}\left(\frac{\partial V_i(.)}{\partial K_i} - \theta_i\right), i, j = 1, 2, i \neq j. \tag{16}$$

Substituting for outputs and investments in (14) yields

$$\begin{aligned} rV_i(K_i, K_j) = \left(A - \omega q_i^C - \eta q_j^C\right) q_i^C - (c_i + \sigma q_i^C - \psi(K_i + \frac{\eta}{\omega}\beta K_j))q_i^C \\ -\frac{\theta_i}{\phi}\left(\frac{\partial V_i(.)}{\partial K_i} - \theta_i\right) - \frac{1}{2\phi}\left(\frac{\partial V_i(.)}{\partial K_i} - \theta_i\right)^2 + \frac{\partial V_i(.)}{\partial K_i}\left[\frac{1}{\phi}\left(\frac{\partial V_i(.)}{\partial K_i} - \theta_i\right) - \delta K_i\right] \\ + \frac{\partial V_i(.)}{\partial K_j}\left[\frac{1}{\phi}\left(\frac{\partial V_j(.)}{\partial K_j} - \theta_j\right) - \delta K_j\right], i, j = 1, 2, i \neq j \end{aligned}$$

where $q_i^C$ and $q_j^C$ are given by (10) and (11).

Straightforward (however long) algebraic manipulations show that the following quadratic value functions are solutions to the above system of partial differential equations

$$V_i^C(K_i, K_j) = z_{1i}^C + z_{2i}^C K_i + z_{3i}^C K_j + z_4^C K_i K_j + \frac{z_5^C}{2} K_i^2 + \frac{z_6^C}{2} K_j^2 \;, i, j = 1, 2, i \neq j, \tag{17}$$

where the coefficients $z_{ki}^C$ solve the system of equations given in Appendix 1. ■

The following comments apply to the results in the above proposition:

- The value functions are quadratic and the strategies linear in the state. This result is a by-product of the (linear-quadratic) structure of the game.
- Under the assumptions made before, namely that $0 \leq \beta \leq 1$ and $\eta \leq \omega$, it is easy to verify that $m_1$ and $m_6$ are positive and hence $\frac{\partial q_i^C}{\partial K_i} = \frac{m_1}{m_6} > 0$, i.e. each player's output is an increasing function of her own capital stock.
- A condition for having $\frac{\partial q_i^C}{\partial K_j} = \frac{m_2}{m_6} > 0, i, j = 1, 2, i \neq j$, is $\beta > \frac{\omega}{2(\sigma+\omega)}$,which means that the spillover parameter has to be above a certain threshold value for the capital stock to have a positive impact on competitor's output.

- Each player's value function in (13) involves six coefficients, three of them ($z_5^C$, $z_6^C$ and $z_4^C$) being common to both players. The system of equations providing them is nonlinear and its solution is a priori not unique. In the numerical simulations, we chose one solution satisfying the condition of global asymptotic stability of the steady state (to be given below) and check if everything makes sense (e.g., quantities and costs are nonnegative). Note that in all numerical experiments, it turned out that only one solutions satisfy these requirements.

The following proposition characterizes the steady state.

**Proposition 3.2** *If following condition holds*

$$\delta > \frac{z_5^C \pm z_4^C}{\phi}, \tag{18}$$

*then Cournot feedback equilibrium steady state is given by*

$$\bar{K}_i^C = \frac{-z_4^C(z_{2j}^C - \theta_j) + \left(z_5^C - \delta\phi\right)\left(z_{2i}^C - \theta_i\right)}{\left(z_4^C\right)^2 - \left(z_5^C - \delta\phi\right)^2}, i, j = 1, 2, i \neq j. \tag{19}$$

**Proof.** Substituting for the equilibrium values of $I_i^C$ (from (12)) into the state dynamics (5) yields a system of first order differential equations. This system is globally asymptotically stable and yields the steady state capital stocks in (19) if condition (18) holds. ■

Given (19), we can evaluate the output and investment strategies (using (10)-(12)) as well as consumer surplus ($CS^C$), profits ($\pi_i^C$) and total welfare ($TW^C$), given by (7)-(9), at the steady state. Again the superscript $C$ refers here to Cournot solution.

# 4 Bertrand Feedback Equilibrium

In the non-cooperative Bertrand game firms independently choose their price and R&D levels. The representative consumer's demand for product $i$ is derived from equation (2) and given by

$$q_i = \frac{(\omega - \eta)\, A - \omega P_i + \eta P_j}{\omega^2 - \eta^2}, \;\; i, j = 1, 2, i \neq j. \tag{20}$$

The production cost and investment costs, profit function and state equation are as defined previously.

The firms' feedback equilibrium price and R&D strategies are given in the following proposition, where the superscript $B$ stands for Bertrand equilibrium.

**Proposition 4.1** *The firms' Bertrand feedback equilibrium price and R&D investment strategies are given by*

$$P_1^B = \frac{1}{y_6}\left[y_1 K_1 + y_2 K_2 + y_3 A + y_4 c_1 + y_5 c_2\right] \tag{21}$$

$$P_2^B = \frac{1}{y_6}\left[y_2 K_1 + y_1 K_2 + y_3 A + y_5 c_1 + y_4 c_2\right] \tag{22}$$

$$I_i^B = \frac{1}{\phi}[z_{2i}^B + z_5^B K_i + z_4^B K_j - \theta_i], i, j = 1, 2, i \neq j, \tag{23}$$

*where the constants $y_1, \ldots, y_6$ are given by*

$$\begin{aligned}
y_1 &= \psi\left[\left(\eta^2\beta + 2\omega^2\right)\left(\eta^2 - \sigma\omega - \omega^2\right) - \sigma\omega\eta^2\beta\right] \\
y_2 &= \omega\psi\eta\left[(2\beta+1)\left(\eta^2 - \omega^2 - 2\sigma\omega\right)\right] \\
y_3 &= -\left[\left(\omega^2 - \eta^2\right)^2 + \omega(1-\eta)\left(\omega^2 - \eta^2\right)\right] - 2\sigma\omega\left[\left(2\sigma\omega - \eta^2\right) + \omega(3\omega - \eta)\right] \\
y_4 &= 2\omega^2\left(\sigma\omega - \eta^2 + \omega^2\right) \\
y_5 &= \eta\omega\left(2\sigma\omega - \eta^2 + \omega^2\right) \\
y_6 &= \left(\eta^2 - 2\omega^2\right)^2 + 4\sigma\omega\left(2\omega^2 - \eta^2\right) + \omega^2\left(4\sigma^2 - \eta^2\right)
\end{aligned}$$

*and the coefficients $z_{ij}^B$ solve the system of equations given in Appendix 2.*

*The value functions are quadratic and given by*

$$V^i(K_i, K_j) = z_{1i}^B + z_{2i}^B K_i + z_{3i}^B K_j + z_4^B K_i K_j + \frac{z_5^B}{2} K_i^2 + \frac{z_6^B}{2} K_j^2, i, j = 1, 2, i \neq j. \tag{24}$$

**Proof.** Similar to that of Proposition 3.1 and it is hence omitted. ■

The following comments apply to the results in the above proposition:

- Under the assumptions made before, namely that $0 \leq \beta \leq 1$ and $\eta \leq \omega$, it is easy to verify that $y_1$ is negative and $y_6$ is positive and hence $\frac{\partial P_i^B}{\partial K_i} = \frac{y_1}{y_6} < 0$, i.e., each player's price is a decreasing function of her own capital stock. Similarly, it is easy to check that $\frac{\partial P_i^B}{\partial K_j} = \frac{x_2}{x6} < 0, i, j = 1, 2, i \neq j$, and hence each player's price is a decreasing function of competitor's capital stock.
- Each player's value function in (24) involves six coefficients, three of them ($z_5^B, z_6^B$ and $z_4^B$) being common to both players. The same comment on the chosen solution in the numerical experiments made in the Cournot case applies here.

The following proposition characterizes the steady state.

**Proposition 4.2** *If the following condition holds*

$$\delta > \frac{z_5^B \pm z_4^B}{\phi}, \tag{25}$$

*then Bertrand feedback equilibrium steady state is given by*

$$\bar{K}_i^B = \frac{-z_4^B(z_{2j}^B - \theta_j) + \left(z_5^B - \delta\phi\right)(z_{2i}^B - \theta_i)}{\left(z_4^B\right)^2 - \left(z_5^B - \delta\phi\right)^2}, i, j = 1, 2, i \neq j. \tag{26}$$

**Proof.** Substituting for the equilibrium values of $I_i^B$, from (23), into the dynamics (5) yields a system of first order differential equations. This system is globally asymptotically stable and yields the steady state capital stocks in (26), provided that condition (25) is satisfied. ■

Given (26), we can evaluate the price and investment strategies (using (21)-(23)) as well as consumer surplus ($CS^B$), profits ($\pi_i^B$) and total welfare ($TW^B$), given by (7)-(9), at the steady state. Again the superscript $B$ refers here to Bertrand solution.

## 5 First-Best Equilibrium

We determine in this section the socially optimal allocation, referred to as first-best solution, which will serve as a benchmark for Bertrand and Cournot equilibria. To obtain this allocation, it is well known that the social planner solves an optimization problem where the objective is defined as the difference between consumer's utility and producers' costs, i.e.,

$$J = \max_{(q_1,q_2,I_1,I_2)} \int_{t=0}^{\infty} e^{-rt}[U(q_1,q_2) - \sum_{i=1}^{2} C_i(q_i,K_1,K_2) - \sum_{i=1}^{2} F_i(I_i)]dt \tag{27}$$

$$\text{subject to (5).} \tag{28}$$

where $U(q_i,q_j)$, $C_i(q_i,K_i,K_j)$ and $F_i(I_i)$ are given by (1), (3), and (4).

The firms' socially optimal price and R&D strategies are given in the following proposition, where the superscript $S$ stands for social optimum.

**Proposition 5.1** *The socially optimal output and investment strategies are given by*

$$q_1^S = \frac{1}{v_6}\left[v_1K_1 + v_2K_2 + v_3A + v_4c_1 + v_5c_2\right] \tag{29}$$

$$q_2^S = \frac{1}{v_6}\left[v_1K_2 + v_2K_1 + v_3A + v_4c_2 + v_5c_1\right] \tag{30}$$

$$I_i^S = \frac{z_{2i}^S + z_5^S K_i + z_4^S K_j - \theta_i}{\phi}, i,j = 1,2, i \neq j \tag{31}$$

*where the constants* $v_1,\ldots,v_6$ *are given by*

$$\begin{aligned} v_1 &= \psi\left(2\sigma\omega + \omega^2 - \eta^2\beta\right) \\ v_2 &= \eta\psi\left(2\beta\sigma + \omega\eta - \omega\right) \\ v_3 &= \left(2\sigma + \omega - \eta\right)\omega \\ v_4 &= -\left(2\sigma + \omega\right)\omega \\ v_5 &= \eta \\ v_6 &= \left(4\sigma(\omega+\sigma) + \omega^2 - \eta^2\right)\omega. \end{aligned}$$

*where the coefficients* $z_{ij}^S$ *solve the system of equations given in Appendix 3.*

*The value function is quadratic and given by*

$$V(K_i, K_j) = z_1^S + z_2^S K_i + z_3^S K_j + z_4^S K_i K_j + \frac{z_5^S}{2} K_i^2 + \frac{z_6^S}{2} K_j^2 \; . \tag{32}$$

**Proof.** The Hamilton-Jacobi-Bellman equation of the optimization problem is

$$rV(K_i, K_j) = \max_{q_i, q_j, I_i, I_j} \left\{ U(q_i, q_j) - \sum_{i=1}^{2} C_i(q_i, K_i, K_j) \right. \\ \left. - \sum_{i=1}^{2} C_i(I_i) + \sum_{i=1}^{2} \frac{\partial V(.)}{\partial K_i}(I_i - \delta K_i) \right\}, i, j = 1, 2, i \neq j \tag{33}$$

Differentiating the right-hand side with respect to $q_i, i = 1, 2$, and equating to zero leads to the results in (29)-(30). The maximization with respect to $I_i$ yields

$$I_i(K_i, K_j) = 1/\phi(\partial V/\partial K_i - \theta_i), i, j = 1, 2, i \neq j$$

Substituting for outputs and investments in (33) yields

$$rV(K_i, K_j) = U(q_i, q_j) - \sum_{i=1}^{2} C_i(q_i, K_i, K_j) - \sum_{i=1}^{2} [\frac{\theta_i}{\phi}(\frac{\partial V}{\partial K_i} - \theta_i)] \\ - \sum_{i=1}^{2} [\frac{1}{2\phi}(\frac{\partial V}{\partial K_i} - \theta_i)^2] + [\frac{\partial V(.)}{\partial K_i}] \left[ \frac{1}{\phi}(\frac{\partial V}{\partial K_i} - \theta_i) - \delta K_i \right] \\ + [\frac{\partial V}{\partial K_j}] \left[ \frac{1}{\phi}(\frac{\partial V}{\partial K_j} - \theta_j) - \delta K_j \right].$$

The following quadratic value function solves the above partial differential equation

$$V(K_i, K_j) = z_1 + z_2 K_i + z_3 K_j + z_4 K_i K_j + \frac{z_5}{2} K_i^2 + \frac{z_6}{2} K_j^2, i, j = 1, 2, i \neq j \tag{34}$$

where the coefficients are given in Appendix 3. ■

The following comments apply to the results in the above proposition:

- Under the assumptions made before, namely that $0 \leq \beta \leq 1$ and $\eta \leq \omega$, it is easy to verify that $v_1$ and $v_6$ are positive and hence $\frac{\partial q_i^S}{\partial K_i} = \frac{v_1}{v_6} > 0$, i.e. each player's output is an increasing function of her own capital stock.
- A condition for having $\frac{\partial q_i^S}{\partial K_j} = \frac{v_2}{v_6} > 0, i, j = 1, 2, i \neq j$, is $\beta > \frac{\omega(1-\eta)}{2\sigma}$ which means, as in the Cournot scenario, that the spillover rate has to be above a certain threshold value to increase own output when the capital stock of the other player increases.

- Each player's value function in (13) involves six coefficients, three of them ($z_5^C$, $z_6^C$ and $z_4^C$) are common to both players. These coefficients satisfy the system of equations given in Appendix 3. As one can expect, this solution is not unique and the previous remarks regarding the numerical simulations applies here.

The following proposition characterizes the steady state.

**Proposition 5.2** *If the following condition holds*

$$\delta > \frac{1}{2\phi}[z_5 + z_6 \pm \sqrt{z_5 - 2z_5 z_6 + 4z_4^2}], \tag{35}$$

*then social optimum steady state capital stocks are given by*

$$\bar{K}_1^S = \frac{-z_4^S(z_3^S - \theta_2) + \left(z_6^S - \delta\phi\right)\left(z_2^S - \theta_1\right)}{\left(z_4^S\right)^2 - \left(\delta\phi - z_6^S\right)\left(\delta\phi - z_5^S\right)}, \tag{36}$$

$$\bar{K}_2^S = \frac{-z_4^S(z_2^S - \theta_1) + \left(z_5^S - \delta\phi\right)\left(z_3^S - \theta_2\right)}{\left(z_4^S\right)^2 - \left(\delta\phi - z_6^S\right)\left(\delta\phi - z_5^S\right)}. \tag{37}$$

**Proof.** Substituting for the equilibrium values of $I_i^P$, from (31), into the dynamics (5) yields a system of first order differential equations. This system is globally asymptotically stable and yields the steady state capital stocks in (36)-(37), provided that condition (35) is satisfied. ■

Given (36)-(37), we can evaluate the output and investment strategies (using (29)-(31)) as well as consumer surplus ($CS^S$), profits ($\pi_i^S$) and total welfare ($TW^S$), given by (7)-(9), at the steady state. Again the superscript $S$ refers here to Social optimum.

## 6 Comparison

As highlighted before, the proposed value functions for Cournot and Bertrand equilibria as well as for the first-best optimum have multiple solutions. Further, these value functions are not amenable to analytical comparison. For this reason, we shall compare numerically the strategies, profits, consumer surplus and total welfare at the steady state. We shall check that each derived solution is an interior one, satisfies the conditions for asymptotic stability and the costs of production and investment are positive.

Recall that the model includes the following thirteen parameters

| | |
|---|---|
| Demand function: | $A, \omega, \eta$ |
| Investment cost functions: | $\theta_1, \theta_2, \phi$ |
| Production cost functions: | $c_1, c_2, \sigma, \psi, \beta$ |
| Depreciation and discount rates: | $\delta, r.$ |

Varying all these parameters in an ordered manner, e.g., one at a time, poses no conceptual difficulty but will induce a huge number of tables to report. Instead, we choose to fix once for all the values of the following parameters which do not play an essential role with respect to the objectives of the paper:

$$A = 100, \quad \psi = 0.4, \quad \delta = 0.09, \quad r = 0.6, \quad \omega = 2, \quad \sigma = 1.$$

Further, we organize the simulations into two sets where in the first one we consider a symmetric setting, i.e., the players have the same investment and production costs, and in the second an asymmetric one.

## 6.1 Symmetric Games

Assume that the players have the same cost parameters, i.e., $\theta_1 = \theta_2 = 3$ and $c_1 = c_2 = 5$. Three parameters, namely $\beta, \eta$ and $\phi$ will be varied in turn to assess the impact of spillover, products substitutability and investment cost on steady state and payoffs.

Table 1 provides the results for various values of the spillover parameter $\beta$ for a given "high" $\phi$ ($\phi = 6$) and a given high level of products substitutability ($\frac{\eta}{\omega} = \frac{1.8}{2}$). Note that a high value for $\phi$ corresponds to a low productivity of investment in process R&D. In Table 2 the level of products substitutability is given a low value ($\frac{\eta}{\omega} = \frac{0.3}{2}$). From these results we note the following:

- Prices are the lowest under first-best followed by Bertrand and Cournot. The same ordering applies to consumer surplus and total welfare. Hence, if consumer has a say with respect to the solution to be chosen, his preference would be clear cut. This is by no mean a surprising result.
- Firms make the highest profits in the Cournot equilibrium. The ordering between first-best and Bertrand depends on the values of $\beta$ and $\eta$. If the spillover parameter is high ($\beta = 0.9$) and the degree of products substitutability is also high ($\frac{\eta}{\omega} = \frac{1.8}{2}$), then first-best leads to higher profits than Bertrand. In all other cases, we obtain the reverse. Note that when $\frac{\eta}{\omega}$ is low, profits under Bertrand and Cournot are very close. This could be explained by the fact that when the products are only far substitutes, each firm faces little competition and therefore in this near monopolistic situation it does not matter to play Bertrand or Cournot from profits perspective.
- In all cases, first-best leads to the highest steady-state levels of R&D capital, followed by Cournot and Bertrand. Note that when the degree of products substitutability is low, Cournot and Bertrand steady-state stocks are very close to each other and almost do not vary with the degree of spillover.

Table 1: Equilibrium values for symmetric games with low R&D productivity and high products substitutability

| Values of parameters | $A = 100,\ \phi = 6,\ \delta = .09,\ \psi = .4,\ \theta_1 = 3,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 5, \omega = 2,\ \sigma = 1$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | |
| Values of variables at equilibria | Cournot | Bertrand | First Best | Cournot | Bertrand | First Best | Cournot | Bertrand | First Best |
| $K_1$ | 5.8349 | 2.99634 | 34.68377 | 7.8366 | 5.4376 | 21.2713 | 9.3949 | 7.3678 | 12.9899 |
| $K_2$ | 5.8349 | 2.99634 | 34.68377 | 7.8366 | 5.4376 | 21.2713 | 9.3949 | 7.3678 | 12.9899 |
| $I_1$ | 0.52514 | 0.26967 | 3.12154 | 0.7053 | 0.4894 | 1.9144 | 0.8455 | 0.6631 | 1.16910 |
| $I_2$ | 0.52514 | 0.26967 | 3.12154 | 0.7053 | 0.4894 | 1.9144 | 0.8455 | 0.6631 | 1.16910 |
| $q_1$ | 12.7211 | 15.7232 | 20.70880 | 12.7260 | 15.8508 | 18.3744 | 12.6613 | 15.8490 | 17.2752 |
| $q_2$ | 12.7211 | 15.7232 | 20.70880 | 12.7260 | 15.8508 | 18.3744 | 12.6613 | 15.8490 | 17.2752 |
| $P_1$ | 51.6599 | 40.25186 | 21.30655 | 51.6411 | 39.7669 | 30.1772 | 51.8871 | 39.7736 | 34.3543 |
| $P_2$ | 51.6599 | 40.25186 | 21.30655 | 51.6411 | 39.7669 | 30.1772 | 51.8871 | 39.7736 | 34.3543 |
| $TC_1$ | 174.094 | 292.7529 | 50.97538 | 174.9381 | 285.8015 | 233.6083 | 180.7152 | 287.037 | 302.6534 |
| $TC_2$ | 174.094 | 292.7529 | 50.97538 | 174.9381 | 285.8015 | 233.6083 | 180.7152 | 287.037 | 302.6534 |
| $\pi_1$ | 483.075 | 340.1349 | 390.2578 | 482.2479 | 344.5363 | 320.8807 | 476.2423 | 343.337 | 290.8239 |
| $\pi_2$ | 483.075 | 340.1349 | 390.2578 | 482.2479 | 344.5363 | 320.8807 | 476.2423 | 343.337 | 290.8239 |
| $\pi_1 + \pi_2$ | 966.150 | 680.2698 | 780.5156 | 964.4958 | 689.0726 | 641.7615 | 952.4846 | 686.674 | 581.6478 |
| $CS$ | 614.939 | 939.4318 | 1629.646 | 615.4178 | 954.7443 | 1282.9519 | 609.1701 | 954.531 | 1134.039 |
| $TW$ | **1581.089** | **1619.7016** | **2410.161** | **1579.9136** | **1643.8169** | **1924.7134** | **1561.654** | **1641.20** | **1715.687** |

Table 2: Equilibrium values for symmetric games with low R&D productivity and low products substitutability

| Values of parameters | $A = 100,\ \phi = 6,\ \delta = .09,\ \psi = .4,\ \theta_1 = 3,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 5, \omega = 2,\ \sigma = 1$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{0.3}{2}$ | | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | | |
| Values of variables at equilibria | Cournot | Bertrand | First Best | Cournot | Bertrand | First Best | Cournot | Bertrand | First Best |
| $K_1$ | 11.4465 | 11.4386 | 24.5182 | 11.4516 | 11.4442 | 22.0605 | 11.4564 | 11.4496 | 20.1771 |
| $K_2$ | 11.4465 | 11.4386 | 24.5182 | 11.4516 | 11.4442 | 22.0605 | 11.4564 | 11.4496 | 20.1771 |
| $I_1$ | 1.0302 | 1.0293 | 2.2066 | 1.0306 | 1.0299 | 1.9854 | 1.3011 | 1.0305 | 1.8159 |
| $I_2$ | 1.0302 | 1.0293 | 2.2066 | 1.0306 | 1.0299 | 1.9854 | 1.3011 | 1.0305 | 1.8159 |
| $q_1$ | 15.9042 | 16.0181 | 24.6817 | 15.8501 | 15.9636 | 24.2683 | 15.8068 | 15.92 | 23.9699 |
| $q_2$ | 15.9042 | 16.0181 | 24.6817 | 15.8501 | 15.9636 | 24.2683 | 15.8068 | 15.92 | 23.9699 |
| $P_1$ | 63.4202 | 63.1584 | 43.2321 | 63.5448 | 63.2837 | 44.1829 | 63.6445 | 63.3839 | 44.8691 |
| $P_2$ | 63.4202 | 63.1584 | 43.2321 | 63.5448 | 63.2837 | 44.1829 | 63.6445 | 63.3839 | 44.8691 |
| $TC_1$ | 256.0905 | 259.753 | 479.083 | 259.7943 | 263.4663 | 501.0764 | 262.7342 | 266.413 | 516.2919 |
| $TC_2$ | 256.0905 | 259.753 | 479.083 | 259.7943 | 263.4663 | 501.0764 | 262.7342 | 266.413 | 516.2919 |
| $\pi_1$ | 752.5604 | 751.923 | 587.958 | 747.3959 | 746.7699 | 571.167 | 743.2784 | 742.660 | 559.2184 |
| $\pi_2$ | 752.5604 | 751.923 | 587.958 | 747.3959 | 746.7699 | 571.167 | 743.2784 | 742.660 | 559.2184 |
| $\pi_1 + \pi_2$ | 1505.1208 | 1503.84 | 1175.916 | 1494.791 | 1493.539 | 1142.335 | 1486.556 | 1485.32 | 1118.4368 |
| $CS$ | 581.7734 | 590.131 | 1401.127 | 577.8172 | 586.1243 | 1354.585 | 574.6635 | 582.929 | 1321.4859 |
| $TW$ | **2086.8942** | **2093.97** | **2577.143** | **2072.609** | **2079.664** | **2496.920** | **2061.220** | **2068.25** | **2439.9226** |

In Tables 3 and 4, we provide the same type of results with a lower value for $\phi$ ($\phi = 2$)[2]. The main difference with the above results is that now Cournot produces a higher total welfare than Bertrand when the degree of products substitutability is high ($\frac{\eta}{\omega} = \frac{1.8}{2}$) provided that the spillover rate is high ($\beta = 0.9$) or moderate ($\beta = 0.4$)[3]. When the spillover rate is near zero, then we recover again the result that total welfare is higher under Bertrand. This shows that the investment cost has a significant effect on the results, not only quantitatively speaking but also qualitatively.

Table 3: Equilibrium values for symmetric games with high R&D productivity and high products substitutability

| Values of parameters | $A = 100,\ \phi = 2,\ \delta = .09,\ \psi = .4,\ \theta_1 = 3,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 5, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 21.6643 | 10.52185 | 27.93956 | 18.75377 | 32.47696 | 25.07075 |
| $K_2$ | 21.6643 | 10.52185 | 27.93956 | 18.75377 | 32.47696 | 25.07075 |
| $I_1$ | 1.94979 | 0.94697 | 2.51456 | 1.68784 | 2.92293 | 2.25637 |
| $I_2$ | 1.94979 | 0.94697 | 2.51456 | 1.68784 | 2.92293 | 2.25637 |
| $q_1$ | 14.19038 | 16.60482 | 14.12809 | 17.02299 | 13.84497 | 16.99487 |
| $q_2$ | 14.19038 | 16.60482 | 14.12809 | 17.02299 | 13.84497 | 16.99487 |
| $P_1$ | 46.07655 | 36.90167 | 46.31325 | 35.31265 | 47.38911 | 35.41949 |
| $P_2$ | 46.07655 | 36.90167 | 46.31325 | 35.31265 | 47.38911 | 35.41949 |
| $TC_1$ | 59.39401 | 235.98945 | 69.37557 | 209.13992 | 98.36335 | 215.23062 |
| $TC_2$ | 59.39401 | 235.98945 | 69.37557 | 209.13992 | 98.36335 | 215.23062 |
| $\pi_1$ | 594.44981 | 376.75624 | 584.94229 | 391.98689 | 557.7375 | 386.71906 |
| $\pi_2$ | 594.44981 | 376.75624 | 584.94229 | 391.98689 | 557.7375 | 386.71906 |
| $\pi_1 + \pi_2$ | 1188.8996 | 753.75624 | 1169.88458 | 783.97378 | 1115.4750 | 773.43812 |
| $CS$ | 765.19442 | 1047.73679 | 758.49138 | 1101.17174 | 728.3963 | 1097.53737 |
| $TW$ | **1954.0940** | **1801.24927** | **1928.37596** | **1885.14552** | **1843.8713** | **1870.97549** |

## 6.2 Asymmetric Games

We now turn to the case where the players differ in terms of their production and investment costs. We adopt the following parameters' values

$$c_1 = 5, \quad c_2 = 7, \quad \theta_1 = 2, \quad \theta_2 = 3,$$

assuming hence that player 1 has both a lower investment and production cost.

In Tables 5 and 6, the results are reported for a low investment productivity ($\phi = 6$) and various spillover rates $\beta$ and products substitutability parameter $\frac{\eta}{\omega}$. It is straightforward to

[2]We present from now on only the results for Bertrand and Cournot equilibria. The results for the first-best are clear-cut without any surprise.

[3]Our numerical experiments show that for even lower values for $\beta$ one still has the same qualitative result. For instance for $\beta = 0.2$, whereas total welfare is 1892.593 in Cournot equilibrium, it is equal to 1887.084 in Bertrand equilibrium. Profits are, as in all other experiments, higher under Cournot than under Bertrand and consumer surplus is higher under Bertrand.

see that the results are qualitatively similar to those obtained in the symmetric case. The main (expected) difference is that the cost-advantaged player is doing better, profit-wise, than her competitor.

Table 4: Equilibrium values for symmetric games with high R&D productivity and low products substitutability

| Values of parameters | $A = 100,\ \phi = 2,\ \delta = .09,\ \psi = .4,\ \theta_1 = 3,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 5, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{0.3}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 41.23081 | 41.19829 | 40.68892 | 40.69951 | 40.28109657 | 40.25403817 |
| $K_2$ | 41.23081 | 41.19829 | 40.68892 | 40.69951 | 40.28109657 | 40.25403817 |
| $I_1$ | 3.71077 | 3.70785 | 3.6620 | 3.65936 | 3.625298693 | 3.622863433 |
| $I_2$ | 3.71077 | 3.70785 | 3.6620 | 3.65936 | 3.625298693 | 3.622863433 |
| $q_1$ | 18.0506 | 18.17810 | 17.81779 | 17.94399 | 17.63689502 | 17.76204880 |
| $q_2$ | 18.0506 | 18.17810 | 17.81779 | 17.94399 | 17.63689502 | 17.76204880 |
| $P_1$ | 58.48362 | 58.190367 | 59.01907 | 58.72778 | 59.43514145 | 59.14728775 |
| $P_2$ | 58.48362 | 58.190367 | 59.01907 | 58.72778 | 59.43514145 | 59.14728775 |
| $TC_1$ | 103.0940 | 106.20189 | 123.56461 | 126.72778 | 139.0898391 | 142.2966752 |
| $TC_2$ | 103.0940 | 106.20189 | 123.56461 | 126.72778 | 139.0898391 | 142.2966752 |
| $\pi_1$ | 952.5704 | 951.58849 | 928.0251 | 927.10157 | 909.1615109 | 908.2803368 |
| $\pi_2$ | 952.5704 | 951.58849 | 928.0251 | 927.10157 | 909.1615109 | 908.2803368 |
| $\pi_1 + \pi_2$ | 1905.1408 | 1903.17698 | 1856.0502 | 1854.20312 | 1818.3230218 | 1816.5606736 |
| $CS$ | 749.3956 | 760.01957 | 730.1897 | 740.56973 | 715.438152 | 725.627869 |
| $TW$ | **2654.5364** | **2663.19655** | **2586.2399** | **2594.77285** | **2533.761174** | **2542.188542** |

Table 5: Equilibrium values for asymmetric games with low R&D productivity and high products substitutability

| Values of parameters | $A = 100,\ \phi = 6,\ \delta = .09,\ \psi = .4,\ \theta_1 = 2,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 7, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 7.8774498 | 5.046532165 | 9.947360497 | 7.58726556 | 11.57937346 | 9.627724094 |
| $K_2$ | 5.5616956 | 2.736031195 | 7.433123903 | 5.021198298 | 8.859348795 | 6.784316611 |
| $I_1$ | 0.7089704 | 0.4541878949 | 0.8952624456 | 0.68285390 | 1.042143611 | 0.8664951688 |
| $I_2$ | 0.5005526 | 0.2462428079 | 0.6689811520 | 0.45190784 | 0.7973413915 | 0.6105884951 |
| $q_1$ | 12.934043 | 16.08785539 | 12.97209033 | 16.28019724 | 12.94297207 | 16.34950605 |
| $q_2$ | 12.415948 | 15.24460026 | 12.34265115 | 15.25038592 | 12.20773163 | 15.13347395 |
| $P_1$ | 51.783204 | 40.38400876 | 51.83904727 | 39.98891073 | 52.14013893 | 40.06073481 |
| $P_2$ | 51.886823 | 40.55265978 | 51.96493511 | 40.19487299 | 52.28718701 | 40.30394125 |
| $TC_1$ | 170.82363 | 294.0489846 | 171.8304307 | 288.0300744 | 177.6292606 | 290.0758969 |
| $TC_2$ | 184.01010 | 298.4207067 | 187.7115049 | 294.0032162 | 195.5210960 | 296.8384604 |
| $\pi_1$ | 498.94258 | 355.6431084 | 500.6303731 | 362.9972797 | 497.2191013 | 364.8973292 |
| $\pi_2$ | 460.21404 | 319.7883811 | 453.6735612 | 318.9841089 | 442.7868507 | 313.1001846 |
| $\pi_1 + \pi_2$ | 959.15663 | 675.4314895 | 954.3039343 | 681.9813886 | 940.0059520 | 677.9975138 |
| $CS$ | 610.50443 | 932.6721922 | 608.8141389 | 944.5218209 | 600.9570303 | 941.6930640 |
| $TW$ | **1569.6610** | **1608.103682** | **1563.118073** | **1626.503210** | **1540.962982** | **1619.690578** |

Decreasing the value of $\phi$ to 2 (Tables 7 and 8), leads also to the same qualitative results as in the symmetric case. Again, we obtain that Cournot is more efficient, i.e., produces a higher total welfare, than Bertrand when the products substitutability rate is high and the spillover rate is high ($\beta = 0.9$) or moderate ($\beta = 0.4$). Actually, this last reult still holds

Table 6: Equilibrium values for asymmetric games with low R&D productivity and low products substitutability

| Values of parameters | $A = 100,\ \phi = 6,\ \delta = .09,\ \psi = .4,\ \theta_1 = 2,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 7, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{0.3}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 13.45664526 | 13.44885444 | 13.46503742 | 13.45782922 | 13.47276976 | 13.46602092 |
| $K_2$ | 11.07515611 | 11.06718015 | 11.06695735 | 11.05953989 | 11.06114481 | 11.05416641 |
| $I_1$ | 1.211098073 | 1.210396900 | 1.211853368 | 1.211204630 | 1.212549279 | 1.211941883 |
| $I_2$ | 0.9967640503 | 0.9960462136 | 0.9960261611 | 0.9953585897 | 0.9955030331 | 0.9948749771 |
| $q_1$ | 16.05227744 | 16.16738402 | 16.00068918 | 16.11550760 | 15.95954636 | 16.07413095 |
| $q_2$ | 15.55683967 | 15.66799246 | 15.49162284 | 15.60237642 | 15.43943233 | 15.54986184 |
| $P_1$ | 63.22839322 | 62.96483425 | 63.35113479 | 63.08827189 | 63.44907758 | 63.18677955 |
| $P_2$ | 64.07063743 | 63.81379991 | 64.21654757 | 63.96059489 | 64.33327143 | 64.07803703 |
| $TC_1$ | 248.7553487 | 252.4020073 | 252.4251266 | 256.0811606 | 255.3330603 | 258.9968003 |
| $TC_2$ | 276.6617598 | 280.3875060 | 280.8116824 | 284.5476563 | 284.1005663 | 287.8448856 |
| $\pi_1$ | 766.2043613 | 765.5746477 | 761.2366904 | 760.6183644 | 757.2854347 | 756.6757687 |
| $\pi_2$ | 720.0748743 | 719.4466298 | 714.0068526 | 713.3896212 | 709.1686245 | 708.5597372 |
| $\pi_1 + \pi_2$ | 1486.279236 | 1485.021278 | 1475.243543 | 1474.007986 | 1466.454059 | 1465.235506 |
| $CS$ | 574.6076839 | 582.8634282 | 570.3754250 | 578.5757995 | 567.0050912 | 575.1610432 |
| $TW$ | **2060.886920** | **2067.884706** | **2045.618968** | **2052.583786** | **2033.459150** | **2040.396549** |

Table 7: Equilibrium values for asymmetric games with high R&D productivity and high products substitutability

| Values of parameters | $A = 100,\ \phi = 2,\ \delta = .09,\ \psi = .4,\ \theta_1 = 2,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 7, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{1.8}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 28.506168 | 17.21645979 | 35.35324635 | 26.2590336 | 40.78967653 | 33.92397885 |
| $K_2$ | 21.312721 | 10.04574688 | 26.61204263 | 17.22540662 | 29.84343264 | 21.87807479 |
| $I_1$ | 2.5655551 | 1.549481381 | 3.181792171 | 2.36331303 | 3.671070887 | 3.053158094 |
| $I_2$ | 1.9181449 | 0.9041172196 | 2.395083836 | 1.55028659 | 2.685908939 | 1.969026730 |
| $q_1$ | 14.666568 | 17.30048053 | 14.71661672 | 17.96001118 | 14.62173236 | 18.33763553 |
| $q_2$ | 14.060210 | 16.31405645 | 13.70762907 | 16.28845737 | 13.10304247 | 15.69485974 |
| $P_1$ | 45.358484 | 36.03373760 | 45.89303423 | 34.76075441 | 47.17105883 | 35.07398119 |
| $P_2$ | 45.479755 | 36.23102242 | 46.09483176 | 35.09506515 | 47.47479681 | 35.60253635 |
| $TC_1$ | 31.641810 | 215.8576892 | 42.14115715 | 189.4798534 | 69.15632896 | 194.5509210 |
| $TC_2$ | 55.820086 | 227.3198838 | 81.07524960 | 212.5656151 | 122.2669467 | 228.6274608 |
| $\pi_1$ | 633.61149 | 407.5432866 | 633.2490378 | 434.8236844 | 620.5662684 | 448.6229626 |
| $\pi_2$ | 583.63486 | 363.7550612 | 550.7756062 | 359.0788575 | 499.7973322 | 330.1493536 |
| $\pi_1 + \pi_2$ | 1217.2463 | 771.2983478 | 1184.024644 | 793.9025419 | 1120.363601 | 778.7723162 |
| $CS$ | 783.98482 | 1073.488884 | 767.5917642 | 1114.449421 | 730.3453041 | 1100.649418 |
| $TW$ | **2001.2311** | **1844.787232** | **1951.616408** | **1908.351963** | **1850.708905** | **1879.421734** |

true provided that the spillover parameter is not close to zero. Indeed, if we let $\beta = 0.2$, then total welfare would be equal to 1907.136 in Cournot equilibrium and to 1902.416 in Bertrand equilibrium. Ranking of profits and consumer surplus are the same as in all other experiments.

Table 8: Equilibrium values for asymmetric games with high R&D productivity and low products substitutability

| Values of parameters | $A = 100,\ \phi = 2,\ \delta = .09,\ \psi = .4,\ \theta_1 = 2,\ \theta_2 = 3,\ r = 0.6,\ c_1 = 5,\ c_2 = 7, \omega = 2,\ \sigma = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\beta = 0.9,\ \frac{\eta}{\omega} = \frac{0.3}{2}$ | | $\beta = 0.4,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | | $\beta = 0,\ \frac{\eta}{\omega} = \frac{.3}{2}$ | |
| Values of variables at equilibria | Cournot | Bertrand | Cournot | Bertrand | Cournot | Bertrand |
| $K_1$ | 48.45285085 | 48.41961795 | 47.82626857 | 47.79746190 | 47.45453796 | 47.42820490 |
| $K_2$ | 40.15112275 | 40.11682688 | 39.33904842 | 39.30891718 | 38.80647554 | 38.77862366 |
| $I_1$ | 4.360756564 | 4.357765622 | 4.304364168 | 4.301771569 | 4.270908419 | 4.268538444 |
| $I_2$ | 3.613601060 | 3.610514424 | 3.540514357 | 3.537802548 | 3.492582801 | 3.490076130 |
| $q_1$ | 18.54059412 | 18.67188777 | 18.30918498 | 18.43929980 | 18.13795917 | 18.26713483 |
| $q_2$ | 17.68578747 | 17.81021391 | 17.39844905 | 17.52122857 | 17.18020041 | 17.30164719 |
| $P_1$ | 57.61307552 | 57.31316030 | 58.16209532 | 57.86503187 | 58.57002154 | 58.27523622 |
| $P_2$ | 59.06624682 | 58.77800587 | 59.71034641 | 59.42575297 | 60.19821143 | 59.91656519 |
| $TC_1$ | 64.65746901 | 67.62114333 | 86.35807756 | 89.37647226 | 102.1664456 | 105.2304209 |
| $TC_2$ | 130.1707686 | 133.3792247 | 153.9061164 | 157.1760098 | 171.4153611 | 174.7357597 |
| $\pi_1$ | 1003.523180 | 1002.523754 | 978.5424844 | 977.6141987 | 960.1742134 | 959.2911761 |
| $\pi_2$ | 914.4623194 | 913.4696333 | 884.9613036 | 884.0361912 | 862.8019759 | 861.9195123 |
| $\pi_1 + \pi_2$ | 1917.985499 | 1915.993387 | 1863.503788 | 1861.650390 | 1822.976189 | 1821.210688 |
| $CS$ | 754.912211 | 765.608206 | 733.497710 | 743.924982 | 717.628981 | 727.850666 |
| $TW$ | **2672.897710** | **2681.601593** | **2597.001498** | **2605.575372** | **2540.605170** | **2549.061354** |

# 7 Concluding Remarks

The numerical experiments (both symmetric and asymmetric) suggest the following:

- Bertrand prices (quantities) are lower (higher) than their Cournot counterparts. Consumer surplus under Bertrand competition is always higher than under Cournot competition. These results are expected in view of those in, e.g., Refs. 1–3.
- Firms' profits under Bertrand competition are always lower than those of the Cournot competition. Note however that they are very close when the rate of products substitutability is low.
- Firms' capital stocks and process R&D investments under Cournot competition are higher than their Bertrand counterparts. The same result has been obtained by Qiu (Ref. 11).
- Bertrand competition is more efficient than Cournot if either R&D productivity is low, or products are very different. Cournot competition is more efficient provided that R&D productivity is high, products are close substitutes and R&D spillovers are not close to zero.

Comparing our results to Qiu's ones, we obtain that his conclusions still hold true with one important difference however. Indeed, our simulations show that for any but not close to zero spillover rate (and not only for strong spillovers as in his case), Cournot is more efficient than Bertrand provided that R&D productivity is high and the products are close substitute.

Given the important role played by the spillover rate, a natural extension to this work is to allow for endogenous R&D spillovers by letting, e.g., the firm invests in acquiring knowledge from its competitor's R&D.

# Appendix 1

The coefficients of the value functions in Cournot game are the solutions of the following nonlinear system of equations

$$
\begin{aligned}
-z_5^2 - 2z_4^2 + nz_5 + C1 &= 0\\
-z_4^2 - 2z_6 z_5 + nz_6 + C2 &= 0\\
-2z_4 z_5 - z_6 z_4 + nz_4 + C3 &= 0\\
(-z_{21} + \theta_1)z_5 + (-z_{22} + \theta_2 - z_{31})z_4 + mz_{21} + C4 &= 0\\
(\theta_2 - z_{22})z_6 + (-z_{21} + \theta_1)z_4 - z_{31}z_5 + mz_{31} + C5 &= 0\\
(-2z_{22} + 2\theta_2)z_{31} - z_{21}^2 - \theta_1^2 + 2\phi r z_{11} + 2z_{21}\theta_1 + C6 &= 0\\
(\theta_2 - z_{22})z_5 + (\theta_1 z_{21} - z_{32})z_4 + mz_{22} + C7 &= 0\\
(-z_{21} + \theta_1)z_6 + (\theta_2 - z_{22})z_4 + mz_{32} - z_{32}z_5 + C8 &= 0\\
(2\theta_1 - 2z_{21})z_{32} + 2z_{22}\theta_2 + 2\phi r z_{12} - z_{22}^2 - \theta_2^2 + C9 &= 0
\end{aligned}
$$

where

$$C1 = -\frac{2\phi\psi^2(\sigma+\omega)(-2\omega^2 - 2\sigma\omega + \eta^2\beta)^2}{\omega^2(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C2 = -\frac{2\phi\eta^2\psi^2(\sigma+\omega)(-\omega + 2\omega\beta + 2\beta\sigma)^2}{\omega^2(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C3 = \frac{2\phi\psi^2\eta(\sigma+\omega)(-\omega + 2\omega\beta + 2\beta\sigma)(\eta^2\beta - 2\omega^2 - 2\sigma\omega)}{\omega^2(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C4 = -\frac{2\phi\psi(\sigma+\omega)(-2\omega A + 2\omega c_1 + A\eta - 2A\sigma - \eta c_2 + 2\sigma c_1)(\eta^2\beta - 2\omega^2 - 2\sigma\omega)}{\omega(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C5 = \frac{2\phi\eta\psi(\sigma+\omega)(-2\omega A + 2\omega c_1 + A\eta - 2A\sigma - \eta c_2 + 2\sigma c_1)(-\omega + 2\omega\beta + 2\beta\sigma)}{\omega(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C6 = -\frac{2\phi(\sigma+\omega)(-2\omega A + 2\omega c_1 + A\eta - 2A\sigma - \eta c_2 + 2\sigma c_1)^2}{(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C7 = \frac{2\phi\psi(\sigma+\omega)(-2\omega c_2 + 2\omega A - 2\sigma c_2 - A\eta + c_1\eta + 2A\sigma)(\eta^2\beta - 2\omega^2 - 2\sigma\omega)}{\omega(2\sigma+2\omega+\eta)^2(2\sigma+2\omega-\eta)^2}$$

$$C8 = -\frac{2\phi\eta\psi(\sigma+\omega)(-2\omega c_2 + 2\omega A - 2\sigma c_2 - A\eta + c_1\eta + 2A\sigma)(-\omega + 2\omega\beta + 2\beta\sigma)}{\omega(2\sigma + 2\omega + \eta)^2(2\sigma + 2\omega - \eta)^2}$$

$$C9 = -\frac{2\phi(\sigma+\omega)(-2\omega c_2 + 2\omega A - 2\sigma c_2 - A\eta + c_1\eta + 2A\sigma)^2}{(2\sigma + 2\omega + \eta)^2(2\sigma + 2\omega - \eta)^2}$$

$$n = \phi(r + 2\delta), \qquad m = \phi(r + \delta)$$

# Appendix 2

The coefficients of the value functions in Bertrand game are the solutions of the following nonlinear system of equations

$$\begin{aligned}
-z^2 + nz_5 - 2z_4^2 + C_1 &= 0\\
-z_4^2 + nz_6 - 2z_6z_5 + C_2 &= 0\\
-2z_5z_4 - z_6z_4 + nz_4 + C_3 &= 0\\
(\theta_1 - z_{21})z_5 + (-z_{22} + \theta_2 - z_{31})z_4 + mz_{21} + C_4 &= 0\\
(\theta_2 - z_{22})z_6 + mz_{31} + (-z_{21} + \theta_1)z_4 - z_{31}z_5 + C_5 &= 0\\
-2z_{31}z_{22} + 2z_{31}\theta_2 - \theta_1^2 - z_{21}^2 + 2\phi r z_{11} + 2\theta_i z_{21} + C_6 &= 0\\
(\theta_1 - z_{21} - z_{32})z_4 + mz_{22} + (-z_{22} + \theta_2)z_5 + C_7 &= 0\\
(\theta_2 - z_{22})z_4 - z_{32}z_5 + (-z_{21} + \theta_1)z_6 + mz_{32} + C_8 &= 0\\
2z_{32}\theta_1 + 2\theta_2 z_{22} - \theta_2^2 - z_{22}^2 + 2\phi r z_{12} - 2z_{32}z_{21} + C_9 &= 0
\end{aligned}$$

where

$$C_1 = \frac{2(\eta^2 - \sigma\omega - \omega^2)(\eta^2\beta + \eta^2 - 2\sigma\omega - 2\omega^2)^2\psi^2\omega\phi}{(-2\omega^2 - 2\sigma\omega - \eta\omega + \eta^2)^2(-2\omega^2 - 2\sigma\omega + \eta\omega + \eta^2)^2}$$

$$C_2 = \frac{2(\eta^2 - \sigma\omega - \omega^2)(\eta^2\beta + \omega^2 - 2\sigma\omega\beta - 2\omega^2\beta)^2\psi^2\eta^2\phi}{\omega(-2\omega^2 - 2\sigma\omega - \eta\omega + \eta^2)^2(-2\omega^2 - 2\sigma\omega + \eta\omega + \eta^2)^2}$$

$$C_3 = \frac{2(\omega^2 + \omega\sigma - \eta^2)(-\omega^2 + 2\omega^2\beta + 2\beta\omega\sigma - \eta^2\beta)(-2\omega\sigma - 2\omega^2 + \eta^2 + \eta^2\beta)\psi^2\eta\phi}{(2\omega\sigma - \eta^2 + 2\omega^2 + \omega\eta)^2(2\omega\sigma - \eta^2 + 2\omega^2 - \omega\eta)^2}$$

$$C_4 = -\frac{2(\omega^2 + \omega\sigma - \eta^2)\Phi(-2\omega\sigma - 2\omega^2 + \eta^2 + \eta^2\beta)\omega\phi\psi}{(2\omega\sigma - \eta^2 + 2\omega^2 + \omega\eta)^2(2\omega\sigma - \eta^2 + 2\omega^2 - \omega\eta)^2}$$

$$C_5 = \frac{2(\omega^2 + \omega\sigma - \eta^2)\Phi(-\omega^2 + 2\omega^2\beta + 2\sigma\beta\omega - \eta^2\beta)\omega\phi\psi}{(2\omega\sigma - \eta^2 + 2\omega^2 + \omega\eta)^2(2\omega\sigma - \eta^2 + 2\omega^2 - \omega\eta)^2}$$

$$C_6 = -\frac{2(\omega^2 + \omega\sigma - \eta^2)\Phi^2\omega\phi}{(2\omega\sigma - \eta^2 + 2\omega^2 + \omega\eta)^2(2\omega\sigma - \eta^2 + 2\omega^2 - \omega\eta)^2}$$

$$C_7 = -\frac{2(\omega^2 + \omega\sigma - \eta^2)\Psi(-2\omega^2 - 2\omega\sigma + \eta^2\beta)\omega\phi\psi}{(2\omega\sigma - \eta^2 + 2\omega^2 + \omega\eta)^2(2\omega\sigma - \eta^2 + 2\omega^2 - \omega\eta)^2}$$

$$C_8 = \frac{2(\omega^2+\omega\sigma-\eta^2)\Psi(2\omega^2\beta-\omega^2+2\sigma\beta\omega-\eta^2\beta)\phi\eta\psi}{(2\omega\sigma-\eta^2+2\omega^2+\omega\eta)^2(2\omega\sigma-\eta^2+2\omega^2-\omega\eta)^2}$$

$$C_9 = -\frac{2(\omega^2+\omega\sigma-\eta^2)\Psi^2\omega\phi}{(2\omega\sigma-\eta^2+2\omega^2+\omega\eta)^2(2\omega\sigma-\eta^2+2\omega^2-\omega\eta)^2}$$

$$n = \phi(r+2\delta), \qquad m = \phi(r+\delta)$$

$$\Phi = (-2\omega^2 A + 2\omega^2 c_1 + \omega\eta A + 2\omega\sigma c_1 - \omega\eta c_2 - 2\omega\sigma A - c_1\eta^2 + A\eta^2)$$

$$\Psi = (2\omega^2 c_2 - 2\omega^2 A - \omega c_1\eta + 2\omega\sigma c_2 + \omega\eta A - 2\omega\sigma A + A\eta^2 - c_2\eta^2)$$

# Appendix 3

The coefficients of the value function of social optimum are the solutions of the following nonlinear system of equations

$$-z_5^2 - z_4^2 + nz_5 + C_1 = 0$$
$$-z_4^2 - z_6^2 + nz_6 + C_2 = 0$$
$$(-z_6 + n - z_5)\,z_4 + C_3 = 0$$
$$(-z_2+\theta_1)z_5 + (-z_3+\theta_2)z_4 + mz_2 + C_4 = 0$$
$$(-z_3+\theta_2)z_6 + mz_3 + (-z_2+\theta_1)z_4 + C_5 = 0$$
$$-z_2^2 - z_3^2 - \theta_2^2 - \theta_1^2 + 2z_2\theta_1 + 2\phi r z_1 + 2z_3\theta_2 + C_6 = 0$$

where

$$C_1 = -\frac{(\omega^3+2\sigma\omega^2+\omega\eta^2\beta^2-2\omega\eta^2\beta+2\sigma\eta^2\beta^2)\psi^2\phi}{\omega^2(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$C_2 = -\frac{(\omega^3+2\sigma\omega^2+\omega\eta^2\beta^2-2\omega\eta^2\beta+2\sigma\eta^2\beta^2)\psi^2\phi}{\omega^2(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$C_3 = \frac{(\omega^2-2\omega^2\beta-4\sigma\beta\omega+\eta^2\beta^2)\phi\eta\psi^2}{\omega^2(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$C_4 = \frac{(-\Psi+\omega^2 c_1+c_2\eta\beta\omega-c_2\eta\omega-\eta\beta A\sigma\omega+2c_1\sigma\omega+2c_2\sigma\eta\beta-c_1\eta^2\beta)\phi\psi}{\omega(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$C_5 = -\frac{(-\omega^2 c_2+\Psi+\eta\beta A\omega-\eta\beta c_1\omega+c_1\eta\omega-2c_2\sigma\omega-2\eta\beta c_1\sigma+c_2\eta^2\beta)\phi\psi}{\omega(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$C_6 = \frac{2A(\omega+2\sigma-\eta)(A-c_1-c_2)+(\omega+2\sigma)(c_1^2+c_2^2)-2\eta c_1 c_2}{(2\sigma+\omega+\eta)(2\sigma+\omega-\eta)}$$

$$n = \psi(r+2\delta), \qquad m = \phi(r+\delta)$$

$$\Psi = -\omega^2 A + A\eta\omega - 2A\sigma\omega - 2\eta\beta A\sigma + \eta^2\beta A$$

## References

1. SINGH, N., and VIVES, X., *Price and Quantity Competition in a Differentiated Duopoly*, Rand Journal of Economics, Vol. 15, pp. 546–554, 1984.
2. CHENG, L., *Comparing Bertrand and Cournot Equilibria: A Geometric Approach*, Rand Journal of Economics, Vol. 16, No. 1, pp. 146–152, 1985.
3. VIVES, X., *On the Efficiency of Bertrand and Cournot Equilibria with Product Differentiation*, Journal Economic Theory, Vol. 36, pp. 166–175, 1985.
4. OKUGUCHI, K.,*Equilibrium Prices in Bertrand and Cournot Oligopolies*, Journal of Economic Theory, Vol. 42, pp. 128–139, 1987.
5. DASTIDAR, K. G., *Comparing Cournot and Bertrand in a Homogeneous Product Market*, Journal of Economic Theory, Vol. 75, pp. 205–212, 1997.
6. HACKNER, J., *A note on Price and Quantity Competition in Differentiated Oligopolies*, Journal of Economic Theory, Vol. 93, pp. 233–239, 2000.
7. AMIR, R., AND JIN, J. Y., *Cournot and Bertrand Equilibria Compared: Substitutability, Complementarity, and Concavity*, International Journal of Industrial Organization, Vol. 19, pp. 303–317, 2001.
8. TANAKA, Y., *Profitability of Price and Quantity Strategies in an Oligopoly*, Journal of Mathematical Economics, Vol. 35, pp. 409–418, 2001.
9. DELBONO, F., AND DENICOLO, V., *R&D in a Symmetric and Homogeneous Oligopoly*, International Journal of Industrial Organization, Vol. 8, pp. 297–313, 1990.
10. MOTTA, M., *Endogenous Quality Choice: Price Versus Quality Competition*, Journal of Industrial Economics, Vol. 41, pp. 113–131, 1993.
11. QIU, L., *On the Dynamic Efficiency of Bertrand and Cournot Equilibria*, Journal of Economic Theory, Vol. 75, pp. 213–229, 1997.
12. SYMEONIDIS, G., *Comparing Cournot and Bertrand Equilibria in a Differentiated Duopoly with Product R&D*, International Journal of Industrial Organization, Vol. 21, pp. 39–55, 2003.
13. REYNOLDS, S., *Dynamic Oligopoly with Capacity Adjustment Costs*, Journal of Economic Dynamics and Control, Vol. 15, pp. 491–514, 1991.
14. REYNOLDS, S., *Capacity Investment, Preemption, and Commitment in an Infinite Horizon Model*, International Economic Review, Vol. 28, pp. 69–88, 1987.
15. D'ASPREMONT, C., AND JACQUEMIN, A., *Cooperative and non Cooperative R&D in Duopoly with Spillovers*, American Economic Review, Vol. 78, pp. 1133–1137, 1988.
16. DOCKNER, E., JØRGENSEN, S., VAN LONG, N., AND SORGER, G., *Differential Games in Economics and Management Science*, Cambridge University Press, 2000.